On Rephasing Invariants $\Delta_{ia}$ in Three Generations of Quarks

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Assuming that there are three generations of quarks, we express $|\text{Im}(\Delta_{ia})|^2$ and the ratio of $|U_{ij}|$ in terms of $\text{Re}\Delta_{ia}$ to find relations among $\Delta_{ia}$ and further we examine a restriction on $\Delta_{ia}$ according to the present data, where $\Delta_{ia} = U_{ia} U_{as}^* U_{sa}^*$ are rephasing invariants for the quark mixing matrix $U$.

The six quark mixing scheme serves as a useful parametrization of the connection between generations of quarks. In this scheme, the weak currents are given by

$$ (\bar{u}c\frac{1}{\sqrt{2}}(1-\gamma_5)U_{i\alpha}(d). $$

The $3 \times 3$ matrix $U$ must be unitary, due to the constraint of universality of the charged weak interaction. As usual $U$ is defined by

$$ U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}, $$

where $U_{11} = U_{ud}$, $U_{12} = U_{us}$, and others. Then the unitarity relations read

$$ \sum_a U_{ia} U_{ia}^* = \delta_{ii}, \quad \sum_i U_{ia} U_{ib}^* = \delta_{ab}. $$

The original Kobayashi-Maskawa matrix

$$ U = \begin{pmatrix} c_1 & s_1 c_3 & -s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\alpha} & c_1 s_2 s_3 + s_2 c_3 e^{i\alpha} \\ s_1 s_2 & c_1 s_2 c_3 + s_2 s_3 e^{i\alpha} & c_1 c_2 s_3 - c_2 c_3 e^{i\alpha} \end{pmatrix} $$

is well known as one of parametrizations of $U$, where $s_i = \sin \theta_i$ and $c_i = \cos \theta_i$.

On the other hand, $|U_{12}|$ in (2) is known from measurements on weak decays of light and heavy quarks and the unitarity relation (3), although a $t$ quark has not been discovered. For example, Particle Data Group gives the quark mixing matrix in absolute values as follows,

$$ |U| = \begin{pmatrix} 0.9748 \text{ to } 0.9761 & 0.217 \text{ to } 0.223 & 0.003 \text{ to } 0.010 \\ 0.217 \text{ to } 0.223 & 0.9733 \text{ to } 0.9754 & 0.030 \text{ to } 0.062 \\ 0.001 \text{ to } 0.023 & 0.029 \text{ to } 0.062 & 0.9980 \text{ to } 0.9995 \end{pmatrix}, $$

which is almost the same as in others. Since the quark mixing matrix $U$ suffers from a rephasing ambiguity, it is necessary to examine rephasing invariants defined
by
\[ \Delta_{ia} = U_{ij} U_{kj} U_{ki}^* U_{ji}^* \]  \hfill (6)

with \(i, j, k\) and \(\alpha, \beta, \gamma\) cyclic. Directly from (3) and (6), we have
\[
0 = U_{11} U_{12}^* (U_{12}^* U_{21} + U_{22}^* U_{32} + U_{32}^* U_{13})
= |U_{11} U_{12}|^2 + \Delta_{33} + \Delta_{33}^*
\]
\hfill (7)

and others, which leads to that \(\text{Im}(\Delta_{ia})\) are the same up to a sign.

