The Modular Invariant Regularization Method and One-Loop Corrected Effective Action in the Closed Bosonic String in $D=26$

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We show the modular invariant regularization of 1-loop string amplitudes in the S-matrix approach and the 1-loop 2-point Green function in the closed bosonic string in $d=26$. Then we show that there is no mass-shift for massless antisymmetric tensor but that there are mass-shifts for graviton and dilaton, by using the 1-loop (torus-topology) 2-point string amplitudes in S-matrix approach. The graviton mass-shift and the dilaton mass-shift are due to the existence of the cosmological constant $C/\sqrt{G}$. These results agree with the result of Tseytlin who uses the factorization analysis or in the zero momentum limit.

§ 1. Introduction

One possible way to understand compactification and phenomenology of string theories is to study a low-energy effective action for massless modes. The most thorough study about tree-level (sphere topology) corrections to the effective Lagrangian in the bosonic string has been done by Tseytlin. Furthermore many attempts concerning the string loop effect have been made. There are three ways to derive the effective action. The first is the non-linear $\sigma$ model approach, the second is the Fischler-Klevanov-Susskind mechanism which corresponds to the limit of all the vertices coming close to each other in the string amplitude and the third is S-matrix approach in which the string tension $\rightarrow \infty$ limit. But it was pointed out that a non-linear $\sigma$ model approach will be insensitive to the string loop correction in the lower order of $\alpha'$. However, it was indicated that the regularization parameter in the string 2-dimensional world sheet has loop effect, namely, 2-dimensional metric dependence or moduli dependence. Thus it is necessary to examine the modular invariant regularization method to obtain loop corrected effective action.

We expect that the consideration of this moduli dependent part of observables will be useful for deriving the effective Lagrangian.

The bosonic string loop amplitudes are given by integrals over the moduli $\{r\}$ parameters and the Koba-Nielsen variables $\{\nu\}$. Singularities appear from the boundaries of the integration region and they are classified into two types; one is the quadratic (tachyon) infinity and the other is the logarithmic (dilaton) infinity. In the regularization we require that observables such as the string amplitudes should keep modular invariance and that their 2-dimensional metric dependence is determined by the modular invariance.

The purpose of this paper is to show how to regularize the observables and add local counterterms to them to keep modular invariance in the string one-loop level (torus-topology). For example, we take the 2-point Green function and regularize it in a modular invariant way. As the consistency check of the above procedure we
similarly regularize the 3-point string amplitude in S-matrix approach, whose modular invariance is ascertained before the \( \nu \)-integration. Then we show how to derive the effective Lagrangian in the S-matrix approach. The result is that there is the mass-shift for the massless dilaton and graviton, and that the no-mass shift for the antisymmetric tensor analyzing one-loop two-point “gravitons” (graviton, dilaton and antisymmetric tensor) amplitudes in the closed bosonic string by S-matrix approach. This agrees with the result which is stated by Tseytlin by the factorization analysis and non-linear \( \sigma \) model approach in the zero momentum limit.\(^7\) The plan of this paper is as follows. In \( \S \) 2 we explain why we should consider the modular invariant regularization in non-linear \( \sigma \) model approach. The moduli dependent counterterm is necessary in the point-splitting regularization method by requiring the 2-point Green function to be modular invariant and by comparing the \( \xi \)-functional regularization. In \( \S \) 3 we show regularization of the 3-point string amplitude. In \( \S \) 4 we consider one-loop 2-point amplitudes. In \( \S \) 5 we present our conclusion.

\section{2. Modular invariant 2-point Green function on the torus}

In this section we analyze a non-linear \( \sigma \) model in a curved Riemann space. A partition function on a curved Riemann surface is given by

\[
Z = \sum_{\text{2-dim. topology}} \int [d\gamma] \int [dX] \exp(-S[X, \gamma]) \frac{1}{N}.
\]

