Relativistic Effects on Charmonium

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For the charmonium system we reexamine the relativistic effects due to the 1/(quark mass)² corrections to the nonrelativistic model. Although these effects are often treated as small perturbations to the nonrelativistic model, we emphasize the importance of their nonperturbative treatment, in particular, with respect to the fine and hyperfine structures of the energy spectrum and to the leptonic decay processes and electromagnetic transitions.

§ 1. Introduction

There is little doubt that heavy quarkonia such as charmonium are the best objects that illustrate the usefulness of the quark potential model. This model assumes that a hadron consists of massive quarks which interact with each other through a “QCD inspired” quark-quark interaction. If the quarks involved are very heavy, which is the case for heavy quarkonia, one would expect that nonrelativistic treatment is sufficient. For describing gross features of the energy spectra, the nonrelativistic treatment is indeed successful. However, if one looks into some details of the spectra, and also the leptonic decay processes and electromagnetic transitions, one realizes that the nonrelativistic treatment has much to be improved. The purpose of this paper is to critically reexamine relativistic corrections to the nonrelativistic model. These relativistic effects are often treated as small perturbations, i.e., the first order perturbations, to the nonrelativistic model. However, we emphasize some subtle aspects which require nonperturbative treatment. We focus on charmonium, which is the lightest among the heavy quarkonia. We do not intend to construct an optimal potential model which reproduces gross features of all mesons, from the π to the Y, at the expense of details for individual mesons. We prefer to confine ourselves to charmonium so that the charmonium spectra are described as accurately as possible.

It would be in order to clearly define what we mean by the “relativistic effects” in this paper. The Hamiltonian that we assume for the charmonium system is

\[ H = 2(m^2 + p^2)^{1/2} + V_c(r) + V_{\text{conf}}(r) + V_s + V_0. \] (1)

The first term on the RHS is the relativistic kinetic energy; \( m \) is the quark mass and \( p \) the relative momentum. We have taken the center-of-mass system. The interaction terms are: the color Coulomb, confinement, spin-dependent and spin-independent potentials, in that order. We will specify these potentials explicitly in § 2, but let us mention here that \( V_s \) and \( V_0 \) are proportional to \( 1/m^2 \). For the kinetic energies we
use the fully relativistic form as such. This means that the kinetic energy contains effects of higher orders with respect to \((p/m)^2\). This is inconsistent with our treatment of the interaction for which we retain only the \(1/m^2\) terms. However, we will see that the expansion of the kinetic energy in the power of \((p/m)^2\) converges rapidly, which justifies our treatment of the kinetic energy.

We define the nonrelativistic counterpart of our model by

\[
H_{NR} = 2 \left( m + \frac{p^2}{2m} \right) + V_c(r) + V_{\text{conf}}(r). \tag{2}
\]

What we mean by the relativistic effects are those due to the difference \(H - H_{NR}\). Note that, unlike the nonrelativistic quark model of De Rujula, Georgi and Glashow,\(^1\) our \(H_{NR}\) does not include the Fermi-Breit interaction. The Fermi-Breit interaction is of relativistic origin and is contained in \(V_s\) and \(V_0\) of \(H\).

Stanley and Robson,\(^2\) and Godfrey and Isgur\(^3\) considered models very similar to that of Eq. (1). They diagonalized the Hamiltonian using harmonic oscillator bases. However, their treatments are unsatisfactory in the following respect. The behavior of wave functions at short distance, which is governed by the color Coulomb potential,\(^4\) is of crucial importance in describing leptonic decays and electromagnetic transitions. Since the harmonic oscillator bases are not very suitable for describing such short-range behavior, a huge truncation space is required. In place of the harmonic oscillator bases, we propose to use a set of eigenstates of \(H_{NR}\) as the bases of expansion.

There appears another type of relativistic correction in decay and transition processes. This is related to the difference between the relativistic and nonrelativistic expressions of the quark current. It seems to us that this correction has not been carefully treated in Refs. 2) and 3).

