Canonical Formulation of Supermechanics

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The canonical formulation of a theory of dynamical systems with both Grassmann even and odd variables is investigated. The sufficient condition for the system being analytically solvable is given. The geodesic motion of a particle in the super Poincaré upper half plane is solved as an example.

We investigate here the canonical theory of supermechanics, a dynamical system described by both Grassmann even and odd variables. As is well known, this theory is a rather straightforward extension of the conventional theory and well-formulated by simply replacing the ordinary Poisson brackets with the $\mathbb{Z}_2$-graded type. This fact implies the well-behaved mathematical object (and the formulation on it) for supermechanics exists. A supermanifold is known as the one of these. In fact, the canonical theory of supermechanics is supposed to be a simple application of a theory of supermanifolds. Here we intend to see how the theory goes well through some explicit calculations in supermechanics.

After reviewing the Hamilton formalism of supermechanics in brief, we consider the canonical mapping a little, and finally we give an example, a free particle motion in the super Poincaré upper half plane.

As Berezin and Marinov pointed out, a theory of dynamical systems requires merely the algebraic construction of the ring and the Lie algebra (Poisson bracket). In such a theory, the motion of the system will be represented by a continuous map of one-parameter (time) from the ring into itself, and the notion of trajectories does not appear. However it would be convenient that there exists such a notion in a physical theory. To this end, we introduce the geometrical object called a supermanifold. Therefore, let us begin with a $(2m,2n)$-dimensional (real) supermanifold $\mathcal{M}$ in the sense of DeWitt. We call it the phase space of a dynamical system, and every point of $\mathcal{M}$ represents a certain physical state of the dynamical system. Let $\mathcal{F}$ be the set of (differentiable) functions on $\mathcal{M}$. It constitutes a Grassmann algebra. We usually consider only the functions of type-definite (i.e., whether $\mathbb{R}$ or $\mathbb{R}^*$-valued), and call them physical quantities. The coordinate functions on $\mathcal{M}$ are called canonical variables, consist of two types of variables, $p_i$ and $q^i$, where $i=1, 2, \cdots, m, -1, -2, \cdots, -n$, and we call them canonical momentum and generalized coordinate respectively. Now we introduce the structure of the graded Lie algebra, the extended

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** We use $\mathbb{R}(\mathbb{R}^*)$ for (the set of) real Grassmann even (odd) numbers.

*** Throughout this paper, notations follow Ref. 2), except for the definition of a real odd Grassmann number: We define it as $a \in \mathbb{R}^* \iff \bar{a} = ia$.

**** All functions with a suffix of positive (negative) integer are supposed to be $\mathbb{R}(\mathbb{R}^*)$-valued.
Poisson bracket, into $\mathcal{F}$, defined by\(^3\)

$$\{f, g\} = \sum_i f \frac{\partial}{\partial q^i} \frac{\partial}{\partial p_i} g - \left( - \right) g \frac{\partial}{\partial q^i} \frac{\partial}{\partial p_i} f$$

(1)

for any $f, g \in \mathcal{F}$. It satisfies the axioms of $\mathbb{Z}_2$-graded Lie algebra. For the canonical variables,

$$\{q^i, p_i\} = \delta^i_j, \quad \{q^i, q^j\} = 0, \quad \{p_i, p_i\} = 0.$$  

(2)

The motion of the system is described as a trajectory in the phase space $\mathcal{M}$. The variation of $f \in \mathcal{F}$ along the trajectory is characterized as

$$\dot{f} = \{f, H\},$$

(3)

where $H \in \mathcal{F}$ and is called Hamiltonian. In the case of $f$ being the canonical variables,

$$\dot{q}^i = \{q^i, H\} = \frac{\partial}{\partial p_i} H, \quad \dot{p}_i = \{p_i, H\} = -H \frac{\partial}{\partial q^i}.$$  

(4)

These are the canonical equations of motion and determine the trajectory in $\mathcal{M}$ completely. If the Hamiltonian does not depend on a certain generalized coordinate $q^i$, then the equations of motion can be integrated right away with respect to $q^i$. Such a coordinate is called a cyclic coordinate. If all the coordinates $q^i$ are cyclic, then the system can be solved trivially.