The action functional is given by\(^5\)

\[
S = \frac{1}{4\pi\alpha'} \int d^2\sqrt{\gamma} \left[ \partial_a X^i \partial_b X^j \gamma^{ab} g_{ij}(X) - \frac{\alpha'}{2} R_2(\gamma) \phi(X) \right],
\]

where the string coordinates \( X^i(\sigma), i=1, \cdots, 26, \sigma=(\sigma_1, \sigma_2) \) label a point on the world sheet and \( R_2 \) is a world sheet Ricci scalar corresponding to the world sheet metric \( \gamma^{ab}(\sigma), a, b=1, 2 \). We use coordinate gauge for the Weyl and reparametrization invariance for the path integration, where the modular invariance remains

\[
\gamma = \tilde{\gamma}(\tau) \exp 2\rho,
\]

where \( \tau = \tau_1 + i\tau_2 \) denotes the moduli parameter. Thus we obtain

\[
Z = \sum_{\text{2-dim. topology}} \int [d\tau] \int [d\rho] \exp(-W(\rho, \tau)) \frac{1}{N},
\]

where \( e^{-W} = \text{const} \left( \Pi d\xi \right) e^{-S} \) and \( X^i \) is expanded by the geodesic coordinates; \( X^i(\sigma) = X_0^i(\sigma) + \sqrt{2\pi a^2} \xi^i(\sigma) \). \( W(\rho, \tau) \) is given in terms of Green functions and their derivatives in the coincident limit of the Riemann surface. The usual regularization method against the infrared divergences in the coincident-limit of 2-dimensional surface does not keep the modular invariance. However, it has been pointed out that the conformal invariance condition \( (\delta W(\rho, \tau))/\delta \rho = 0 \) is insensitive to the string loop effect in the lower order of \( \alpha' \).\(^5\) But the \( \tau \)-integration will yield a new infinity. It has been suggested that the regularization of this infinity may be one possible way of incorporating the string loop effect in the low energy field theory. Thus it is neces-
sary to examine \( \tau \)-integral to get loop correction. Therefore \( W(\rho, \tau) \) should maintain modular invariance. Now we treat the string one-loop only. Green function on a torus is expressed in a manifestly modular invariant form,

\[
G(\nu_i - \nu_j | \tau |_{\nu_i + \nu_j}) = \frac{1}{2\tau_2} \left( \text{Im}(\nu_i - \nu_j) \right)^2 - \frac{1}{2\pi} \ln \left| \theta_1(\nu_i - \nu_j | \tau) \right| \\
= \frac{\tau_2}{4\pi^2} \sum_{k,m} |k\tau - m|^2 \exp \left[ 2\pi i k \left( \text{Re} \nu - \text{Re} \frac{\text{Im} \nu}{\text{Im} \tau} \right) \right] \\
\times \exp \left( 2\pi i m \frac{\text{Im} \nu}{\text{Im} \tau} \right) + \text{constant},
\]

(2.4)

where the prime omits \( k=m=0 \) case in a summation, \( \nu_i - \nu_j = \nu \), \( \nu = \sigma^1 + \tau \sigma^2 \) and \( 0 \leq \sigma^1, \sigma^2 \leq 1 \),

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \Delta_{12}^{1/24}, \]

where \( q = \exp 2\pi i \tau \). \( \Delta_{12} \) is a cusp form with weight 12.

In the coincident limit we regularize it by two methods; one is the \( \zeta \)-functional regularization\(^8\) and the other is the point-splitting regularization (\( \nu_i - \nu_j = \epsilon \)),

\[
G(\nu_i - \nu_j | \tau |, s) = \frac{\tau_2}{4\pi^2} \lim_{s \to 0} \sum_{k,m \neq 0} \frac{\tau_2^s}{|k\tau - m|^{2+2s}} + \text{constant} \\
= \frac{1}{4\pi^2} E(\tau, 1) + \text{constant}.
\]

(2.5)

\( E(\tau, 1) \) is given by "Kronecker's first limit formula",\(^{13}\)

\[
E(\tau, 1) = \lim_{s \to 0} \frac{1}{s + 2\pi \left( \gamma - \frac{1}{2} \ln \tau - \frac{1}{12} \ln |\Delta_{12}| \right)},
\]