This paper is organized as follows. In § 2, the Hamiltonian is specified in detail, and the method of calculation is presented. The relativistic effects on the mass spectrum are examined in § 3, and thereby the parameters of the model are fixed. The relativistic effects on leptonic decay widths and electromagnetic transition rates are discussed in § 4. Section 5 is devoted to a summary, conclusion and discussion.

§ 2. The model Hamiltonian

In this section we specify the Hamiltonian for the charmonium system, and explain our method of calculation. It is understood that the charmonium system is composed of only two quarks, \(c\) and \(\bar{c}\), and the Hamiltonian contains no channels other than that of \(cc\). We now give the details of the Hamiltonian \(H\) of Eq. (1). As we already stated, the first term of \(H\) is the relativistic kinetic energy. The second term is the color Coulomb potential given by

\[
V_c(r) = -\frac{4}{3} \frac{a_s(r)}{r}, \tag{3}
\]

where the factor \(a_s(r)\) is to take into account the correction related to the running coupling constant. Many forms of \(a_s(r)\) have been proposed. For example, a loga-
Rithmic shape is often assumed; it reflects the nature of the renormalized coupling constant in QCD. However, the results of our calculation are quite insensitive to the details of the form. We use the error function devised by McClary and Byers,

\[ \alpha_s(r) = a \text{erf} \left( \frac{r}{\sigma} \right), \]

where \( a \) and \( \sigma \) are constants. For the confinement potential, we assume

\[ V_{\text{cont}}(r) = br + C, \]

where \( b \) and \( C \) are constants. This form of \( V_{\text{cont}} \) is consistent with what the lattice QCD calculation suggests.

Apart from the kinetic energy, relativistic corrections are all in the other two terms of the potential, \( V_s \) and \( V_0 \). They can be derived, for example, by applying the Chraplyvy transformation\(^7\) to the Salpeter equation with an appropriate interaction and by retaining terms up to the order of \((p/m)^2\).\(^8\) It is known that there is some ambiguity in the nonrelativistic reduction of a scalar interaction. We follow the convention of Barnes and Ghandour,\(^9\) assuming that the confinement part of interaction is a Lorentz scalar, we obtain

\[ V_s = \frac{\partial^2 V_c}{6m^2} \sigma_1 \cdot \sigma_2 + \frac{3V_c - V_{\text{cont}}}{2m^2r} L \cdot S \]

\[ + \frac{V_c - V_{\text{cont}}}{12m^2r} \frac{3(\sigma_1 \cdot r)(\sigma_2 \cdot r)}{r^2} - \sigma_1 \cdot \sigma_2, \]

\[ V_0 = \frac{\partial^2 (V_c + V_{\text{cont}})}{4m^2} - \frac{p V_{\text{cont}} p}{m^2} \]

\[ + \frac{1}{2m^2} \left\{ \left\{ p V_c - \frac{p \cdot r}{r} V_c (r \cdot p) \right\} \right\}, \]

where the double curly brackets are defined by

\[ \{ p_i F_{ij} p_j \} = \frac{1}{4}(p_i p_j F_{ij} + p_i F_{ij} p_j + p_j F_{ij} p_i + F_{ij} p_i p_j). \]

Let us point out that, when the factor \( \alpha_s(r) \) is incorporated, we are free from intractable singularities, which would otherwise emerge through derivatives of \( V_c \).

We diagonalize the Hamiltonian \( H \) using appropriate bases. We choose a set of eigenfunctions of the nonrelativistic Hamiltonian \( H_{\text{NR}} \) of Eq. (2). In spite of the handicap that one has to solve the eigenvalue equation for \( H_{\text{NR}} \) numerically, this choice has two important advantages which distinguish our method from those of Refs. 2) and 3). First, the short range part of the color Coulomb potential can accurately and easily be taken into account. Secondly, since the Hamiltonian \( H_{\text{NR}} \) is used to generate the bases of usual perturbation approaches, one can readily compare our results with those obtained by perturbation theory. The effects beyond the lowest order perturbation are brought in through the off-diagonal elements of
\[ H' = H - H_{\text{NR}} \]
\[ = 2(m^2 + p^2)^{1/2} - 2\left( m + \frac{p^2}{2m} \right) + V_s + V_0, \]  
which contains all relativistic corrections.

