Information Aspects of Type IX Models

G. FRANCISCO

Instituto de Física Teórica, Universidade Estadual Paulista
Rua Pamplona, 145, 01405-São Paulo-S.P.

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We present a study about amounts of information necessary to localize the orbit of type IX models when the system is evolved towards the cosmological singularity. We discuss how this affects the estimate of the dimension of the associated space of Kasner configurations.

In order to study information functions associated to the Mixmaster diagonal type IX model we need to find a suitable family of probability distributions that describe the statistical behaviour of the orbits. The chaotic properties of the system\(^1\)\(^-\)\(^3\) are usually described by evolving the model towards the initial singularity and by finding the distribution \(\mu(z) = [(1+z)\ln 2]^{-1}\) giving the asymptotic probability \(p(z, \Delta z) = \int_{z}^{z+\Delta z} \mu(z') \, dz'\) that a Kasner configuration, parametrized by a Kasner parameter \(z \in (0, 1)\), be found in \((z, z+\Delta z)\). As is well known the measure \(\mu\) is preserved by the Poincaré map of the model,\(^3\) here denoted by \(T\), responsible for the dynamical evolution in terms of a sequence of Kasner configurations: Starting from a given \(z_0\) the orbit can be approximated by a sequence \(\{z_0, Tz_0, \ldots, T^n z_0, \ldots\}\). It is possible to generate from an initial \(\mu_0\) a family \(\mu_n\) under the iteration of \(T\) such that \(\mu_{Loo} = \mu\). A theorem of Szüssz\(^4\) asserts that \(\mu_n(z) = \mu(z) + O(q^n)\) with \(0 < q < 1\). Here we will use the normalization factor \(N_n\) and redefine \(\mu_n(z) = N_n(\mu(z) + O(q^n))\) with \(O(q^n) = e^{-\lambda n z}\), \(\lambda > 0\)\(^3\)\(^,\)\(^5\). From the relationship\(^6\) \(n \propto \ln \ln |\Lambda|\) we see that \(n\) can be related to the synchronous time \(t\) and the singularity is located at \(n = \infty\). From Refs. 2) and 7) \(\Lambda\) is the trace of the momentum tensor of the model during the Kasner stages of the evolution. An interpretation of \(z\) can also be given: It is related to the direction of the orbit of the system inside Misner's equipotential walls.\(^*\)

Due to a symmetry of the problem all distributions compatible with this system must be peaked towards \(z=0\) (e.g., \(\mu(0) > \mu(z)\), \(z > 0\)). Any initial distribution \(\mu_0\) satisfying this condition also fulfils a hypothesis in Szüssz theorem requiring \(\mu_0\) to be \(\theta\)-Lipschitz. Consequently the family \(\mu_n\) discussed, \(n \geq 1\), will indeed be generated from all \(\mu_0\) compatible with the Mixmaster model (where \(\lambda\) depends only on \(\theta\)).

Another remark that should be made is the compatibility of \(\mu_n\) under the evolution of average values of measurable functions: \(\langle \phi \rangle_{n+1} = \langle T \phi \rangle_n\) where \(\langle \phi \rangle_n = \int \phi(z) \mu_n(z) \, dz\).

Consider an experiment whose outcomes belong to a finite family \(\xi = \{A_i\}\) and have probability \(p(A_i)\). The information function defined over these outcomes

\(^*\) In Misner's approach\(^2\) the pair \((A, z)\) can be used to parametrize the pair of canonical momenta of the model during Kasner stages.\(^7\)
\[ I(A_i) = -\ln p(A_i) \] (1)
gives a quantitative measure of how unexpected the outcome is.\(^8\) The average information of the experiment is
\[ S(\xi) = -\sum_{A_i \in \xi} p(A_i) \ln p(A_i). \] (2)

We want to perform measurements on the Mixmaster model in order to determine the average information for obtaining Kasner configurations as outcomes that is, the information about directions of the orbit in Misner's Hamiltonian treatment. Since the distributions \( \mu_n, \mu \) are continuous we might consider extending (2) to objects like
\[ -\int \mu(z) \ln \mu(z). \] However integrals of this kind are not necessarily positive, not necessarily finite and not always interpretable as average information.\(^9\) Another alternative is to introduce a partition on the space of Kasner parameters, \( \xi = \{ A_i \}_{i=1}^N, \) \( A_i \cap A_j = \emptyset, A_i \cup \cdots \cup A_n = (0, 1), \) and define \( p_n(A_i) = \int_{A_i} \mu_n(z) dz \) with size \( A_i, \) size \( A_i+1 \) on \( (0, 1). \)

Define the following average:
\[ S(\xi, \mu_n) = -\sum_{A_i \in \xi} p_n(A_i) \ln p_n(A_i). \] (3)

In order to compute \( \Delta S(\xi, n) = S(\xi, \mu_{n+1}) - S(\xi, \mu_n), \) we need the following inequality.\(^{10}\)
Suppose the numbers \( \{ \alpha_i \}_{i=1}^N, \) \( \{ \beta_i \}_{i=1}^N; \) satisfy (i) \( \alpha_i \geq \alpha_{i+1}, \beta_i \geq \beta_{i+1}; \) (ii) \( \sum_{i=1}^N \alpha_i = 1 = \sum_{i=1}^N \beta_i; \) (iii) \( \sum_{i=1}^k \alpha_i \leq \sum_{i=1}^k \beta_i \) for any \( 1 \leq k < N. \) Then \( -\sum_{i=1}^N \alpha_i \ln \alpha_i \geq -\sum_{i=1}^N \beta_i \ln \beta_i. \)