(2.6)

where \( \gamma \) is the Euler constant. \( E(\tau, 1) \) is apparently modular invariant since

\[
\sqrt{\tau_2} \to \frac{1}{|\tau|} \sqrt{\tau_2},
\]

\[ |\Delta_{12}| \to |\tau|^{12} |\Delta_{12}| \]

under \( \tau \to -1/\tau \). For the point splitting method case, we obtain

\[
G(\nu_i - \nu_j | \tau |, \epsilon) = -\lim_{\epsilon \to 0} \frac{1}{2\pi} \ln \epsilon - \frac{1}{2\pi} \ln |\eta(\tau)|^2 - \frac{1}{2\pi} \ln |\epsilon|,
\]

(2.7)

where we use \( \theta_i(\epsilon | \tau) = 2\pi \epsilon \eta(\tau)^3 \). By comparing Eqs. (2.5) and (2.7), we get

\[
\lim_{\epsilon \to 0} \ln \epsilon = \lim_{s \to 0} \frac{1}{2\pi s} \left( \gamma - \frac{1}{2} \ln \tau_2 \right) + \text{constant},
\]

(2.8)

where \( \tau_2 \) represents an imaginary part of the moduli parameter (period matrix in the one-loop case). Thus when we normalize logarithmic infinity which is regularized by the point-splitting method in the observables, we should add \( \ln \sqrt{\tau_2} \) to 2-point Green function to keep the modular invariance. The 2-dimensional metric dependence of
the 2-point Green function in the regularization is determined by requiring that the 2-point Green function should maintain the modular invariance. The extension of this to the higher loop case will be easy by using the prime form. In the next section the same relation will be derived in the string 3-point amplitude.

§ 3. The one-loop 3-point amplitude

In this section we derive the same relation as Eq. (2.8) by requiring that the bosonic closed string 3-point graviton amplitude should be modular invariant in the S-matrix approach.

We review the $\nu$-integration in S-matrix approach. We follow mostly the convention of Ref. 9. One technique for $\nu$ integration of the modular invariant string amplitude is to find a weight of integrand under the Jacobi modular transformation,

$$\nu \rightarrow \frac{\nu}{\tau}, \quad \tau \rightarrow -\frac{1}{\tau}.$$  \hspace{1cm} (3.1)

For example, if integrand with an invariant measure is a holomorphic function with weight $n$ ($n$ is a positive integer) and has no poles or branch point contributing to $\nu$-integral, the result of $\nu$-integral becomes a modular form. The $\nu$-integrand for closed bosonic string consists of functions which are doubly periodic in $\nu$ with period 1 and $\tau$. They are given in terms of the Jacobi theta function $\theta_i(\nu | \tau)$,

$$\chi(\nu) = 2\pi \exp\left(\frac{-\pi (\text{Im} \nu)^2}{\text{Im} \tau}\right) \left| \frac{\theta_1(\nu | \tau)}{\theta_1'(0 | \tau)} \right|,$$  \hspace{1cm} (3.2)

(The relation between $\chi(\nu - \nu')$ and $G(\nu - \nu' | \tau)$ is given by $G(\nu - \nu' | \tau) = (-1/2\pi) \ln \chi(\nu - \nu') - (1/\pi) \ln|\eta(\tau)|$.)

$$K(\nu | \tau) = \frac{1}{i\pi} \frac{\partial}{\partial \nu} \ln \chi(\nu) = \frac{1}{2\pi i} \frac{\partial}{\partial \nu} \ln \theta_1(\nu | \tau) + \frac{\text{Im} \nu}{\text{Im} \tau},$$

$$T(\nu | \tau) = \frac{i}{2\pi} \frac{\partial^2}{\partial \nu^2} K(\nu | \tau) = \frac{1}{4\pi^2} \frac{\partial^2}{\partial \nu^2} \ln \theta_1(\nu | \tau) + \frac{1}{4\pi^2 \text{Im} \tau}.$$  \hspace{1cm} (3.3)

Their transformation properties are as follows:

$$K(\nu | \tau) \rightarrow \tau K(\nu | \tau),$$

$$T(\nu | \tau) \rightarrow \tau^2 T(\nu | \tau),$$

$$\chi(\nu) \rightarrow |\tau| \chi(\nu).$$  \hspace{1cm} (3.4)