There are five parameters to be determined in the model. They are the quark mass \( m \), two constants of the \( V_{\text{cont}} \) \( (b \text{ and } C) \), the coupling strength \( (a) \) and the range parameter \( (\sigma) \) of the Coulomb potential. We determine them such that the following six levels below the \( D\bar{D} \) threshold (3734 MeV); \( \eta_c, J/\psi, \psi' \) and three \( \chi_c \) states, are all fixed. In the input we do not include the recently reported masses of \( \eta'_c \) and \( h_c \) because they are not as well established as the other six.

§ 3. Mass spectrum

First we fix the parameters by least chi-square fitting the six low lying levels. The size of the truncation space is determined such that the mass spectrum below \( \psi \) (4415) is accurately calculated. We find that the basis functions with radial quantum numbers up to 5 yield the masses with the four-digit accuracy. The increase of radial quantum numbers from 5 to 10, for example, gives no appreciable change to the spectrum below \( \psi \) (4160) and a change of only a few MeV even to \( \psi \) (4415). This small size of the truncation space is what was expected when we incorporated the Coulomb potential into our diagonalization bases. The Hamiltonian \( H \) of Eq. (1) is diagonalized in the matrix space of either \( 5 \times 5 \), or \( 10 \times 10 \) in the case that two angular momentum states are coupled by the tensor force. In the following we designate a state in terms of the quantum numbers of the basis state with the largest amplitude, which we will refer to as the dominant-basis-state (DBS).

The values of the parameters are listed in Table I. Two features should be noted. First, the value of \( C \) is very large.*) This large value of \( C \) is instrumental in obtaining correct ordering of levels. This was also pointed out in Ref. 5). We tried with much smaller values of \( C \); in that case the effect of \( V_0 \) becomes very large, making it difficult to obtain the correct level-ordering. Secondly, the tensor force in \( V_s \) with the parameters in Table I is very weak; consequently the coupling between different angular momentum states is negligible in most cases of the spectrum calculation.

The calculated masses are given in Table II together with the experimental values. The values under \( M_0 \) are for the masses with no relativistic corrections, that is, the eigenvalues of \( H_{\text{NR}} \). Column \( \Delta M_{\text{pert}} \) gives the diagonal elements of \( H' \) for the DBS's. They are nothing but the first order perturbation energies when \( H_{\text{NR}} \) (with the parameters in Table I) is

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
\( m \text{(GeV)} \) & \( b \text{(GeV/fm)} \) & \( C \text{(GeV)} \) & \( a \) & \( \sigma \text{(fm)} \) \\
\hline
2.021 & 0.865 & -1.061 & 0.614 & 0.101 \\
\hline
\end{tabular}
\caption{Parameters determined by fitting the spectrum \( J/\psi, \psi', \chi_c, \chi_{c1}, \chi_{c2} \) and \( \eta_c \).}
\end{table}

*) Actually the values of \( C \) in other works with similar potential models spread over wide range, depending on the details of models applied. Our value belongs to the side of large values. An extremely small values of \( C \) is seen in Ref. 11), for example.
Table II. Calculated masses in MeV below and near the $D\bar{D}$ threshold (3739 MeV). The states marked with a star were used for input. The experimental values in parentheses were quoted from Ref. 10. Explanations of columns are given in the text.

