Purely Quantum Derivation of Density Fluctuations
in the Inflationary Universe

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We calculate the amplitude of density fluctuations generated during de Sitter phase of the inflationary universe without introducing the concept of the classical scalar field. This is done by directly evaluating the quantum two-point function of a certain combination of the components of the scalar field energy-momentum tensor which is invariant under coordinate gauge transformations. We find the amplitude agrees with that obtained in literature in which the existence of the homogeneous classical field is assumed, provided that the de Sitter-invariant quantum state of the field is manifestly unstable.

§ 1. Introduction

It is commonly believed that cosmological objects such as galaxies were formed through gravitational instability of initially small density fluctuations. One of the great successes of the inflationary universe scenario is that it has a possibility to explain the origin of these initial fluctuations naturally by quantum fluctuations of the scalar field which drives inflation.1)

During the de Sitter phase, the wavelength of vacuum fluctuations of the scalar field is stretched rapidly by the exponential expansion of the universe. When the wavelength exceeds the Hubble horizon size, the fluctuation stops oscillating and is frozen to a constant value. What has been done commonly in the literature1) to evaluate the density fluctuation is that one first decomposes the scalar field into the homogeneous classical part and the quantum fluctuation part, solves each part separately, regards the quantum part as classical fluctuations when the wavelength exceeds the horizon size, and solves the classical evolution of the fluctuations which are consequently interpreted as density fluctuations that give rise to galaxies. However, the decomposition into classical and quantum parts is rather artificial. In particular, recent investigations by means of the stochastic approach to inflation2) indicate that the scalar field behaves highly stochastic initially and the notion of a homogeneous classical scalar field appears only a posteriori as a result of the stochastic process (see, e.g., Ref. 3 and references cited therein). It is hence very much desirable to evaluate the density fluctuation amplitude based purely on quantum theoretical arguments, and see if the result is in accordance with the one obtained previously.

In this paper, we shall calculate the amplitude of density fluctuations without introducing any concept of the classical scalar field. We shall only use purely quantum correlation functions of the scalar field and evaluate the two-point corre-

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tion function of the energy density of the scalar field.

Contents of this paper are as follows. In § 2, we present the method for calculating density fluctuations based on the gauge-invariant cosmological perturbation theory. In § 3, we apply the formulas derived in § 2 and calculate the amplitude of density fluctuations by directly evaluating the two-point correlation of the (quantum) field energy density. We find that the explicit instability of the de Sitter-invariant vacuum state is essential to recover the result that has been obtained in the literature. In particular we find the approximate treatment of the scalar field, as a massless field does not give the correct answer. The derivation of the quantum two-point function for an unstable scalar field in de Sitter space is given in the Appendix. Finally, conclusion is given in § 4.

§ 2. Gauge-invariant formalism

It is well known that the behavior of density fluctuations depends on the choice of the coordinate gauge. It is however possible to construct gauge-invariant quantities out of various gauge-dependent perturbation variables and write down the Einstein equations only in terms of gauge-invariant quantities.\(^6\) In this section, we briefly review the gauge-invariant formalism and mention the previous result based on the assumed existence of the homogeneous classical scalar field.\(^*)\)

For simplicity, we assume that the background geometry is given by a spatially flat Friedmann-Robertson-Walker metric:
\[ ds^2 = -dt^2 + a^2(dx^2) = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2), \]
where \( \eta \) is the conformal time. Then the background equations are given by
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho, \quad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \]
where the dot (‘) denotes \( d/dt \). The metric is perturbed when density fluctuations are present. In general, perturbations can be classified into three types, scalar, vector and tensor, with respect to spatial coordinate transformations which preserve the symmetry of the background 3-space. In the case of density perturbations, perturbations of the metric components as well as matter variables are known to be of scalar-type and can be expressed in terms of scalar functions.

For scalar-type perturbations, the perturbed metric components are expressed as
\[ \bar{g}_{\mu\nu} = -a^2(1+2A), \]
\[ \bar{g}_{\mu\nu} = a^2 \delta_{\mu\nu}B, \]
\[ \bar{g}_{\mu\nu} = a^2[(1+2\mathcal{R})\delta_{\mu\nu} + 2\delta_{\mu}^{\alpha}\partial_{\nu}\mathcal{H}_{\alpha}], \]
where \( \eta \) is adopted as the time coordinate. Similarly, the perturbed components of the energy-momentum tensor are expressed as
\(\text{\textsuperscript{\textdagger}}\) In this paper, we follow the notation of Ref. 5).
\[\begin{align*}
\tilde{T}^0_0 &= -\rho[1 + \delta], \\
\tilde{T}^0_j &= -(\rho + p)\partial_j(v - B), \\
\tilde{T}^i_0 &= (\rho + p)\partial^i v, \\
\tilde{T}^i_j &= \rho \left[ (1 + \pi_L)\delta^i_j + \left( \partial^i \partial_j - \frac{1}{3} \delta^i_j A \right) \Pi \right].
\end{align*}\]