Then, in this paper, we express \([\text{Im}(\Delta_{ia})]^2\) and the ratio of \(|U_{ij}|\) in terms of \(\text{Re}\Delta_{ia}\) to find relations among \(\Delta_{ia}\) and further we examine a restriction on \(\Delta_{ia}\) according to the present analysis (5). If we combine (7) with \(|U_{11}/U_{22}| = (U_{11} U_{12})/(U_{22} U_{13})| = |(U_{11} U_{21})/(U_{22} U_{21})|\) and others, we have
\[
|U_{11}/U_{22}|^2 = \text{Re}(\Delta_{23} + \Delta_{33})/\text{Re}(\Delta_{31} + \Delta_{33}) = \text{Re}(\Delta_{32} + \Delta_{33})/\text{Re}(\Delta_{33} + \Delta_{33})
\]
\[
|U_{12}/U_{21}|^2 = \text{Re}(\Delta_{31} + \Delta_{33})/\text{Re}(\Delta_{31} + \Delta_{33}) = \text{Re}(\Delta_{23} + \Delta_{33})/\text{Re}(\Delta_{23} + \Delta_{33})
\]
\[
|U_{13}/U_{31}|^2 = \text{Re}(\Delta_{23} + \Delta_{32})/\text{Re}(\Delta_{32} + \Delta_{32}) = \text{Re}(\Delta_{21} + \Delta_{22})/\text{Re}(\Delta_{21} + \Delta_{22})
\]
\[
|U_{23}/U_{32}|^2 = \text{Re}(\Delta_{23} + \Delta_{31})/\text{Re}(\Delta_{23} + \Delta_{31}) = \text{Re}(\Delta_{12} + \Delta_{11})/\text{Re}(\Delta_{12} + \Delta_{11})
\]
\[
|U_{33}/U_{11}|^2 = \text{Re}(\Delta_{21} + \Delta_{22})/\text{Re}(\Delta_{21} + \Delta_{22}) = \text{Re}(\Delta_{23} + \Delta_{22})/\text{Re}(\Delta_{23} + \Delta_{22})
\]
\hfill (8)

The relations among \(\Delta_{ia}\) are written from (8),
\[
\Delta_{21} \Delta_{31} - \Delta_{12} \Delta_{13} = (\Delta_{12} + \Delta_{13} - \Delta_{21} - \Delta_{31}) \Delta_{31}^*,
\]
\[
\Delta_{12} \Delta_{32} - \Delta_{21} \Delta_{23} = (\Delta_{21} + \Delta_{23} - \Delta_{12} - \Delta_{32}) \Delta_{32}^*,
\]
\[
\Delta_{13} \Delta_{23} - \Delta_{31} \Delta_{32} = (\Delta_{31} + \Delta_{32} - \Delta_{13} - \Delta_{23}) \Delta_{33}^*.
\]
\hfill (9)

Next we express \([\text{Im}(\Delta_{ia})]^2\) in terms of only \(\text{Re}\Delta_{ia}\). According to Ref. 4), \([\text{Im}(\Delta_{ia})]^2\) is written by
\[
[\text{Im}(\Delta_{ia})]^2 = |\Delta_{31}|^2 - [\text{Re}(\Delta_{31})]^2
= |U_{12} U_{13} U_{22} U_{23}|^2 - |U_{11} U_{21}|^2 - |U_{12} U_{22}|^2 - |U_{13} U_{23}|^2)/4.
\]
\hfill (10)

A further simplification can now be made from (7),
\[
[\text{Im}(\Delta_{ia})]^2 = |U_{11} U_{12} U_{21} U_{22}|^2 + |U_{12} U_{13} U_{22} U_{23}|^2 + |U_{13} U_{11} U_{23} U_{21}|^2
- (|U_{11} U_{21}|^2 + |U_{12} U_{22}|^2 + |U_{13} U_{23}|^2)/4
= |\Delta_{31}|^2 + |\Delta_{32}|^2 + |\Delta_{33}|^2 - [\text{Re}(\Delta_{31} + \Delta_{32} + \Delta_{33})]^2
= 3[\text{Im}(\Delta_{ia})]^2 - 2(\text{Re}\Delta_{31}\text{Re}\Delta_{32} + \text{Re}\Delta_{32}\text{Re}\Delta_{33} + \text{Re}\Delta_{33}\text{Re}\Delta_{31}).
\]
\hfill (11)