<table>
<thead>
<tr>
<th>State Assignment</th>
<th>$M_0$</th>
<th>$\Delta M_{pert}$</th>
<th>$\Delta M_{diag}$</th>
<th>$M$</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$*_{J/\psi}$</td>
<td>1$^3S_1$</td>
<td>2986</td>
<td>128</td>
<td>115</td>
<td>3101</td>
</tr>
<tr>
<td>$*\phi'$</td>
<td>2$^1S_0$</td>
<td>2986</td>
<td>52</td>
<td>55</td>
<td>3693</td>
</tr>
<tr>
<td>$\phi(3770)$</td>
<td>1$^3D_1$</td>
<td>3777</td>
<td>24</td>
<td>24</td>
<td>3801</td>
</tr>
<tr>
<td>$\phi(4040)$</td>
<td>3$^3S_1$</td>
<td>4063</td>
<td>21</td>
<td>23</td>
<td>4086</td>
</tr>
<tr>
<td>$\phi(4160)$</td>
<td>2$^3D_1$</td>
<td>4152</td>
<td>-4</td>
<td>-4</td>
<td>4148</td>
</tr>
<tr>
<td>$\phi(4415)$</td>
<td>4$^3S_1$</td>
<td>4411</td>
<td>-15</td>
<td>-12</td>
<td>4400</td>
</tr>
<tr>
<td>$*\eta_0$</td>
<td>1$^1S_0$</td>
<td>2986</td>
<td>-4</td>
<td>-7</td>
<td>2980</td>
</tr>
<tr>
<td>$\eta_0$</td>
<td>2$^1S_0$</td>
<td>2986</td>
<td>-9</td>
<td>-11</td>
<td>3638</td>
</tr>
<tr>
<td>$h_c$</td>
<td>1$^1P_1$</td>
<td>3472</td>
<td>50</td>
<td>48</td>
<td>3520</td>
</tr>
<tr>
<td>$*\chi_{co}$</td>
<td>1$^1P_0$</td>
<td>3472</td>
<td>-41</td>
<td>-57</td>
<td>3415</td>
</tr>
<tr>
<td>$*\chi_{c1}$</td>
<td>1$^1P_1$</td>
<td>3472</td>
<td>34</td>
<td>34</td>
<td>3505</td>
</tr>
<tr>
<td>$*\chi_{c2}$</td>
<td>1$^3P_0$</td>
<td>3472</td>
<td>90</td>
<td>78</td>
<td>3550</td>
</tr>
</tbody>
</table>

Our main concern in this section is to see whether or not the relativistic corrections are perturbative, i.e., first order perturbations. As seen in Table II, the difference between $\Delta M_{pert}$ and $\Delta M_{diag}$ is relatively small except for $J/\psi$, $\chi_{co}$ and $\chi_{c2}$ which are related to the fine and the hyperfine structures. In order to see the effects due to higher order perturbations on the fine structure in $^3P$ states and on the hyperfine structure in $S$ states, let us define

$$R_F = \frac{M(\chi_{c2}) - M(\chi_{c1})}{M(\chi_{c1}) - M(\chi_{co})}$$  \hspace{1cm} (10)$$

and

$$R_{HF} = \frac{M(\psi') - M(\eta_c')}{M(J/\psi) - M(\eta_c)}.$$  \hspace{1cm} (11)$$

When evaluated with the recently reported mass of $\eta_c$, the experimental values of these ratios are 0.48 and 0.78 for $R_F$ and $R_{HF}$, respectively. The perturbative calculation yields
while the effects beyond the lowest order perturbation give
\[ R_t = 0.48, \quad R_{HF} = 0.54. \] (13)

Therefore the effects of higher order perturbations must be reckoned with in estimating the fine and the hyperfine structures. The deviation of \( R_{HF} \) from the experimental value is mainly due to the calculated mass of \( \eta_c \) being 34 MeV larger than its experimental mass.

All states above the \( DD \) threshold and \( h_c \) are predictions. The predicted mass of \( h_c \) almost agrees with the lately reported experimental value.\(^\text{10} \) Note that \( \Delta M_s \) is very small for \( h_c \), for which only the spin-spin part of \( V_s \) contributes in the first order perturbation. The smallness of the effect of the spin-spin interaction in the \( P \) states is crucial in achieving good agreement with the empirical spectrum. If it were not for the factor \( \alpha_s(r) \), then the spin-spin interaction would be a contact interaction which has no effect on the \( P \) states. Because of \( \alpha_s(r) \), this contact interaction is smeared out, and one may wonder if the effect of the spin-spin interaction becomes too large. With our choice of the parameters, however, the effect of the spin-spin interaction in the \( P \) states remains sufficiently small.