(2.4)

It is often more convenient to expand the perturbation variables in terms of scalar harmonics of the background 3-geometry:

\[\begin{align*}
A &= A_k Y_k, & B &= k^{-1} B_k Y_k, \\
R &= R_k Y_k, & H &= k^{-2} H_{Tk} Y_k, \\
\delta &= \delta_k Y_k, & v &= k^{-1} v_k Y_k, \\
\pi_L &= \pi_{Lk} Y_k, & \Pi &= k^{-2} \Pi_k Y_k,
\end{align*}\]

(2.5)

where \( Y_k = e^{ik \cdot x} \).

Combining the above perturbation variables, we can construct the following mutually independent gauge-invariant variables:

\[\begin{align*}
\Phi &= \Phi_k Y_k = \left[ R_k - \frac{a'}{ak} H'_{Tk} \right] Y_k, \\
\Psi &= \Psi_k Y_k = \left[ A_k + \frac{1}{ak} \left( a \left( B_k - \frac{1}{k} H'_{Tk} \right) \right) \right] Y_k, \\
\Delta &= \Delta_k Y_k = \left[ \delta_k + 3(1 + \omega) \frac{a'}{k\omega} v_k \right] Y_k, \\
V &= V_k Y_k = \left[ v_k - \frac{1}{k} H'_{Tk} \right] Y_k, \\
\Gamma &= \Gamma_k Y_k = \left[ \pi_{Lk} - \frac{c_s^2}{\omega} \delta_k \right] Y_k,
\end{align*}\]

(2.6)

and \( \Pi \) which is already gauge-invariant by itself. Here the prime ( ') denotes \( d/d\eta \), \( w = \rho/\rho \) and \( c_s^2 = \rho'/\rho' \). The metric variables \( \Phi \) and \( \Psi \) represent the intrinsic spatial curvature perturbation and a relativistic version of the Newton potential, respectively. The variables \( \Delta, V, \Gamma \) and \( \Pi \) represent the energy density, velocity, entropy and anisotropic stress, respectively, of the matter perturbations. It is important to note that because of the gauge invariance the right-hand sides of Eq. (2.6) can be evaluated in any gauge.

The full Einstein equations written in terms of these variables can be found in Chapter II of Ref. 5). Here we only mention that the gauge-invariant matter variables form a closed set of evolution equations by themselves and the Fourier components of the metric variables are algebraically expressed in terms of the matter variables. For our purpose, we only need one of such algebraic equations which is essentially the Hamiltonian constraint,
\[
-\frac{1}{a^2} \Delta \Phi = 4\pi G \rho \Delta ,
\]
(2.7)

or in terms of the Fourier components,
\[
\frac{k^2}{a^2} \Phi_k = 4\pi G \rho \Delta_k ,
\]
(2.8)

Using the background Friedmann equation (2.2), this can be rewritten as
\[
\Delta_k = \frac{2}{3} \frac{k^2}{H^2 a^2} \Phi_k ,
\]
(2.9)

where \( H = \dot{a}/a \). Thus \( \Phi_k \) gives the amplitude of density fluctuations when the wavelength crosses the Hubble horizon, i.e., when \( k/a = H \).

Unfortunately, however, \( \Phi \) is not always a good representative of the density fluctuation amplitude. Instead, there exists a better gauge-invariant quantity defined by
\[
\mathcal{R}_m = \mathcal{R}_{mk} Y_k = \left[ \Phi_k - \frac{a'}{ka} V_k \right] Y_k .
\]
(2.10)

When the wavelength is greater than the Hubble horizon size, \( \mathcal{R}_{mk} \) is known to remain constant in time for adiabatic (growing mode) perturbations, which are the ones that directly induce gravitational instability, and represents the true physical (=geometrical) amplitude of density fluctuations.\(^{3,4}\) Furthermore, quite generally for adiabatic perturbations, it can be shown from the perturbed Einstein equations that \( \mathcal{R}_{mk} \) and \( \Phi_k \) are related to each other on super-horizon scales as
\[
\mathcal{R}_{mk} = \left( 1 + \frac{2\rho}{3(\rho + p)} \right) \Phi_k .
\]
(2.11)

Under the assumption that there exists a homogeneous classical background of the scalar field \( \varphi(t) \), the quantum theory of the fluctuation of the scalar field \( \delta \varphi \) coupled with the metric perturbation \( \delta g_{\mu\nu} \) has been solved and the amplitude of \( \mathcal{R}_m \) on super-horizon scales has been evaluated in Ref. 7) without an ad hoc switchover to the classical treatment of the fluctuation at the horizon crossing in de Sitter phase. The result is
\[
\langle \mathcal{R}_m^2 \rangle_k = \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}_{mk}|^2 \approx \frac{H^4}{(2\pi)^2 \dot{\varphi}^2(t_k)} .
\]
(2.12)

where \( t_k \) is approximately the time at which the wavelength of a (quantum) fluctuation exceeds the horizon size in de Sitter phase.