The integration region is restricted to the fundamental region,

$$F_1 = \left\{ 0 \leq \text{Im} \nu \leq \text{Im} \tau, \quad -\frac{1}{2} \leq \text{Re} \nu \leq \frac{1}{2} \right\}.$$  \hspace{1cm} (3.5)

We precisely define the $\nu$-integral of functions with poles by the following regularization: If $f(\nu)$ has a pole at $\nu = a$, we define
\[
\int_{F_1} f(\nu) d^2 \nu = \lim_{\varepsilon \to 0} \int_{F_1}, f(\nu) d^2 \nu, \quad F_1' = \{ \nu \in F_1, \nu - a \geq \varepsilon \}.
\]

This \( \varepsilon \) is identified with the point-splitting parameter to regularize the divergences which appear where the vertices are close to each other in the string amplitude.

Now we examine the string amplitude. In the S-matrix approach we need the following integration in the expansion of the 3-graviton 1-loop amplitude:

\[
\int_{F_1} K(\nu|\tau) K(\nu|\tau) \frac{d^2 \nu}{\tau_2}.
\]

We integrate it by regularizing its logarithmic divergence by two methods; the \( \zeta \)-functional regularization and the point splitting regularization as before. By the \( \zeta \)-functional regularization, we obtain

\[
\int_{F_1} K(\nu|\tau) K(\nu|\tau) \frac{d^2 \nu}{\tau_2} = \frac{1}{4 \pi^2 \lim_{s \to 0} \sum_{k,m \in Z} s \zeta \left( \frac{1}{(k \tau - m)^{1+2s}} \right)}
\]

\[
= \frac{1}{4 \pi^2 \tau^2} E(\tau, 1).
\]

To prove Eq. (3.8), we use the following expansion derived by Schellekens.\(^{11}\)

\[
K(\nu|\tau) = \lim_{s \to 0} \frac{1}{2\pi i} \sum_{k,m} \frac{1}{(k \tau - m)^{1+s}}
\]

\[
\times \exp \left[ 2\pi ik (\text{Re}\nu - \text{Re}\tau) \right] \exp \left( 2\pi i m \frac{\text{Im}\nu}{\text{Im}\tau} \right).
\]

In the case of point splitting regularization

\[
i\pi \int_{F_1}, K(\nu|\tau) K(\nu|\tau) \frac{d^2 \nu}{\tau_2} = -\frac{1}{2i\tau} \int_{F_1}, \ln \chi(\nu) \frac{d^2 \nu}{\tau_2} - \int_{F_1}, \frac{\partial}{\partial \nu} [K(\nu|\tau) \ln \chi(\nu)] \frac{d^2 \nu}{\tau_2},
\]

where

\[
\int_{F_1}, \frac{\partial}{\partial \nu} [K(\nu|\tau) \ln \chi(\nu)] \frac{d^2 \nu}{\tau_2} = -\frac{1}{2i\tau} \ln \varepsilon.
\]

If \( \ln \varepsilon \) has no moduli dependent counterterm, the first term on the right-hand side and the left-hand side of Eq. (3.10) transform in the same way. Apparently there is a difference in their transformation properties,

\[
\int_{F_1} K(\nu|\tau) K(\nu|\tau) \frac{d^2 \nu}{\tau_2} \rightarrow |\tau|^2 \int_{F_1} K(\nu|\tau) K(\nu|\tau) \frac{d^2 \nu}{\tau_2},
\]

\[
\int_{F_1} \frac{1}{\tau_2} \ln \chi(\nu) \frac{d^2 \nu}{\tau_2} \rightarrow |\tau|^2 \int_{F_1} \frac{1}{\tau_2} \ln \chi(\nu) \frac{d^2 \nu}{\tau_2} + |\tau|^2 \ln |\tau|,
\]

where \( \rightarrow \) denotes the Jacobi modular transformation. This difference is due to the singular term which is given by \( \ln \varepsilon \). The result of \( \nu \)-integral of Eq. (3.10) is given by

\[
\int_{F_1} \ln \chi(\nu) \frac{d^2 \nu}{\tau_2} = \frac{1}{2} \ln 2 - 2 \ln |\tau| |\tau|^{1/24}.
\]
In this case the expansion of \( \ln \chi(\nu) \) which we use is derived from the following formula:\(^{14}\)