Suppose that there are two dynamical systems defined on $\mathcal{M}$ and $\mathcal{M}'$ respectively and there is a one-to-one mapping (correspondence) between $\mathcal{F}$ and $\mathcal{F}'$ with preserving the structures of the algebra and the Lie algebra on $\mathcal{F}$. Let $\phi$ be such a mapping between $\mathcal{F}$ and $\mathcal{F}'$.

$$\phi : f \in \mathcal{F} \to f' \in \mathcal{F}', \quad \phi : \{f, g\} \in \mathcal{F} \to \{f', g'\} \in \mathcal{F'},$$

(5)

where $\{,\}$ is a Poisson bracket on $\mathcal{F}'$. We call such a mapping a canonical mapping, the isomorphism of the ring $\mathcal{F}$ and $\mathcal{F}'$. The canonical mapping brings an equivalence relation into the set of dynamical systems. Needless to say, we are interested in the trivial class, the class containing the trivially solvable system.

Finally we write the relation between the Lagrange and the Hamilton formalism. Let $L(q, \dot{q})$ be the Lagrangian. We define the canonical momentum $p_i$ and the Hamiltonian $H$ as

$$p_i = \frac{\partial}{\partial \dot{q}^i} L, \quad H = \sum_i p_i \dot{q}^i - L.$$  

(6)

then we can work in the Hamilton formalism.

As we have seen the main points of the Hamilton formalism, we will see the canonical mapping in detail.

First, we point out that for each canonical mapping, there exists the generator function $W$, which connects the old canonical variables $(p, q)$ (and the Hamiltonian $H$) with the new ones, $(P, Q)$ (and $H'$). Taking $W$ as a function of the $q^i$'s and the $P_i$'s, then,
They are the same as the conventional ones\textsuperscript{9} except for the distinction between right and left derivatives.

Now we propose the condition whether the system is in the trivial class or not and the way to construct the generator function. Analogous to the ordinary mechanics, if we can find \( m \mathbb{R}_c \) and \( n \mathbb{R}_c \)-valued \( f_i \)'s \( \in \mathcal{F} \), the integrals of motion, satisfying

\[
\begin{align*}
\left[ \text{sdet} \left( \frac{\partial f_i}{\partial p_j} \right) \right]_{\theta} &= 0, \\
\{ f_i, f_i \} &= 0, \\
\{ f_i, H \} &= 0,
\end{align*}
\]

then the dynamics on \( \mathcal{M} \) is to be canonically trivial. We can see it as follows: (8) indicates the mapping \( p_i \to f_i \) is regular (invertible) and (9) shows all \( f_i \)'s can be regarded as the new canonical momenta (see (2) and (5)). (10) represents the new system is trivial. The generator function \( W \) is constructed by inverting the equations; \( f_i (p, q) = c_i \) (all \( c_i \)'s are constant) in terms of \( p_i \), and combining with (7),

\[
p_i = p_i (q, c) = W \frac{\partial}{\partial q_i}.
\]

Therefore \( W \) is obtainable if the integrability conditions

\[
W \frac{\partial}{\partial q^i} \frac{\partial}{\partial q^j} - (-)^i W \frac{\partial}{\partial q^j} \frac{\partial}{\partial q^i} = p_i \frac{\partial}{\partial q^i} - (-)^i p_i \frac{\partial}{\partial q^i} = 0
\]

are satisfied. In fact, it can be shown as follows.\textsuperscript{5} We regard each \( f_i \) as a function of time \( t \) defined along a trajectory in \( \mathcal{M} \).

\[
f_i (p(q, c), q) = c_i.
\]

Differentiating both sides of (13) by \( t \), we obtain

\[
\left( f_i \frac{\partial}{\partial p_i} \right) \left( p_i \frac{\partial}{\partial q^k} \right) + \left( f_i \frac{\partial}{\partial q^k} \right) = 0.
\]

We apply \( (\partial / \partial p_k) f_i \) and do the summation over \( k \). Then

\[
M_{ij} = \sum_{k} \left( f_i \frac{\partial}{\partial p_i} \right) \left( p_i \frac{\partial}{\partial q^k} \right) \left( \frac{\partial}{\partial p^k} f_j \right) + \left( f_i \frac{\partial}{\partial q^k} \right) \left( \frac{\partial}{\partial p^k} f_j \right) = 0.
\]

And

\[
0 = M_{ij} - (-)^i M_{ji} = \sum_{k} \left( f_i \frac{\partial}{\partial p_i} \right) \left( p_i \frac{\partial}{\partial q^k} \right) \left( \frac{\partial}{\partial p^k} f_j \right) - (-)^i p_i \frac{\partial}{\partial q^i} \left( \frac{\partial}{\partial p^i} f_j \right).
\]