\section{Derivation of the density fluctuation amplitude}

In this section, we calculate the amplitude of density fluctuations without introducing any \textit{a priori} "classical background field". Thus even the background space-time is defined through the expectation value of the quantum energy-momentum
tensor. As a scalar field which drives inflation, we assume a minimally coupled scalar field with the potential given by

\[ U(\phi) = \frac{1}{2} m^2 \phi^2 + U_0 \]  

(3.1)

with \( m^2 < 0 \). This should be a good approximation until the final stage of inflation when \( \phi^2 \sim U_0/|m^2| \). As usual, we also assume \( |m^2| \ll H^2 \) in order for our model to give a successful inflation. We consider the case when quantum fluctuations give rise to small metric fluctuations. In particular, we focus on those metric fluctuations which are directly due to the density fluctuation mode. Hence the metric has the form given by Eq. (2.3).

3.1. Basic formulas

With these assumptions, the Einstein equations split into the part which describes the homogeneous background and the fluctuating part. The homogeneous part satisfies the background equations (2.2) with \( \rho \) and \( p \) given by

\[ \rho = -\langle \tilde{T}_0 \rangle + U_0, \]

\[ p = \frac{1}{3} \langle \tilde{T}^i_i \rangle - U_0, \]  

(3.2)

where the energy-momentum tensor \( \tilde{T}^{\mu\nu} \) is given by

\[ \tilde{T}^{\mu\nu} = \tilde{\nabla}^\mu \phi \tilde{\nabla}_\nu \phi - \frac{1}{2} g^{\mu\nu} (\tilde{\nabla}_a \phi \tilde{\nabla}^a \phi + m^2 \phi^2). \]  

(3.3)

As for the fluctuating part, one can construct gauge-invariant quantities out of gauge-dependent ones, write down the Einstein equations in terms of the gauge-invariant ones only, and solve the equations. However, in our case, it is more convenient to solve the scalar field equation in a specific gauge, construct appropriate gauge-invariant quantities out of the solution, and solve for the rest of the quantities with the help of gauge-invariant equations. Specifically, we evaluate \( \tilde{T}^0_\nu \)-components of the energy-momentum tensor in synchronous gauge and construct the gauge-invariant quantity \( \Delta \).

We now work in the synchronous gauge (i.e., \( A = B = 0 \) in the metric (2.3)). From the expressions for \( \tilde{T}^{(3)}_0 \) and \( \tilde{T}^{(3)}_0 \) in Eq. (2.4) and that for \( \Delta \) in Eq. (2.6), one finds

\[ \Delta(\rho \Delta) = \Delta(-\tilde{T}^0_0) + 3H\partial^i(-\tilde{T}^0_0). \]  

(3.4)

Here it is important to note that, even if there exist vector- or tensor-type perturbations in \( \tilde{T}^{\mu\nu} \), such components do not appear on the right-hand side of the above equation; tensor-type perturbations do not contribute to \( \tilde{T}^0_\nu \) from the beginning and vector-type perturbations which can be contained in \( \tilde{T}^0_0 \) are automatically eliminated by taking the divergence of it. In addition, the spatial differentiations on the right-hand side of Eq. (3.4) automatically eliminate the homogeneous components in the energy-momentum tensor.

In the synchronous gauge, \( -\tilde{T}^0_0 \) and \( -\tilde{T}^0_0 \) are given by
\[- \nabla_0^a = \frac{1}{2} \left[ \dot{\phi}^2 + \frac{1}{a^2} (\delta^{ij} - h^{ij}) \partial_i \phi \partial_j \phi + m^2 \phi^2 \right], \]

\[- \nabla_0^a = \frac{1}{a} \dot{\phi} \partial_j \phi, \quad (3.5)\]

where

\[h^{ij} = 2 (\Re \delta^{ij} + \delta^i \delta^j H_1). \quad (3.6)\]

The field equation for \(\phi\) is

\[\dot{\phi} + 3 \left( H + \frac{1}{6} \dot{h}^i \right) \phi - \frac{1}{a^2} \Delta \phi + m^2 \phi = 0. \quad (3.7)\]

Inserting Eq. (3.5) into Eq. (3.4) and using the field equation, we obtain

\[\Delta(\rho \Delta) = \partial^i (\dot{\phi} \partial_i \phi - \ddot{\phi} \partial_i \phi) + \frac{1}{a^2} \partial^i \partial^j (\partial_i \phi \partial_j \phi) \]

\[- \Delta(\dot{h}^i \partial_i \phi \partial_j \phi) - \frac{1}{2} \partial^i (\dot{h}^i \partial_i \phi \partial_j \phi). \quad (3.8)\]