For \( \phi \) (4160) and \( \phi \) (4415), no clear assignment of quantum numbers has been done. However, our model assigns \( 2D \) and \( 2S \) to \( \phi \) (4160) and \( \phi \) (4415), respectively. The mass of \( \phi \) (4160) lies within the experimental range, and the mass of \( \phi \) (4415) is slightly below the experimental value. The calculation overestimates the masses of \( \phi \) (3770) and \( \phi \) (4040). This suggests that the states near and above the \( DD \) threshold may not be well described by a simple potential model with the quark component alone. There are attempts to describe these states by incorporating other possible mechanisms such as the coupling to the continuum \( DD \) channel.\(^\text{13} \)

The amplitudes of the basis states are tabulated for the input states in Table IV. The off-diagonal elements of \( H' \) are so small that the admixture of states other than the DBS's gives negligible contributions to the masses. This supplements the aforementioned statement that, as far as the mass spectroscopy is concerned, \( H' \) can be

<table>
<thead>
<tr>
<th>States</th>
<th>Amplitudes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_c )</td>
<td>0.9986, 0.00275, 0.00300, 0.00265, 0.00226</td>
</tr>
<tr>
<td>( J/J \phi )</td>
<td>0.9914, -0.01186, -0.00410, -0.00236, -0.00168</td>
</tr>
<tr>
<td>( \phi' )</td>
<td>0.1121, 0.9873, -0.01030, -0.00394, -0.00159</td>
</tr>
<tr>
<td>( \psi' )</td>
<td>-0.0111, -0.00091, -0.00076, -0.00068, -0.00059</td>
</tr>
<tr>
<td>( \chi_{c0} )</td>
<td>0.9870, 0.1207, 0.0756, 0.0583, 0.0457</td>
</tr>
<tr>
<td>( \chi_{c1} )</td>
<td>0.9995, -0.0268, 0.0029, 0.0096, 0.0093</td>
</tr>
<tr>
<td>( \chi_{c2} )</td>
<td>0.8983, -0.1363, -0.0443, -0.0209, -0.0134</td>
</tr>
<tr>
<td>-0.0085, -0.0056, -0.0043, -0.0036, -0.0030</td>
<td></td>
</tr>
</tbody>
</table>
regarded as small perturbation. In the next section, however, we show that states with small amplitudes significantly affect leptonic decay widths and electromagnetic transition rates.

§ 4. Leptonic decay and electromagnetic transition

The spectroscopic calculations only reflect the global features of wave functions. On the contrary, the local behavior becomes important in the calculations of leptonic decay widths and electromagnetic transition rates. Leptonic decay widths are sensitive to the wave functions near the origin, while \( E1 \) and \( M1 \) transition rates are sensitive to the location of nodes in the initial and the final wave functions of the transition. In this section we examine the relativistic effects related to the behavior of the wave function. Let us start with electron-positron decays as a typical leptonic decay process, focusing on the cases for which experimental values are available.

4.1. Electron-positron decay widths

We have two types of relativistic effects in describing \( e^+e^- \) decays within the order of \( (p/m)^2 \). One of them enters through the wave functions and the other through the quark current.\(^{14}\) The \( e^+e^- \) decay width for the \( 1^- \) state is given by

\[
\Gamma(1^- \to e^+e^-) = \left( \frac{4\alpha}{3M} \right) \left[ R_s(0) + \frac{R''_s(0)}{2m^2} + \frac{5R''_D(0)}{2\beta^2 m^2} \right]^2 ,
\]

where \( \alpha \) is the fine structure constant, \( M \) the charmonium mass, and \( R_s(r) \) and \( R_D(r) \) the radial parts of the \( S \)- and \( D \)-wave parts of the charmonium wave function, respectively. When the second derivative terms are ignored, Eq. (14) becomes the well-known van Royen-Weisskopf (vRW) formula.\(^{15}\) The second term with \( R''_s \) turns out to be very important in the decays of \( J/\psi \) and \( \psi' \).