\[
\ln \left[ \frac{\theta(\nu | r)}{\theta(0 | r)} \right] = \ln(\sin \pi \nu) + 4 \sum_{m=1}^{\infty} \frac{q^m}{1-q^m} \frac{\sin 2m\pi \nu}{m},
\]

\[
\ln \chi(\nu) = \frac{1}{2} \ln 2 - \frac{\pi \nu^2}{\tau_2} + \frac{1}{2} \ln(\cosh \pi y - \cos 2\pi x)
\]

\[+ \sum_{m=1}^{\infty} \text{Re} \left( \frac{q^m}{m(1-q^m)} \right) \times 1 - \frac{\cos 2m\pi xcosh 2m\pi y}{2}
\]

\[+ 4 \sum_{m=1}^{\infty} \text{Im} \left( \frac{q^m}{m(1-q^m)} \right) \times \frac{\sin 2m\pi xsinh 2m\pi y}{2},
\]

where \( \nu = x + iy \). It is straightforward to integrate \( \ln \chi(\nu) \) by using the following formula:

\[
\int_{0}^{\pi} \ln(a + bcot t) \, dt = \pi \ln(a + \sqrt{a^2 - b^2}) \quad (a \leq |b|)
\]  

\[(3.15)\]

Substituting these results into Eq. (3.10), and comparing them with Eq. (3.8) we again obtain

\[
\lim_{\nu \to 0} \ln \epsilon = \lim_{s \to 0} \frac{1}{2\pi s} - \left( \nu - \frac{1}{2} \ln t \right) + \frac{1}{2} \ln 2.
\]  

\[(3.16)\]

This agreement of the local counterterms in the point-splitting regularization shows the consistency of the two regularization methods.

### § 4. The one-loop 2-point amplitude

The one-loop \( N \)-point massless "graviton" (including antisymmetric tensor and dilaton) scattering amplitude for the closed bosonic string in the operator formalism is given by\(^{15,9}\)

\[
T_{1-loop}(1, 2, \cdots, N) = C_{\text{torus}} \left[ \frac{k}{2\alpha'} \right]^N \int_{F_3} d^2 \tau \int_{F_1} \prod_{i=1}^{N-1} d^2 \nu_i \cdot t_2^{-12} |\eta(\tau)|^{-48}
\]

\[\times t_2^N \prod_{i,j} \text{exp} \left[ \frac{1}{2} \left( ik_i + \xi_i \frac{2}{\sqrt{2d}} \partial_{\nu_i} + \bar{\xi}_i \frac{2}{\sqrt{2d}} \bar{\partial}_{\nu_i} \right) \right]
\]

\[\times \left( ik_j + \xi_j \frac{2}{\sqrt{2d}} \partial_{\nu_j} + \bar{\xi}_j \frac{2}{\sqrt{2d}} \bar{\partial}_{\nu_j} \right) | - \alpha' \ln \chi(\nu_i) \right] \right.
\]

\[
(4.1)
\]

with the understanding that "gravitons" are given by terms linear in \( \xi \) and \( \bar{\xi} \). The vertex operator is given by

\[
V_0(k, \xi, z, \bar{z}) = 4 \xi_\mu(k) : \partial_\rho X^\sigma \bar{\partial}_z X^\sigma \exp ikX(z, \bar{z}) :.
\]

where \( k^2 = 0 \) and \( k^\mu \xi_\nu = k^\nu \xi_\mu = 0 \). \( C_{\text{torus}} \) is the overall normalization and can be determined by the factorization analysis of tree amplitude and by the sewing procedure in the operator formalism,\(^9\) \( C_{\text{torus}} = (2\alpha')^N/2^{24} \). Rewriting it in terms of
\( \chi(\nu), K(\nu) \) and \( T(\nu) \), we have