Because of the quantum nature of \(\phi\), the energy-momentum tensor is of order \(\hbar\). Since the perturbation we are dealing with is of quantum origin, the metric fluctuation \(h_{ij}\) should be at least of order \(\hbar\). This implies that the last two terms which contain the metric perturbation \(h_{ij}\) are of higher order in the amplitude of the perturbation. At the same time, one can consistently neglect the term involving \(\dot{h}^i\) in the field equation (3.7) when evaluating Eq. (3.8). Thus our basic formula for \(\Delta\) reduces to

\[\Delta(\rho \Delta) = \partial^i (\dot{\phi} \partial_i \phi - \ddot{\phi} \partial_i \phi) + \frac{1}{a^2} \partial^i \partial^j (\partial_i \phi \partial_j \phi) \quad (3.9)\]

with \(\phi\) being the quantum field satisfying

\[\dot{\phi} + 3H \phi - \frac{1}{a^2} \Delta \phi + m^2 \phi = 0. \quad (3.10)\]

Now we define the two-point function,

\[D(x, x') = \langle \Delta(\rho \Delta(x)) \Delta(\rho \Delta(x')) \rangle. \quad (3.11)\]

From Eq. (3.9), \(D\) can be expressed in the form,

\[D(x, x') = f^{\mu \nu}_i (t)f^{\sigma \tau}_{i'} (t') \partial^i \partial^j \left[ (\partial_\sigma \partial_\tau \partial_\rho G(x, x'))(\partial_{\sigma'} \partial_{\tau'} \partial_{\rho'} G(x, x')) \right] \]

\[+ (\partial_\rho G(x, x'))(\partial_{\rho'} \partial_{\tau'} \partial_{\rho'} G(x, x')) \], \quad (3.12)\]

where

\[G(x, x') = \langle \phi(x) \phi(x') \rangle, \quad (3.13)\]

and the coefficients \(f^{\mu \nu}_i\) are given by
\[
\begin{align*}
    f_i^{\delta\delta} &= f_i^{\delta\delta} = f_i^{\delta\delta} = f_i^{\delta\delta} = 0 , \\
    f_i^{\delta\delta} &= f_i^{\delta\delta} = \frac{1}{2} \delta_i^j , \\
    f_i^{\delta\delta} &= -\delta_i^j , \\
    f_i^{\delta\delta} &= \frac{1}{2} \left[ \delta_i^j \delta_i^j + \frac{1}{2} (\delta_i^k \delta_i^k + \delta_i^j \delta_i^j) \right] 
\end{align*}
\] (3.14)

with the coordinate \( x^\delta \) being the cosmic proper time \( t \). Thus the evaluation of \( D \) is straightforward, though tedious, once the explicit form of \( G \) is given. We note that since what we are interested in is the space-like correlation, the two-point function \( G \) is equal to a half of the symmetric two-point function \( G^{(1)} \),

\[
    G(x, x') = \frac{1}{2} \langle \phi(x) \phi(x') + \phi(x') \phi(x) \rangle .
\]

\[
    = \frac{1}{2} G^{(1)}(x, x') .
\] (3.15)

We also note that we have not used any particular property of the de Sitter background up to now. Therefore the formulas given above can be applied equally well to any cosmological model dominated by a scalar field.

3.2. **Two-point function of the field energy density**

In order to evaluate \( G^{(1)} \), we approximate the background space-time to an exact de Sitter space. Thus we assume \( a = 1/( -H \eta) = \exp H(t - \infty < \eta < 0, \infty < t < \infty) \). If \( m^2 \) of the scalar field were positive, the scalar field would have a family of stable de Sitter-invariant vacua. Among them the most natural vacuum that gives the same short-distance behavior of the field as in the Minkowski space is called the Euclidean vacuum. It is known that any physical quantity measured in an excited state with respect to the Euclidean vacuum approaches that measured in the vacuum state as \( t \to \infty \). Therefore we would have been allowed to evaluate \( D \) in terms of \( G^{(1)} \) in the Euclidean vacuum, whose explicit form can be found in the literature. However, in the present case, the scalar field has a negative \( m^2 \), and hence there exists no stable vacuum.