In Table V(a) we give the \( e^+e^- \) decay widths from \( J/\psi \) and \( \psi' \). The values in column vRW are those obtained by the vRW formula with only the DBS's, i.e., those expected from the nonperturbative calculations. Note that, if we multiply the value in column vRW by the square of the amplitude of the corresponding DBS, we can extract the contribution from the DBS to the decay width by the diagonalization method. The values under \( S \) are the decay widths by the vRW formula including all the \( S \)-wave radially-excited-basis-states (REBS's) mixed through diagonalization. They exhibit the relativistic effects which arise through wave functions as compared with those of vRW. The values under \( S+S'' \) and \( S+S''+D'' \) are those obtained

<table>
<thead>
<tr>
<th>Decays</th>
<th>vRW</th>
<th>( S )</th>
<th>( S+S'' )</th>
<th>( S+S''+D'' )</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J/\psi \to e^+e^- )</td>
<td>13.97</td>
<td>10.14</td>
<td>7.59</td>
<td>7.49</td>
<td>5.1 ±0.3</td>
</tr>
<tr>
<td>( \psi' \to e^+e^- )</td>
<td>5.30</td>
<td>5.30</td>
<td>3.52</td>
<td>3.47</td>
<td>2.3 ±0.2</td>
</tr>
</tbody>
</table>

Table V. (a) Calculated \( e^+e^- \) decay widths in keV. Explanations of columns are given in the text. The experimental values in the upper rows were quoted from Ref. 10, and those in the lower rows were from Ref. 12.)
when the terms up to the second and the third terms of Eq. (14) are added, respectively. The values in these two columns give the magnitudes of the relativistic effects which arise through quark current as compared with those in column S.

The important role of the REBS's is to be stressed, even when their amplitudes are very small. In the $J/\psi$ decay, for example, the mixing of the REBS's improves the results reducing the vRW value by 27%, while their probabilities sum up to only 1.7%. The coincidence of the values in columns vRW and S for $\psi'$ decay needs an explanation. As can be seen from Table IV, the contributions from $1S$ and $3S$ states to the $\psi'$ wave function compensate at short distance, resulting in the coincidence. This fact by no means implies that the mixing of basis states is unimportant in general.

The relativistic effects through the quark current involving $R''_S$ are obviously important. Thus the leptonic decay widths are influenced significantly not only by the values of the wave functions at the origin, but also by their behaviors around the origin. Strangely, little attention seems to have been paid to the role of this term so far. The effects through $R''_D$ are negligible for the decay from the $S$ states; this is simply because $R''_D$ itself is very small.

The $R''_D$ term becomes important, however, when Eq. (14) is applied to the decays from $D$ states. Note that the term with $R_S$ is induced by the tensor force in this case. Although it emerges with a small amplitude, the interference with $R''_D$ term yields significant contribution. We calculated the width for the decay from $\phi$ (3770) and obtained 0.06 keV (which is too small as compared with the experimental value of 0.26 ±0.15 keV). Detailed calculation shows that 40% of 0.06 keV comes from the interference of $R''_D$ with $R_S$ induced by the weak tensor force. One might expect that a stronger tensor force gives larger admixture of $S$ basis states which enhance the decay widths further up to the experimental value. However it can easily be demonstrated that such a strong tensor force ruins the spectrum below the $DD$ threshold.

Finally let us remark that the ratio of the decay width for $\psi'$ to that for $J/\psi$ is improved by the relativistic effects. It is 0.46, showing good agreement with the experimental value of 0.45, while the vRW values yield 0.38.*

4.2. $E1$ and $M1$ transition rates

In calculating electromagnetic transition rates, we have two types of corrections to the naive nonrelativistic expression. One is the relativistic effects through wave functions. The other is the effect due to the spatial extension of the interaction source, that is, the finite size effect. Let us give the expressions of the $E1$ transition rates specifically from an $S$ state to a $P$ state and vice versa for which experimental values are available. When the contributions from different angular momentum states coupled to the DBS's are ignored, they are

$$
\Gamma^{(E1)}(\psi' \rightarrow \chi_{cJ}) = \frac{1}{243} \alpha k^3 \sum_{n, n'} B(n'0; n1) 
\times \int_0^\infty d\rho \rho^3 \left[ 1 + \frac{k}{12m} \frac{(kr)^2}{120} \right] R_{n0}(\rho) R_{n1}(\rho)^2,
$$

(15)

*) This ratio has been frequently taken to be an experimental quantity to be compared with the theoretical values.
Table V. (b) Calculated $E1$ and $M1$ transition rates in keV. Explanations of columns are given in the text. The first 6 rows give $E1$ transition rates, and the rest $M1$ transition rates.