We use (9) on the last equality. By virtue of (8) we finally get (12). Once we get \( W = W(q, c) \), then, by differenctiating \( W \) by \( c_i \), we obtain the algebraic equations of \( q^i \)'s. In the conservative system, we can take the Hamiltonian as the one of the

\textsuperscript{9} B abbreviates "Body".
$R_c$-valued integrals, and these equations are $(H(p, q) = E)$

$$\frac{\partial}{\partial c_m} W = \frac{\partial}{\partial E} W = t + t_0, \quad \frac{\partial}{\partial c_i} W = \text{const.} \quad (i \neq m)$$

(17)

$p_i = p_i(t)$ are given by (11), and the trajectory in $\mathcal{M}$ is determined.

As an example, we solve here the free (geodesic) motion of a particle in the super Poincaré upper half plane, $sH$.\(^{5}\) We write the geometry of $sH^0$ at first. It is a $(1, 1)$-dimensional complex Riemannian supermanifold given by a supersymmetrization of the ordinary one.\(^7\) The metric, which is super Möbius invariant, is (with $q = (z, \theta)$)

$$ds^2 = dq_i g_{ij}(q) dq^j$$

$$= \frac{1}{Y^2} \{dzd\bar{z} - i\theta dzd\bar{\theta} + i\bar{\theta} dzd\theta - (2Y + \theta\bar{\theta}) d\theta d\bar{\theta}\},$$

(18)

where $Y = \text{Im}z + (1/2) \theta\bar{\theta}$. The super Möbius transformation is given by

$$(z, \theta) \in sH \leftrightarrow (z', \theta') = \left( \frac{az + b}{cz + d}, \theta + \frac{a\theta + \beta}{cz + d} \right) \in sH,$$

(19)

where $a, b, c, d \in \mathbb{R}_c$, $ad - bc = 1$, $a, \beta \in \mathbb{R}_a$ and each of which generates the isometry mapping, the distance-preserving mapping, $sH \mapsto sH$. Now we consider the free particle motion in $sH$. The Lagrangian is $L = (1/2)(ds/dt)^2$. The Euler-Lagrange equations, which coincide with the super geodesic equations of $sH$, are

$$\ddot{z} + \frac{1}{Y}(i\dot{z}^2 - \dot{z} \dot{\theta}\bar{\theta}) = 0, \quad \ddot{\theta} + \frac{i}{Y} \dot{z} \dot{\theta} = 0.$$

(20)

The particle always moves along the geodesics of $sH$. With the real coordinates, $z = x + iy$, $\theta = \xi + i\eta$, the Hamiltonian is

$$H = \frac{1}{2} (y^2 + y\xi\eta)(p_x^2 + p_y^2) - \frac{1}{2} (y + \xi\eta)p_x p_y$$

$$- \frac{1}{2} y(\xi p_\xi + \eta p_\eta)p_\eta + \frac{1}{2} \gamma(\xi p_\xi - \eta p_\eta)p_x.$$

(21)

Before solving the dynamics generated by $H$, it is worth while seeing the fact that we can translate an arbitrary geodesic by some isometry, with preserving the dynamical properties, into the one which begins at $(z, \theta) = (i, 0)$ and terminates at $(is, \sigma + i\tau)$ $(s \in \mathbb{R}_c; \sigma, \tau \in \mathbb{R}_a)$. This means that we only concentrate on solving the motion “along the y-axis”. The whole dynamics (i.e., all the geodesics in $sH$) can be known through the isometries, the super Möbius transformations. Now let us see these features in dynamics. The coordinate $x$ is cyclic, therefore $p_x = \text{const} = k$. The time-derivative of $x$ is given as

$$\dot{x} = 2k(y^2 + y\xi\eta) + \frac{\gamma}{2}(p_x \eta - p_\eta \xi).$$

(22)
Concerning the last term, we find
\[ \frac{d}{dt} (p_x \eta - p_y \xi) = 0. \] (23)

Therefore under the initial condition as \( k=0 \), \( \xi(0) = \eta(0) = 0 \) (always realizable due to the preceding discussion), \( x(t) = 0 \) and \( p_x(t) = 0 \) hold for all \( t \). The motion in \( x \)-direction is quenched.