In order to find an appropriate form of \( G^{(1)} \) in our case, we must recall the physical picture of the inflationary universe. In the inflationary scenario, the present universe corresponds to a tiny comoving region of a much larger universe. Hence, as far as we are interested in implications of inflation to our universe, we may focus our attention on a fixed comoving region of the whole inflationary universe. This means one may set an infrared cutoff at certain comoving wavenumber \( k = k_e \) of the modes of \( \phi \). Alternatively, one may modify the infrared behavior of \( \phi \) in a way such that all the modes become stable in the limit \( t \to -\infty \) (\( a \to 0 \)) so that there exists a well-defined vacuum initially but the behavior of the modes with \( k \) larger than \( k_e \) is unaffected by the regularization. One of the simplest ways to do so is to add a "comoving" mass term to the potential of \( \phi \); i.e., to replace the proper energy \( \omega_k \) of the mode \( k \) as \( \omega_k^2 = m^2 + k^2/\alpha^2 \to m^2 + (k^2 + \mu^2)/\alpha^2 \) with \( \mu^2 > 0 \). Note that, since the temperature \( T \) is proportional to \( a^{-1} \), this is equivalent to taking into account a
thermal correction to the potential of $\phi$ due to some interactions; $m^2 \to m^2 + gT^2$, where $g$ is a numerical factor which depends on the nature of interactions. This regularization method has been discussed by Linde\(^{(10)}\) to obtain the behavior of $\langle \phi^2 \rangle$. Here we generalize his calculation to the symmetric two-point function $G^{(1)}(x, x')$. The details are given in the Appendix. We find

$$G^{(1)}(x, x') = G_0^{(1)}(Z) + G^{(1)}(\eta') \; ;$$

$$Z = \frac{\eta^2 + \eta'^2 - r^2}{2\eta\eta'}, \quad r^2 = |x - x'|^2, \quad (3.16)$$

where $G_0^{(1)}$ is that for the de Sitter-invariant Euclidean vacuum with $m^2$ analytically continued from $m^2 > 0$ to $m^2 < 0$,

$$G_0^{(1)}(Z) = \frac{H^2}{4\pi \sin c\pi} F\left(c, 3 - c, 2; \frac{1 + Z}{2}\right), \quad (3.17)$$

with $F$ being a hypergeometric function and the constant $c$ given by

$$c = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \approx \frac{m^2}{3H^2} < 0, \quad (3.18)$$

and $G^{(1)}$ is given by

$$G^{(1)}(\eta') \approx -\frac{H^2}{4\pi^2} (\mu^2 \eta')^c, \quad (3.19)$$

which is the part that explicitly breaks de Sitter invariance. We mention that the above $G^{(1)}$ has a well-defined limit for $c \to 0$ ($m^2 \to 0$) and the limit is in agreement with the one obtained by Allen.\(^{(8)}\)

With $G^{(1)}$ given by Eq. (3.16), one can evaluate the correlation function $D$ exactly in principle. However, since we are interested only in the behavior of $D(x, x')$ with $x$ and $x'$ being space-like separated much farther than the horizon size $H^{-1}$, we may expand $G_0^{(1)}(Z)$ by assuming $u \equiv 1 - Z \gg 1$ and retain only a first few terms dominant at $u \to \infty$. Using the asymptotic form of $F$ and noting $|c| \ll 1$, one obtains

$$G(x, x') = G_0(u) + G(u_0) \; ;$$

$$G_0(u) \approx \frac{H^2}{8\pi^2 c} \left( u^{-c} + cu^{1-c-1} \frac{c^2(1+c)}{1-2c} u^{-c-2} + \cdots \right),$$

$$G(u_0) \approx -\frac{H^2}{8\pi^2} u_0^{-c}, \quad (3.20)$$

where

$$u = \frac{r^2 - (\eta - \eta')^2}{2\eta\eta'}, \quad u_0 = \frac{r_0^2}{2\eta\eta'}, \quad r_0 = \frac{1}{\mu}. \quad (3.21)$$

We recall that, since $\mu$ corresponds to an infrared cutoff of comoving wavenumbers, Eq. (3.16) applies to a comoving region of size $\ll r_0 = \mu^{-1}$. This implies that $\bar{G}$, which breaks de Sitter invariance but is spatially homogeneous, is the dominant term in $G$ for $1 \ll u \ll u_0$. In addition, since the only de Sitter-invariant scalar is a constant, the
temporal behavior of $\langle \phi^2 \rangle$ is controlled by $\bar{G}$ with $\eta' = \eta$. Furthermore, since $\bar{G}$ diverges to infinity for $t \to \infty$ ($\eta \to 0$), it becomes the dominant term of $\langle \phi^2 \rangle$ at sufficiently late times. Thus we have

$$
\langle \phi^2(t) \rangle = [\bar{G}]_{\eta' = \eta} = \frac{H^4}{8\pi^2 c} \left( \frac{2H^2}{r_0^2} \right)^c \exp(-2cHt).
$$

(3.22)

Now let us evaluate $D$ at $1 \ll u \ll u_0$. To find terms which contribute dominantly to $D$ in Eq. (3.12), note that spatial derivatives of $G$ are equivalent to those of $G_0$ and that dominant contribution to time derivatives of $G$ comes from those of $\bar{G}$. Then we find

$$
D(x, x') = \Delta_x \Delta_{x'} C(x, x') ;
C(x, x') \approx (\partial \partial_{\nu} \bar{G})(\partial \partial_{\nu} G_0) + (\partial^2 \partial_{\nu} \bar{G})G_0
$$

$$
- (\partial^2 \partial_{\nu} \bar{G})(\partial_{\nu} G_0) - (\partial^2 \partial_{\nu} \bar{G})(\partial_{\nu} G_0).
$$