Through a similar procedure, \([\text{Im}(\Delta_{ia})]^2\) is expressed by
\[
[\text{Im}(\Delta_{ia})]^2 = \text{Re}\Delta_{ia}\text{Re}\Delta_{ia} + \text{Re}\Delta_{ia}\text{Re}\Delta_{ka} + \text{Re}\Delta_{ka}\text{Re}\Delta_{ia} + \text{Re}\Delta_{ka}\text{Re}\Delta_{ka}
= \text{Re}\Delta_{ia}\text{Re}\Delta_{ia} + \text{Re}\Delta_{ia}\text{Re}\Delta_{ia} + \text{Re}\Delta_{ia}\text{Re}\Delta_{ia}.
\]
\hfill (12)
with \( k, \beta = 1, 2, 3 \). It is easily known that (9) is included in (12). From (12), \([\text{Im}(A_{10})]^2\) does not exceed \( \Sigma_i(\text{Re}A_{10})^2 \) and \( \Sigma_x(\text{Re}A_{10})^2 \).

Finally we examine a restriction on \( A_{10} \) from the present theoretical and experimental analysis (5). Before we discuss this restriction, we consider how the magnitudes of \(|U_{ij}| \) in (5) are classified. For simplicity, we deal with a case of

\[
|U_{11}| \approx |U_{22}| \approx |U_{33}|, \\
|U_{12}| \approx |U_{21}|, \quad |U_{13}| \approx |U_{31}|, \quad |U_{23}| \approx |U_{32}|. \tag{13}
\]

Further imposing the condition

\[
|U_{11}|^2 \gg |U_{12}|^2 \gg |U_{23}|^2 \gg |U_{13}|^2 \tag{14}
\]
on (13), we investigate the validity of these summarizations. It is assumed from (13) that the deviations of \(|U_{12}/U_{21}|^2\) and others from unity are very small. For convenience, we introduce the following quantities \( \xi \) and others,

\[
(A_{33} - A_{31})/(A_{32} - A_{23}) = 1 + \xi, \\
(A_{31} - A_{12})/(A_{32} - A_{23}) = 1 + \epsilon, \\
(A_{12} - A_{21})/(A_{32} - A_{23}) = -1 - \Delta, \\
(A_{21} - A_{13})/(A_{32} - A_{23}) = -1 - \eta. \tag{15}
\]

Using these notations together with (8), we rewrite the condition (13) in the following forms,

\[
|U_{22}/U_{11}|^2 = 1 + \Delta - \epsilon + \eta, \\
|U_{33}/U_{11}|^2 = 1 + \Delta, \\
|U_{21}/U_{12}|^2 = (1 + \xi - \Delta + \epsilon - \eta)/(1 + \xi), \\
|U_{31}/U_{13}|^2 = (2 + \xi + \epsilon)/(2 + \xi + \epsilon - \Delta), \\
|U_{32}/U_{23}|^2 = (1 + \eta)/(1 + \epsilon), \tag{16}
\]

which are almost equal to unity. Combining these relations with \(|U_{23}|^2 + |U_{13}|^2 = |U_{33}|^2 + |U_{31}|^2\) from the unitarity relation (3), we have

\[
0 = (\eta - \epsilon)(2 + \xi + \epsilon - \Delta)|U_{23}|^2 + \Delta(1 + \epsilon)|U_{13}|^2. \tag{17}
\]

Similarly we have

\[
0 = (\epsilon - \eta - \Delta)(2 + \xi + \epsilon - \Delta)|U_{12}|^2 + \Delta(1 + \xi)|U_{13}|^2 \tag{18}
\]

from \(|U_{12}|^2 + |U_{13}|^2 = |U_{21}|^2 + |U_{31}|^2\). Substituting

\[
\Delta = (\epsilon - \eta)(2 + \xi + \epsilon)|U_{23}|^2/[((1 + \epsilon)|U_{13}|^2 + (\epsilon - \eta)|U_{23}|^2] \tag{19}
\]

from (17) into (18), we have either

\[
\epsilon = \eta \tag{20}
\]
or
\[(1 + \epsilon)[(2 + \xi + \eta)|U_{23}|^2 - (1 + \epsilon)|U_{13}|^2]|U_{12}|^2\]
\[= (1 + \xi)[(1 + \epsilon)|U_{13}|^2 + (\epsilon - \eta)|U_{23}|^2]|U_{23}|^2.\]  
(21)

