For $1 < D < 2$, the domain extension factor for microcontact size distribution, ψ, can be determined iteratively by solving Eq. (B9) with a bisection method. The numerical results based on an increment of D equal to 0.01 are presented in Table B1. When D increases from 1 to 2, ψ decreases monotonically from 2.618 to 1.718.

Using $a = a'/2$ and Eq. (B5), the real contact area of a fractal domain, A_{rf}, is given by

$$A_{rf} = \int_0^{a_b'} \frac{a'}{2}\, n_e(a')da' = \frac{D}{2(2-D)}\, \psi^{(2-D)/2} a_L' \qquad \text{(B16)}$$

Using the relation $a_L' = 2a_L$ and Eq. (B16), the relationship for the area ratio, a_L/A_{rf}, is found to be

$$\frac{a_L}{A_{rf}} = \frac{2-D}{D}\, \psi^{-(2-D)/2} \qquad \text{(B17)}$$

For the limiting condition $D \to 1$, substituting Eq. (B11) into Eq. (B17) yields

$$\frac{a_L}{A_{rf}} = \frac{\sqrt{5}-1}{2} \approx 0.618034 \qquad \text{(B18)}$$

The number given by Eq. (B18) is the reciprocal of an important mathematical constant, known as the golden ratio.

When $D \to 2$, ψ approaches a finite value equal to $e - 1$ (refer to Eq. (B15)) while the exponent of ψ in Eq. (B17) approaches zero. Thus, the factor, $\psi^{-(2-D)/2}$, approaches a value of unity. Consequently, the area ratio, a_L/A_{rf}, approaches zero, and the asymptotic behavior of this ratio can be represented by the following relationship,

$$\frac{a_L}{A_{rf}} \approx \frac{2-D}{D} \qquad \text{for } D \to 2 \qquad \text{(B19)}$$

Majumdar and Bhushan (1991) derived a relationship for a_L/A_{rf} which takes the same form as Eq. (B19) for the entire range of D, i.e., $1 < D < 2$. Because the conditions given by Eqs. (B1) and (B2) were not satisfied by their size distribution function, as mentioned earlier, the delta function in the originally discrete size distribution was not adequately represented. Consequently, their theory predicts that $a_L/A_{rf} \to 1$ as $D \to 1$. However, it is shown in the following derivation that the limiting value of a_L/A_{rf} must be less than unity.

Using relationship $a = a'/2$ and Eq. (B5), the ratio of the partial real contact area comprising microcontacts with areas between zero and any value of $a' < a_L'$ (denoted by $A(0, a')$) to the largest microcontact area, a_L, can be expressed as

$$\frac{A(0, a')}{a_L} = \frac{2}{a_L'} \int_0^{a'} \frac{a'}{2}\, n_e(a')da' = \frac{D}{2-D}\left(\frac{a'}{a_L'}\right)^{(2-D)/2} \qquad \text{(B20)}$$

For the limiting condition $D \to 1$, Eq. (B20) gives

$$\frac{A(0, a')}{a_L} \to \left(\frac{a'}{a_L'}\right)^{1/2} > 0 \qquad \text{(B21)}$$

The above relationship indicates that the contribution of microcontacts with areas between zero and any value of a' ($a' < a_L'$) to the real contact area is not negligible when $D \to 1$. Therefore, it can be concluded that the limiting value of a_L/A_{rf} for $D \to 1$ must be less than unity by a non-negligible amount. This requirement is satisfied by the relationship for a_L/A_{rf} given by Eq. (B17).

DISCUSSION

B. Bhushan[1] and A. Majumdar[2]

We are pleased to see the fractal analysis of surface roughness of homogeneous and composite surfaces and elastic/plastic contacts developed by us (Majumdar and Bhushan, 1990, 1991; Majumdar et al., 1991; Bhushan and Majumdar, 1992; Oden et al., 1992) is being extended by the subject authors for thermal analysis. The elastic/plastic referred to as Majumdar and Bhushan model (1991) is a significant improvement over the Greenwood and Williamson model. The result that there exists a critical contact spot area, a_c, below which the contact spots deform plastically is a result of our analysis. This is a significant result from our paper since it is radically different from what is obtained from the Greenwood-Williamson theory, which has normally been used. The power-law relation between the elastic load and contact spot area presented by us has been used by the subject authors. We would like to state, although our model predicts a relationship between real area of contact (friction) and the fractal dimensions, it has not been proved experimentally. Before authors use the fractal approach for the temperature calculations should they first validate the model?

Additional References

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Authors' Closure

The authors would like to thank Professors B. Bhushan and A. Majumdar for their discussion. The main objective of the present thermal analysis is to obtain the statistical temperature distribution at sliding interfaces of rough surfaces based on the theory of fractal geometry (Mandelbrot, 1983). In analyzing the frictional temperature rises at elastic microcontacts, some analytical treatments from the elastic-plastic contact model of the discussers (Majumdar and Bhushan, 1991) were adopted, with necessary modifications, to determine the asperity deformation and microcontact load. The present thermal analysis, however, is a self-contained theory that gives the relationship between the statistical distribution of the interfacial temperature rise and the fractional contact area for given fractal parameters, apparent contact area, material properties, friction coefficient, and sliding speed, whereas the discussers' contact model gives the relationship between the external load and the fractional contact area, among other contact mechanics relationships. The introduction of the distribution density function of the temperature rise in the present analysis facilitates a more quantitative understanding of the effects of frictional heating at sliding interfaces than the knowledge of the maximum and average temperature rises given by conventional theories.

Three points were raised in the discussion regarding contact mechanics issues in the present analysis, i.e., the relationship between the load and microcontact area, the critical microcontact area, and the validation of the relationship between the real contact area and the fractal dimension. Before addressing the first point, the fundamental difference between the actual (deformed) microcontact area, a, and the truncated microcontact area, a', must be emphasized (refer to Fig. 2 for the difference in the corresponding radii, r and r'). For an elastically deforming microcontact, the Hertz theory gives $a = a'/2$. This relationship was observed throughout the pres-