\[
T_{\text{1-loop}}(1, 2, \cdots, N) = C_{\text{torus}}\cdot(2\alpha' k)^N \int_{F_1} \frac{d^2 \tau}{\tau_2^2} \int_{F_1} \prod_{i=1}^{N-1} \frac{d^2 \nu_i}{\tau_2} \cdot \tau_2^{-12} |\Delta_{12}|^{-2} \times \prod_{1 \leq i < j \leq N} \chi_{ij}^{a_i k_i} \tau_2^2 \exp\left[ -\pi \left( \frac{1}{\tau_2} - \delta^2(0) \right) \sum_{i=1}^{N} \xi_i \xi_i \right] \times \exp\left[ -\pi \left( \frac{1}{\tau_2} - \delta^2(\nu_{ij}) \right) \sum_{i,j=1,i+j}^{N} \xi_i \xi_j \right] \left\{ + \pi \sum_{i=1}^{N} \xi_i k_i K(\nu_i) + \frac{4\pi^2}{2\alpha'} \sum_{i<j}^{N} \xi_i \xi_j T(\nu_{ij}) \right\} + c.c. \right. , \quad \text{(4.2)}
\]

where we regularize \( \partial_\nu \delta_\nu \ln \chi(\nu_{\nu}) = \lim_{\nu \to \nu} (1/2)(\partial_\nu \delta_\nu + \delta_\nu \partial_\nu) \ln \chi(\nu_{\nu})^{\alpha_1} \) and use \( \delta_\nu \partial_\nu \ln \chi(\nu) = (1/2)(\delta^2(\nu) - (1/\tau_2)) \). In the Polyakov approach, we take the vertex operator as follows \(^7\) to relate the operator formalism:

\[
V_{\nu}(k, \xi, \nu, \bar{\nu}) = \frac{4k}{2\alpha'} \xi \nu(k) \cdot \partial_\nu X^x \delta_\nu X^x \exp ikX(\nu, \bar{\nu}),
\]

where \( k^2 = 0 \) and \( k^\nu \xi_\nu = k^x \xi_\nu = 0 \). We take the 2-point Green function in this case \( -\alpha' \ln \chi(\nu - \nu') \) to make its short distance behavior coincident with that of \( -\alpha' \ln |\nu - \nu'| \) (the 2-point Green function on a sphere). Two-point amplitude is given by\(^5\)

\[
T_{\text{1-loop}}(1, 2) = C_{\text{torus}}\cdot(2\alpha' k)^2 \xi_1 \xi_2 \int_{F_2} \frac{d^2 \tau}{\tau_2^2} I_{F_1} \frac{d^2 \nu_1}{\tau_2} \cdot \tau_2^{-12} |\Delta_{12}|^{-2} (D + G + A), \quad \text{(4.3)}
\]

where

\[
D = \frac{\pi^2}{4\alpha'}(1 - \tau_2 \delta^2(0))^{\eta^{i_1, i_2}},
\]

\[
G = \frac{\pi^2}{8\alpha'}[(1 - \tau_2 \delta^2(\nu_{12}))^2 + 16\pi^2(\tau_2 T(\nu_{12} T(\nu_{12}))^2)(\eta^{i_1, i_2} \eta^{j_1, j_2} + \eta^{i_1, i_2} \eta^{j_1, j_2})]_{|i_1, i_2, j_1, j_2},
\]

\[
A = \frac{\pi^2}{8\alpha'}[1 - (1 - \tau_2 \delta^2(\nu_{12}))^2 + 16\pi^2(\tau_2 T(\nu_{12} T(\nu_{12}))^2)(\eta^{i_1, i_2} \eta^{j_1, j_2} - \eta^{i_1, i_2} \eta^{j_1, j_2})]_{|i_1, i_2, j_1, j_2}.
\]

\( D, G, A \) are the factors contributing to the dilaton, graviton and antisymmetric tensor field one-loop amplitude, respectively. The absence of terms as \( \chi_{12}^{a_i k_i} \) and \( \sqrt{2\alpha'} k^i \xi_i K(\nu_{12}) \) are due to the momentum conservation.