Now we are involved in solving the dynamics generated by reduced Hamiltonian
\[
H' = [H(p, q)]_{p_x=0}
\]
\[
= \frac{1}{2} \left( y^2 + y \xi \eta \right) p_y^2 - \frac{1}{2} \left( y + \xi \eta \right) p_x^2 - \frac{1}{2} y(\xi p_x + \eta p_y) p_y,
\] (24)

under the initial condition
\[
y(0) = 1, \quad \xi(0) = 0, \quad \eta(0) = 0.
\] (25)

We can employ the Hamiltonian itself as an \( R_c \)-valued integral. The residual two \( R_a \)-valued integrals are given as, for example,
\[
Q_1 = p_y + p_x, \quad Q_2 = p_y - p_x.
\] (26)

It is easily seen that \( H' \), \( Q_1 \) and \( Q_2 \) satisfy (8), (9) and (10), e.g., the explicit calculation of \( sdet \) in (8) gives
\[
\text{sdet} \left( \frac{\partial (H', Q_1, Q_2)}{\partial (p_y, p_x, p_x)} \right) = -(y + \xi \eta)^2 p_y.
\] (27)

We solve the equations \( H' = E = (1/2) \omega^2, Q_1 = \mu \) and \( Q_2 = \nu \) (\( \omega, \mu \) and \( \nu \) are all constant) in terms of \( p_i \). The results are
\[
p_y = \frac{\omega}{y + \xi \eta} - \frac{1}{2} \frac{\mu \nu}{\omega}, \quad p_x = \mu - \omega \eta, \quad p_\eta = \frac{1}{2} \frac{\mu \nu}{\omega} \xi - \frac{1}{2} \frac{\mu \nu}{\omega} \xi,
\] (28)

and \( W \) is to be
\[
W = \omega \ln(y + \xi \eta) - \frac{1}{2} \frac{\mu \nu}{\omega} (y + \xi \eta) - \nu \eta + \mu \xi.
\] (29)

By substituting it into (17), and \( \gamma = \mu/(2\omega), \delta = \nu/(2\omega) \), we get
\[
\ln(y + \xi \eta + 2 \gamma \delta (y + \xi \eta)) = \omega(t + t_0),
\]
\[ - \delta(y + \xi \eta) + \xi = \alpha, \quad \gamma(y + \xi \eta) - \eta = \beta,
\] (30)

where \( t_0 \in R_c \) and \( \alpha, \beta \in R_c \) are all constant. Solving the equations in terms of \( y, \xi \) and \( \eta \), and with some redefinitions of constants, we get the general solution for \( H' \).
\[
y(t) = e^{\omega(t+t_0)} - \gamma e^{2\omega(t+t_0)} + \alpha \beta,
\]
\[
\xi(t) = \alpha + \delta e^{\omega(t+t_0)}, \quad \eta(t) = -\beta + \gamma e^{\omega(t+t_0)}.
\] (31)

Taking account of the initial condition (25), we finally get the solution of \( H \) “along the \( y \)-axis”,

\[ z(t) = iy(t) = ie^{\omega t} + i\gamma z - i\gamma e^{2\omega t}, \]
\[ \theta(t) = \xi(t) + i\eta(t) = (\delta + i\gamma)(e^{\omega t} - 1). \]

Notice that it contains the parameters \( \omega \in \mathbb{R}_c \) and \( \gamma, \delta \in \mathbb{R}_a \). The general solution is obtainable from them through the super Möbius transformation, which brings additional three \( \mathbb{R}_c \) and two \( \mathbb{R}_a \)-valued parameters. \(^6\)

The results obtained here imply that the fundamental framework of the canonical theory and the key notions (e.g., the variational principle or the canonical mapping etc.) are still meaningful in supermechanics. The explicit calculations can be also performed along the lines of the canonical theory as we have seen. We can say that a supermanifold is a well-behaved mathematical object in this sense. We have given the sufficient condition of the systems being analytically solvable. Actually all we have to do is to find the adequate number of integrals satisfying the conditions \((8) \sim (10)\). However we still cannot give a general and systematical way to find them. In fact, the two \( \mathbb{R}_a \)-valued integrals (see (26)) were found by trial and error. Generally speaking, the integrals of motion are to be related to the geometrical invariants, therefore it may be meaningful to investigate around these features.

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2) B. DeWitt, Supermanifold (Cambridge Univ. Press, 1984).
5) T. Yamanouchi, Ippan Rikigaku (Iwanami, Tokyo, 1959).