(3.23)

Then from Eq. (3.11), one immediately sees that the function $C(x, x')$ is what we need,

$$
C(x, x') = \langle \rho \Delta(x) \rho \Delta(x') \rangle.
$$

(3.24)

Inserting $G_0$ and $\bar{G}$ given in Eq. (3.20) into the expression for $C$ in Eq. (3.23) and putting $\eta = \eta'$ at the end, we find

$$
C(r, t) = \rho^2 \langle \Delta(r, t) \Delta(0, t) \rangle \approx \frac{H^4}{8\pi^2} \left( \frac{2H^2}{r^2} \right)^c \langle \phi^2(t) \rangle,
$$

(3.25)

where

$$
\langle \phi^2(t) \rangle = [\partial \partial_{\nu} \bar{G}]_{\eta' = \eta} = \frac{cH^4}{8\pi^2} \left( \frac{2H^2}{r_0^2} \right)^c.
$$

(3.26)

We emphasize that the above result would not have been obtained, had the scalar field a positive $m^2$ and were there not the de Sitter symmetry breaking part $\bar{G}$ in $G$. Note also that, because $\langle \phi^2 \rangle = 0$ for $m^2 = 0$, approximating the scalar field by a massless field from the beginning would not give the correct answer. This comes from the fact that even though a massless scalar does not have a de Sitter-invariant vacuum, its energy momentum tensor is well-defined as oppose to the case $m^2 < 0$.

3.3. Power spectrum of the fluctuation

Let us first evaluate the power spectrum of $\Delta$ by Fourier transforming Eq. (3.25). The power spectrum $|\Delta_k|^2$ is defined as

$$
C(r, t) = \rho^2 \int \frac{d^3k}{(2\pi)^3} |\Delta_k|^2 e^{ik \cdot r}.
$$

(3.27)

Hence we have

$$
|\Delta_k|^2 = \frac{1}{\rho^2} \int d^3r C(r, t) e^{-ik \cdot r} = \frac{4\pi}{\rho^2} \int_0^\infty drr^2 j_0(kr) C(r, t),
$$

(3.28)
where \( j_0 \) is a spherical Bessel function of the 0-th order. In general, Fourier transformation of a function \( Q(r) \) can be done only if one knows the precise form of \( Q(r) \) over the entire region of \( r \). However, there exist cases in which the long-distance behavior of \( Q(r) \) determines uniquely the small \( k \) behavior of the Fourier transform, irrespective of the short-distance behavior of \( Q(r) \). In such a case, one can modify or regularize the short-distance behavior arbitrarily to make the Fourier integral convergent without affecting the small \( k \) behavior. Fortunately, our case corresponds to such a case and we may assume that the integral (3.28) converges uniformly. Then replacing \( j_0(kr) \) by the identity,

\[
j_0(kr) = -r \int_0^k dk' j_1(k'r),
\]

we may change the order of integration and perform the \( r \)-integral first to obtain

\[
\Delta_k^2 = \frac{H^4}{2\rho^2} \langle \dot{\phi}^2 \rangle (-k\eta)^{4+\xi} \frac{1}{k^8}; \quad (-k\eta) \ll 1.
\]

Since \( |c| \ll 1 \), this is essentially a Zeldovich spectrum as expected.

From Eq. (2.9), the spectrum of \( \Phi_k \) is given by

\[
|\Phi_k|^2 = \frac{9}{4(k\eta)^4} |\Delta_k|^2 = \frac{9H^4}{8\rho^2} \langle \dot{\phi}^2 \rangle (-k\eta)^{2\xi} \frac{1}{k^4}.
\]

Then from Eq. (2.11), the spectrum of \( \mathcal{R}_{mk} \), which represents the physical amplitude of the perturbation, is found as

\[
|\mathcal{R}_{mk}|^2 = \left(1 + \frac{2\rho}{3(\rho + \rho_p)}\right)^2 |\Phi_k|^2 \approx \frac{H^4}{2\langle \dot{\phi}^2 \rangle} (-k\eta)^{2\xi} \frac{1}{k^8},
\]

where the facts \((\rho + \rho_p)/\rho \ll 1\) and \(\rho + \rho_p = \langle \dot{\phi}^2 \rangle\) have been used.