The string amplitude contains two types of divergences: One is the quadratic (tachyon) infinity and the other is the logarithmic (dilaton) infinity. We assume that loop amplitudes are defined by using some analytic regularization in which the tachyon singularities do not appear. Equation (4.3) contains the quadratic tachyon infinity: For example, \( \int T_{ij} \bar{T}_{ij} d^2 \nu_i = \lim_{n \to 1} \int T_{ij} \bar{T}_{ij} d^2 \nu_i = \lim_{n \to 1} \int_{1/16\pi^2} (\tau_2 \delta^2(\nu_{ij}) - 1) \) where \( T_{ij} = T(\nu_{ij} T(\nu_{ij}) \). We have adopted the similar assumption for this divergent integral in the closed bosonic string as that in the open string\(^2\) to drop the divergent part. The assumption is that the integral in Eq. (4.3) can be represented as.
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\[ \int_{F_1} T_{ij} \tau_i \tau_j d^2 \nu_i = -\lim_{j \to j'} \partial \nu_j \delta_j \int_{F_1} d^2 \nu_i \left( \frac{1}{2 \pi} \right)^2 K_{ij} K_{ij'}, \]

and

\[ \int_{F_1} \partial \nu_i \delta \nu_i \ln \chi \cdot \delta \nu_i \partial \nu_i \ln \chi d^2 \nu_i = -\lim_{j \to j'} \partial \nu_j \delta_j \int_{F_1} d^2 \nu_i \pi^2 K_{ij} K_{ij'} . \]

By using the following equation derived from Eqs. (3.9) and (2.4):

\[ \int_{F_1} K_{ij} K_{ij'} d^2 \nu_i \tau_i = \frac{1}{\tau_2} \left( G_{ij} + \text{constant} \right), \]

we obtain

\[ \tau_2^2 \int_{F_1} T_{ij} \tau_i \tau_j d^2 \nu_i - \frac{1}{16 \pi^2} \quad \text{and} \quad 4 \tau_2^2 \int_{F_1} \partial \nu_i \delta \nu_i \ln \chi \cdot \partial \nu_i \delta \nu_i \ln \chi d^2 \nu_i = -1 . \quad (4.4) \]

Substituting this into Eq. (4.3) we find that there is no mass correction for the antisymmetric tensor. That agrees with the gauge invariance. But for the graviton and dilaton, mass shifts exist (for the dilaton, we drop the divergent part similar to the graviton case).

\[ T_{1 \text{-loop}} = C_{\text{torus}} \cdot \kappa^2 \pi^2 \left[ \xi^4 \xi^2 \xi^2 - 2 \xi^4 \xi^2 \xi^2 \right] \left[ \int_{F_2} d^2 \tau \frac{\tau_2^{-12}}{\tau_2^2 |\partial \tau^2|^2} \right]. \]

The graviton's and dilaton's mass terms come from the weak field expansion of the cosmological constant term,

\[ L_G = C_{\text{torus}} \cdot \sqrt{G} \exp(2 \sqrt{3}) \kappa D \left[ \int_{F_2} d^2 \tau \frac{\tau_2^{-12}}{\tau_2^2 |\partial \tau^2|^2} \right]. \quad (4.5) \]

These results agree with the result given by Tseytlin\textsuperscript{2} who uses the factorization analysis and nonlinear \( \sigma \)-model approach in the zero momentum limit. The value of \( \tau \)-integration in Eq. (4.5) is given by,\textsuperscript{16} where it becomes a finite complex value after being regularized. The imaginary part of the mass gives the decay rate of the particles, namely, to tachyon in this case.

\section{Conclusion}

In this paper, we have shown how to regularize observables in the modular invariant way. The 2-point Green function in the coincident limit becomes modular invariant \( E(\tau, 1) \) in the bosonic closed string one-loop case. The same result is derived in the string 3-point graviton amplitude in the S-matrix approach by imposing the modular invariance. We expect that the consideration of this moduli dependent part of observables will be useful for deriving the low energy effective Lagrangian and the extension of this consideration to the higher loops will be easy.

We also have shown how to derive the field theory effective Lagrangian from one-loop (torus topology) bosonic string by S-matrix approach.

The result of S-matrix approach is that there is no mass shift for massless antisymmetric tensor and that there are mass-corrections for graviton and dilaton.
from one-loop two-point amplitude. They indicated the existence of the cosmological constant. They agree with the result derived by Tseytlin who uses the factorization analysis and non-linear $\sigma$-model approach in the zero momentum limit. The correction from three-point amplitudes will also be derived in the $S$-matrix approach in the same way.

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