For the Mixmaster universe we label the elements of \( \xi \) from \( z=0 \) to \( z=1 \) in increasing order and take \( \alpha_i = p_n(A_i), \) \( \beta_i = p_{n+1}(A_i). \) Clearly (i) and (ii) are satisfied. Since
\[ \int_0^{\mu_{n+1}(z)} dz = \int_0^{\mu_n(z)} dz, \mu_{n+1}(0) > \mu_n(0) \] and \( \mu_n(z), n \geq 1, \) are strictly decreasing (iii) must hold. Consequently \( \Delta S(\xi, n) \leq 0, \) and there is a decrease of \( S(\xi, \mu_n) \) as the system is propagated towards the singularity. Note that this conclusion does not depend on the partition \( \xi \) (see the Appendix).

Any experiment made to determine the Kasner configuration of the system must necessarily involve finite resolution apparatus. This means the introduction of a partition \( \xi \) with diameter \( \varepsilon > 0. \) From Young (Ref. 11), p. 120 and th. 4.4) we are able to use the experiment in order to obtain estimates of the Hausdorff dimensions associated to the model. For sufficiently small \( \varepsilon \)
\[ HD(\varepsilon, \mu_n) = \inf_{\xi, \text{diam } \xi \leq \varepsilon} \frac{S(\xi, \mu_n)}{\ln 1/\varepsilon} \] (4)
gives a good approximation for the Hausdorff measure \( HD(\mu_n). \)\(^{*)\) Thus we necessarily have \( HD(\varepsilon, \mu_{n+1}) \leq HD(\varepsilon, \mu_n) \) that is, the estimates of \( HD \) arising from the measures \( \mu_n \) decrease as \( n \to \infty. \) Such a result is unavoidable when calculating \( HD(\varepsilon, \mu_n) \) using computer simulation since machines can only handle a finite number of digits. Strictly speaking the Mixmaster universe has two degrees of freedom and in order to

\(^{*)\) The definition of the Hausdorff dimension associated to \( \mu_n \) requires arbitrarily fine resolution: \( HD(\mu_n) = \lim_{\varepsilon \to 0} \inf HD(\varepsilon, \mu_n). \) Equation (4) is only an approximation to \( HD(\mu_n). \)
fully specify a Kasner configuration in momentum space\(^*\)) we need a pair \((\Lambda, z)\). However considering distributions \(\mu_n(\Lambda, z)\) would not have altered our conclusions since \(\Lambda\) is a monotonically decreasing variable with asymptotic trivial delta function distribution.\(^{12}\) In summary during the evolution towards the singularity the 1-dimensional approximation given by the Poincaré map gets better while \(HD\), related to the number of parameters needed to specify points on sets, undergoes a monotonic decrease.

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**Appendix**

To see that \(\mu_{n+1}(0) > \mu_n(0)\) just write \(\mu_n(z) = N_n f_n(z)\) where \(f_n(z) = [(1+z)\ln 2]^{-1} + e^{-\lambda n z}\), \(n \geq 1\), and observe that for \(z > 0\), \(f_{n+1}(z) < f_n(z)\). Thus \(\int f_{n+1}(z)dz < \int f_n(z)dz\) and in order that \(\int \mu_{n+1}(z)dz = \int \mu_n(z)dz\) it is evident that we must necessarily have \(N_{n+1} > N_n\); \(n \geq 1\). From \(f_{n+1}(0) = f_n(0)\) one immediately obtains \(\mu_{n+1}(0) > \mu_n(0)\).

In order to prove (iii) holds for any finite partition \(\xi\), note the following facts: \(\int f_{n+1}(z)dz = \int f_n(z)dz\), \(\mu_{n+1}(0) > \mu_n(0)\) and \(\mu_n(z)\) is a strictly decreasing function for any \(n\). From this one readily concludes that \(\mu_{n+1}(z)\) and \(\mu_n(z)\) must cross at one, and only one, point \(z_0 \in (0, 1)\). Using again \(\int f_{n+1}(z)dz = \int f_n(z)dz\) it follows that the area between these functions for \(0 < z < z_0\) (where \(\mu_{n+1}(z) > \mu_n(z)\)) must be equal to the area between these functions for \(z_0 < z < 1\) (where \(\mu_{n+1}(z) < \mu_n(z)\)). In conclusion \(\sum_{i=1}^{n} \mu_{n+1}(A_i) > \sum_{i=1}^{n} \mu_n(A_i)\) for \(k = 1, \ldots, N-1\), no matter how fine one chooses the partition \(\xi\) (fineness of \(\xi\) does not affect the equality of the areas discussed above, which is the key point of our argument).

Note that in this paper we are studying the consequences of assuming that the fluctuation \(O(q_n)\) has the form \(e^{-\lambda n z}\). If this condition is dropped then no conclusion can be derived about the behaviour of the averages (3).

**References**

5) We are thankful to J. Barrow for suggesting this form for \(O(q_n)\).
10) A. Wehr, Rev. Mod. Phys. 50 (1978), 221.

\(^*\) See the footnote on page 251.