<table>
<thead>
<tr>
<th>Transitions</th>
<th>NRC</th>
<th>REW</th>
<th>FSE</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi' \rightarrow \chi_{c0} + \gamma$</td>
<td>41.7</td>
<td>21.3</td>
<td>21.3</td>
<td>20±4</td>
</tr>
<tr>
<td>$\phi' \rightarrow \chi_{c1} + \gamma$</td>
<td>36.7</td>
<td>34.2</td>
<td>34.4</td>
<td>19±4</td>
</tr>
<tr>
<td>$\phi' \rightarrow \chi_{c2} + \gamma$</td>
<td>24.7</td>
<td>30.6</td>
<td>30.8</td>
<td>17±4</td>
</tr>
<tr>
<td>$\chi_{c0} \rightarrow J/\psi + \gamma$</td>
<td>107.0</td>
<td>158.2</td>
<td>155.8</td>
<td>147±38</td>
</tr>
<tr>
<td>$\chi_{c1} \rightarrow J/\psi + \gamma$</td>
<td>232.3</td>
<td>301.4</td>
<td>293.7</td>
<td>&lt;355</td>
</tr>
<tr>
<td>$\chi_{c2} \rightarrow J/\psi + \gamma$</td>
<td>305.2</td>
<td>408.2</td>
<td>395.0</td>
<td>385±212</td>
</tr>
<tr>
<td>$J/\psi \rightarrow \eta_c + \gamma$</td>
<td>1.54</td>
<td>1.52</td>
<td>1.52</td>
<td>0.80±0.25</td>
</tr>
<tr>
<td>$\phi' \rightarrow \eta_c + \gamma$</td>
<td>0.78</td>
<td>0.76</td>
<td>0.75</td>
<td>0.4~2.8</td>
</tr>
<tr>
<td>$\phi' \rightarrow \eta_c + \gamma$</td>
<td>0</td>
<td>5.02</td>
<td>8.14</td>
<td>0.60±0.17</td>
</tr>
</tbody>
</table>

\[
\Gamma^{(E1)}(\chi_{cJ} \rightarrow J/\psi) = \frac{16}{81}ak^3 \sum_{n,n'} B(n'1; n0) \times \int_0^\infty dr r^3 \left[ 1 - \frac{k^2}{12m} \frac{(kr)^2}{120} \right] R_{n'1}(r)R_{n0}(r)^2, \tag{16}
\]

where $k$ is the momentum of the emitted photon. The coefficients $B(n_iL_i; n_f L_f)$ are the products of the amplitudes of the basis states which are specified here by the sets of the radial and the angular momentum quantum numbers as $(n_i, L_i)$ and $(n_f, L_f)$ for the initial and the final states, respectively, and can be calculated from Table IV.

Also we evaluate the $M1$ transition rates only between $S$ states according to

\[
\Gamma^{(M1)}(i \rightarrow f) = \frac{16}{27} (2S+1) \frac{ak^3}{m} \sum_{n,n'} B(n'0; n0) \times \int_0^\infty dr r^2 \left[ 1 - \frac{(kr)^2}{24} \right] R_{n'0}(r)R_{n0}(r)^2. \tag{17}
\]

The results are compiled in Table V(b). The nonrelativistic calculations yield the decay rates under NRC. The values under REW are those obtained when the relativistic effects through wave functions are taken into account. Column FSE gives the values obtained by including the finite size effects. We used the observed masses of mesons to calculate the emitted photon energies.