But it is apparent that (21) does not agree with (13) and (14). Then we must have \(\epsilon = \eta\) and \(\Delta = 0\) or
\[|U_{ii}| = |U_{ji}| \quad \text{and} \quad |U_{ij}| = |U_{ji}|\]  
(22)
from (16). But, from the unitarity relations (3), this means that all of the off-diagonal elements of \(|U|\) must be the same. Judging from (5), the condition (13) with (14) should be modified.

Then, to find another possibility, we decompose condition (13) into the following three cases,
\[|U_{11}| \approx |U_{21}| \quad \text{and} \quad |U_{12}| \approx |U_{22}| \quad \text{for} \quad \Delta_{13} + \Delta_{23} \approx \Delta_{1i} + \Delta_{2i},\]  
(23)
\[|U_{22}| \approx |U_{32}| \quad \text{and} \quad |U_{23}| \approx |U_{33}| \quad \text{for} \quad \Delta_{21} + \Delta_{31} \approx \Delta_{12} + \Delta_{13},\]  
(24)
\[|U_{33}| \approx |U_{11}| \quad \text{and} \quad |U_{31}| \approx |U_{13}| \quad \text{for} \quad \Delta_{31} + \Delta_{32} \approx \Delta_{21} + \Delta_{12}\]  
(25)
from (8). Comparing these relations with (5), we know that only a case of
\[\Delta_{13} + \Delta_{23} \approx \Delta_{31} + \Delta_{32}\]  
(26)
is compatible with the present analysis of weak decays of quarks. This condition (26) is equivalent to
\[\Delta \approx \epsilon - \eta\]  
(27)
from (8) and (16). In a restriction (27), \(\Delta = \epsilon - \eta\) or a combination of \(\Delta = 0\) and \(\epsilon = \eta\) is not included because of (22). But, as is expected from (21), we can show that an approximation (27) does not always mean \(|U_{13}| \approx |U_{31}|\) and \(|U_{23}| \approx |U_{32}|\). Under (27), we can rewrite (16) as follows,
\[|U_{22}/U_{11}|^2 \approx |U_{21}/U_{12}|^2 \approx 1,\]
\[|U_{33}/U_{11}|^2 = 1 + \Delta,\]
\[|U_{31}/U_{11}|^2 = (2 + \xi + \epsilon)/(2 + \xi + \eta),\]
\[|U_{32}/U_{23}|^2 = (1 + \eta)/(1 + \epsilon).\]  
(28)
Since \(\xi\) and others in (28) are arbitrary except (21) and (27), \(|U_{31}/U_{13}|^2\) and others can take the values which are largely deviated from unity. Here we give an example within (5),
\[|U| = \begin{pmatrix}
0.975580 & 0.2195 & 0.007963 \\
0.2188 & 0.974218 & 0.055000 \\
0.019241 & 0.052136 & 0.998455
\end{pmatrix},\]  
(29)
which satisfies the unitarity relation (3) and the positivity\(^4\) of \(|\text{Im}(\Delta_{ii})|^2\). This
example can be explained by

\[ A = +0.0475, \quad \xi = -1.438, \quad e = -0.505, \quad \eta = -0.555. \]  \hspace{1cm} (30)

Of course, this example satisfies (21).

The quark mixing matrix $U$ suffers from a rephasing ambiguity, which makes the reality or complexity of any-specific element of the matrix uncertain. Since $A_{ia}$ are rephasing invariants, it is important for the future analysis to examine a restriction on $A_{ia}$ from the present analysis and the relations among $A_{ia}$. But we leave a definite determination of $A$ and others in (28) for further theoretical investigation of weak decay of a $b$ quark together with the discovery of a $t$ quark.