Noting that \( \langle \dot{\phi}^2 \rangle \) has the same \( \eta \)-dependence as \( \varphi^2 \), where \( \varphi \) is a classical homogeneous scalar field, and that it is proportional to \( (-\eta)^{2\xi} \), it is easily seen that Eq. (3.32) exactly coincides with Eq. (2.12), provided that one interprets \( \dot{\varphi}^2(t) = \langle \dot{\varphi}^2(t) \rangle \) or \( \varphi^2(t) = \langle \varphi^2(t) \rangle \). Thus we have recovered the result obtained in the literature by directly and purely quantum theoretically evaluating the two-point correlation of the field energy density, i.e., without assuming the existence of a homogeneous classical field. To be complete, let us write down our final result for \(|\mathcal{R}_{mk}|^2\),

\[
\langle \mathcal{R}_{mk}^2 \rangle_s = \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}_{mk}|^2 \approx \frac{H^4}{(2\pi)^2 \langle \dot{\phi}^2(t_h) \rangle},
\]

where, as before, \( t_h \) is approximately the time at which the wavelength exceeds the horizon size in de Sitter phase.

In deriving the gauge-invariant perturbation amplitude \( \mathcal{R}_m \), we have first evaluated the density fluctuation \( \Delta \) and then used Eq. (2.11) to obtain \( \mathcal{R}_m \). Although this seems the most natural and comprehensive way to undersand the physical situation,
one might worry about the general validity of Eq. (2·11) which is derived by assuming the perturbation is adiabatic. In order to circumvent such a concern, one can express \( \mathcal{R}_m \) in terms of the components of the energy-momentum tensor by using Eqs. (2·9) and (2·10), and evaluate the two-point correlation function of \( \mathcal{R}_m \) directly. Then repeating essentially the same argument we have given, one can show that the result agrees perfectly with Eq. (3·33).

Finally, we note that Eq. (3·20) is in agreement with the space-like correlation function of the coarse-grained scalar field obtained by Nakao et al.\(^3\) through the stochastic approach to inflation, in which the modes with proper wavenumbers \( k/a \) greater than \( H \) are treated as a noise source. This implies that one can obtain the correct fluctuation amplitude also in the stochastic approach. In other words, one can apply the same argument as given here to the energy-momentum tensor of the coarse-grained scalar field defined on scales greater than the horizon size. An advantage of the stochastic approach is that the systematic understanding of the dynamics of inflation can be rather easily obtained. In particular, there is no need to assume the existence of an \( a \) \textit{priori} classical field. Instead, the development of the homogeneous classical background of the quantum field can be easily visualized and the interpretation of \( \langle \phi^2 \rangle \) as \( \varphi^2 \) arises quite naturally.

§ 4. Conclusion

We have calculated the amplitude of density fluctuations in the inflationary universe by directly evaluating the two-point correlation of the quantum field energy density, without introducing any classical field. This has become possible by a careful construction of the gauge-invariant energy density fluctuation out of the energy-momentum tensor of the scalar field.

We have found that the result is in complete agreement with the one obtained previously where the existence of a homogeneous classical scalar field was assumed \( a \) \textit{priori}, provided that one interprets the expectation value \( \langle \phi^2 \rangle \) as the square of the classical field \( \varphi^2 \). As we have noted at the end of § 3, the same result can be obtained by the stochastic approach to inflation. Since the stochastic approach naturally gives a justification of the above interpretation, our result can be regarded as confirming the physical picture of inflation which has been accepted by many people rather intuitively.

In this paper, we have considered the universe in the midst of inflation when there remains no memory of initial conditions. It would be interesting to extend our analysis and investigate how and when a non-negligible fluctuation arises and how it develops for various cases of initial conditions. It would be of great interest also to carry out a similar analysis on the final stage of inflation when the universe is reheated and turns into a usual Friedmann universe.

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Appendix

In this appendix, we derive the symmetric two-point function \( G^{(1)} \) for a scalar field with negative \( m^2 \) in de Sitter space. The scalar field is expanded in terms of mode functions \( \varphi_k \) as

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left( a_k \varphi_k(\eta)e^{ik\cdot x} + a_k^\dagger \varphi_k^*(\eta)e^{-ik\cdot x} \right), \tag{A.1}
\]

where \( a_k \) and \( a_k^\dagger \) are the annihilation and creation operators, respectively, with respect to a suitably defined vacuum. As discussed in § 3, we regularize the infrared behavior of \( \phi \) by adding a “comoving” mass term \( \mu^2/a^2 \) to the potential. Thus the mode function \( \varphi_k \) satisfies

\[
\varphi_k'' - \frac{2}{\eta} \varphi_k' + \left[ k^2 + \mu^2 + \frac{m^2}{H^2\eta^2} \right] \varphi_k = 0. \tag{A.2}
\]