The mixing of the REBS's again plays a crucial role, depending on the matrix elements involved in the expressions (15) to (17). As seen in Table V(b), the effect is especially noticeable in the $E1$ transitions. We found that 49% of the transition rate from $\phi'$ to $\chi_{c0}$ is produced by constructive interference between the DBS's and other basis states of small amplitudes. The REBS's also contribute to the transition rates from $\phi'$ to $\chi_{c1}$ and $\chi_{c2}$, but the interference is destructive in these cases, and is less important. An exceptionally significant case occurs in the $M1$ transition from $\phi'$ to $\eta_c$ for which the DBS's in the initial and the final states are orthogonal. The relativistic corrections introduce the nonorthogonal basis states into the initial and the final wave functions to yield the nonvanishing transition rate.
Let us add a few words about agreement or disagreement seen in the $E1$ transition rates with the experimental values. In the transitions from $\psi'$ to $\chi_c$ their DBS's have different numbers of nodes, while in the transitions from $\chi_c$ to $J/\psi$ they have the same number of nodes. Consequently subtle cancellations take place in the matrix elements for the transitions from $\psi'$ to $\chi_c$. A slight change of the position of one node can easily give rise to a large variation of the outcome. On the contrary there is no such subtlety in the transitions from $\chi_c$ to $J/\psi$. This is why the $E1$ transition rates for $\chi_c$ states are generally in a better agreement with experiment than those for $\psi'$ are. This clearly shows that the detailed local features of the wave functions associated with the transitions are crucial in calculating the transition rates.

The finite size effect is noticeable only in the $M1$ transitions. When the number of nodes of the DBS's does not change in a transition, the finite size effect is less important. It becomes important, however, once the number of nodes changes. In the transition from $\psi'$ to $\eta_c$, for example, the transition rate is increased by 62% due to the finite size effect. We discuss the large discrepancy regarding the $M1$ transition rates in the next section.

§ 5. Summary

We investigated the relativistic effects on the properties of the charmonium system. We found that the relativistic corrections can be regarded as the first order perturbation to the energies of most of the states, while higher order perturbations are important for the fine and the hyperfine structures. The mass of $h_c$ is in good agreement with the current experimental value, while the mass of $\eta'_c$ is slightly larger than the reported value.

The leptonic decay widths are significantly affected by the mixing of the REBS's in the wave functions involved, even though their amplitudes are small. This feature has recently been pointed out by Lipkin in an approximate calculation based on the first order perturbation theory for wave functions. We also found that the relativistic effects through the quark current are important as much as those through the wave functions. All of these effects are indispensable for improving the decay rates. The $e^+e^-$ decay width from the state $\phi$ (3770) is still underestimated. The strength of the tensor force is too weak to yield large $S$-$D$ mixing which might explain the discrepancy. The coupling to the continuum $D\bar{D}$ channel, for example, may give rise to such a large $S$-$D$ mixing.

For $E1$ and $M1$ transitions the mixing of basis states plays more important role than it does in the leptonic decay, especially when the transitions are such that the number of nodes of the DBS's changes. An extreme case was seen in the $M1$ transitions from $\psi'$, yielding the nonvanishing transition rate.

Let us give comments on two points which we have not touched. First, the convergence of the $(p/m)^2$ expansion of the relativistic kinetic energy is rapid. This can be seen by calculating the ratio of $\Delta M_{\text{RKE}}$, the expectation value of the kinetic energy part of $H'$, to the nonrelativistic kinetic energy. It lies in the range from 0.05 (for $\chi_{co}$) to 0.10 (for $J/\psi$) for the states listed in Table II.

Secondly, after everything is combined, the $M1$ transition rates are still over-
Table VI. Estimated anomalous magnetic moment in the charmed-quark magneton, $e_d/(2m)$ where $e_d=-(2/3)e$. The wide ranges seen in the values from the decay of $\phi'$ to $\eta_c$ are due to the uncertainty in the experimental values of the decay rate (see Table V(b)).

<table>
<thead>
<tr>
<th>Transitions</th>
<th>Ours</th>
<th>Ref. 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi' \rightarrow \eta_c + \gamma$</td>
<td>$-0.73$</td>
<td></td>
</tr>
<tr>
<td>$\phi' \rightarrow \eta'_c + \gamma$</td>
<td>$-0.27 \sim 0.93$</td>
<td></td>
</tr>
<tr>
<td>$J/\phi \rightarrow \eta_c + \gamma$</td>
<td>$-0.27$</td>
<td>$-0.36$</td>
</tr>
</tbody>
</table>

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