As the vacuum, we choose the one that gives the same short-distance behavior as in the Minkowski space. Then the normalized solution for \( \varphi_k \) is given by

\[
\varphi_k = \frac{\sqrt{\pi} H(-\eta)_{3/2} H^{(1)}_{\nu}(\sqrt{k^2 + \mu^2 \eta})}{\nu = \frac{3}{2} - c = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}} \tag{A.3}
\]

where \( H^{(1)} \) is a Hankel function of the first kind. Hence the symmetric two-point function \( G^{(1)} \) is expressed as

\[
G^{(1)}(x, x') = \langle \phi(x) \phi(x') + \phi(x') \phi(x) \rangle
= \int \frac{d^3k}{(2\pi)^3} \left( \varphi_k(\eta) \varphi_k^*(\eta') + \varphi_k(\eta') \varphi_k^*(\eta) \right)
= \frac{\pi H^2}{4} (\eta\eta')^{3/2} \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot r}
\times \{ H^{(1)}(-\sqrt{k^2 + \mu^2 \eta}) H^{(2)}(-\sqrt{k^2 + \mu^2 \eta'}) + c.c. \}. \tag{A.4}
\]

Now we apply the same technique as used in Appendix A of Bunch and Davies.\(^9\) Namely, we use the relation

\[
H^{(1)}(z)H^{(2)}(z') = \frac{4}{\pi^2} K_\nu(-iz)K_\nu(iz'), \tag{A.5}
\]

and the integral formula,

\[
K_\nu(z)K_\nu(z')
= \frac{1}{2} \int_0^\infty d\cosh\nu s \int_0^\infty \frac{dv}{v} \exp\left[ -\frac{v}{2} - \frac{1}{2v}(z^2 + z'^2 + 2zz'\cosh s) \right], \tag{A.6}
\]

where \( K_\nu \) is a modified Bessel function of the second kind, and re-express \( G^{(1)} \) in terms...
of the integrals with respect to $s$, $v$ and $h$. Then, performing the $h$-integral first and the $v$-integral next, we obtain

$$G^{(1)}(x, x') = \frac{H^2}{2\pi^2} \int_0^\infty ds \cosh s \frac{1 + p(2\cosh s - 2Z)^{1/2}}{(2\cosh s - 2Z)^{3/2}} \times \exp[-p(2\cosh s - 2Z)^{1/2}], \quad (A\cdot7)$$

where

$$Z = \frac{\eta^2 + \eta'^2 - r^2}{2\eta \eta'}, \quad p = (\mu^2 \eta \eta')^{1/2}. \quad (A\cdot8)$$

Since we are interested in the modes $k^2 \gg \mu^2$ and the form of $G^{(1)}$ at late times $\eta$, $\eta' \to 0$, we only need to know $G^{(1)}$ at $p \ll 1$. Let us assume that $G^{(1)}$ takes the form

$$G^{(1)}(x, x') \approx G^{(1)}(x, x')_{p=0} + \alpha p^\beta; \quad p \to 0. \quad (A\cdot9)$$

Then it is readily seen that $G^{(1)}|_{p=0} = G_0^{(1)}$, where $G_0^{(1)}$ is the de Sitter-invariant symmetric two-point function for the Euclidean vacuum with $m^2$ analytically continued to $m^2 < 0$,

$$G_0^{(1)}(Z) = \frac{H^2}{4\pi \sin c\pi} F \left( c, 3-c, 2; \frac{1+Z}{2} \right). \quad (A\cdot10)$$

The parameters $\alpha$ and $\beta$ can be determined by examining the behavior of $p\partial / \partial p G^{(1)}$ at $p \sim 0$,

$$p \frac{\partial}{\partial p} G^{(1)} = \frac{H^2}{2\pi^2} p^3 \int_0^\infty ds \cosh s \frac{\exp[-p(2\cosh s - 2Z)^{1/2}]}{(2\cosh s - 2Z)^{1/2}}. \quad (A\cdot11)$$

Since $\nu = 3/2 - c(|c| \ll 1)$, the above integral diverges in the limit $p \to 0$. This means that the dominant contribution to the integral comes from $s \gg 1$ for $p \to 0$. Making the substitution $x = e^{as/2}$, we have

$$\int_0^\infty ds \cosh s \frac{\exp[-p(2\cosh s - 2Z)^{1/2}]}{(2\cosh s - 2Z)^{1/2}}$$

$$\frac{1}{p^c} \int_0^\infty dx x^{2\nu-2} e^{-px} = \frac{\Gamma(2\nu-1)}{p^{2\nu-1}} = \frac{\Gamma(2-2c)}{p^{2-2c}}. \quad (A\cdot12)$$

Hence

$$p \frac{\partial}{\partial p} G^{(1)} \approx -\frac{H^2}{2\pi^2} p^c, \quad (A\cdot13)$$

and $G^{(1)}$ is found to be

$$G^{(1)}(x, x') \approx G^{(1)}(Z) - \frac{H^2}{4\pi^2 c} p^c. \quad (A\cdot14)$$

With $p = (\mu^2 \eta \eta')^{1/2}$, this is the form we quoted in § 3, Eq. (3.16).
References

1) For a review, see, e.g.,


