Research note

Finite integral solution for the point source response of a layered acoustic medium

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The computation of the analytic response for a time-harmonic point source, located within a stratified earth model, is in general performed with what has become known in seismology as the reflectivity method (Harkrider 1964; Fuchs 1968; Fuchs & Müller 1971; Kennett 1983).

One of the basic features of the reflectivity method is that the exact solutions are formulated in terms of infinite solution integrals. As is shown in this paper, the classical solution integral obtained for a time-harmonic point source above a stratified acoustic medium can be easily transformed into a finite integral solution. It results from causality considerations and investigating the reflectivity function.

For instance, for the transient reflected acoustic potential, that results from the following incident acoustic potential

$$\phi_{\text{inc}}(t) = \frac{1}{R} \delta(t - R/c) = \text{Re} \left\{ \frac{1}{\pi} \int_0^\infty d\omega \exp(i\omega t) \hat{\phi}_{\text{inc}}(\omega) \right\}$$

where

$$\hat{\phi}_{\text{inc}}(\omega) = \frac{1}{R} \exp(-i\omega R/c)$$

and where the source is located at point $S$ above the stratified acoustic layers (Fig. 1), the reflectivity method offers for $z < 0$ the well-known transient acoustic potential for the reflection response:

$$\phi_{\text{ref}}(t) = \text{Re} \left\{ \frac{1}{\pi} \int_0^\infty d\omega \exp(i\omega t) \hat{\phi}_{\text{ref}}(\omega) \right\}$$

where

$$\hat{\phi}_{\text{ref}}(\omega) = -i\omega \int_0^\infty dp \frac{P}{P_0(p)} J_0(rp\omega) \exp\left[-i\omega(h - z)P_0(p)\right] R(p, \omega).$$

Equation (1c) is evaluated by a numerical integration over frequency. $J_0(a)$ is the zeroth-
order Bessel function of the first kind and
\[
P_0(p) = \sqrt{1/c_0^2 - p^2} = \begin{cases} |P_0(p)| & \text{if } 0 < p < 1/c_0 \\ -i|P_0(p)| & \text{if } p > 1/c_0 \end{cases}
\]
\(R = \sqrt{r^2 + (z + h)^2}\). \(t\) is time, \(h\) is the distance of the point source \(S\) above the uppermost interface, \(z\) and \(r\) are the cylindrical coordinates, \(p\) is the ray parameter and \(\omega\) the circular frequency. \(\rho_j\) and \(c_j\) designate the density and velocity of the \(j\)th layer. \(R(p, \omega)\) is the reflectivity function. For further details of the theory of the reflectivity method that leads to equations (1), we refer the reader to the above-mentioned publications.

Throughout this note we will ignore the index \(j = 0\). If \(i = 0\), \(P(0) = P_0(p)\), \(c = c_0\).

Let us remark that the infinite integral in (1d) cannot of course be fully taken care of. In practice one usually considers the range \(0 < p < p_{\text{max}}\), where \(p_{\text{max}} = \max\{1/c, 1/c_1, \ldots, 1/c_{N+1}\}\).

Subsequently we show how \(\phi_{\text{ref}}(\omega)\) can be exactly expressed without having to perform the infinite integration over \(p\). For this we make use of the following two results:

(a) Due to the causality of \(\phi_{\text{ref}}(t)\) it is well-known that
\[
\hat{\phi}_{\text{ref}}(\omega) = \text{Re} \hat{\phi}_{\text{ref}}(\omega) + i \text{Im} \hat{\phi}_{\text{ref}}(\omega) = -H[\text{Im} \hat{\phi}_{\text{ref}}(\omega)] + i \text{Im} \hat{\phi}_{\text{ref}}(\omega)
\]
where \(H[f(\omega)]\) denotes the Hilbert transform of \(f(\omega)\).

(b) As subsequently proven we have
\[
\text{Im} \hat{\phi}_{\text{ref}}(\omega) = \text{Im} \left\{ -i\omega \int_0^{p_m} dpp^{-1}_m(p) J_0(rp\omega) \exp \left[ -i\omega(h - z)P(p) \right] R(p, \omega) \right\}
\]
where
\[
p_m = \max \{1/c_0, 1/c_{N+1}\}.
\]
Consequently the exact spectrum $\hat{\phi}_{\text{ref}}(\omega)$ can be computed according to (2) by first evaluating $\text{Im} \hat{\phi}_{\text{ref}}(\omega)$ by the finite integral expression (3a) and then performing a Hilbert transform on $\text{Im} \hat{\phi}_{\text{ref}}(\omega)$ to obtain $\text{Re} \hat{\phi}_{\text{ref}}(\omega)$.

To demonstrate the validity of (3) we must prove that for all $p > p_m$

$$A(p) = \text{Im} \left\{ (-i\omega)p \frac{P^{-1}(p)}{J_0(rp\omega)} \exp \left[ -i\omega(h - z)P(p) \right] R(p, \omega) \right\} = 0.$$  \hspace{1cm} (4a)

Now, for all $p > p_m$, we have from (1c) $P(p) = -i |P(p)|$, so that one can write for $A(p)$ the simpler expression

$$A(p) = \omega p |P(p)|^{-1} J_0(rp\omega) \exp \left[ -\omega(h - z) |P(p)| \right] \text{Im} \{ R(p, \omega) \}. \hspace{1cm} (4b)$$

Our assertion is true if we show that $R(p, \omega)$ is real whenever $p > p_m$. This interesting fact is proven as follows:

Let us recall the well-known recursion formula for the time-harmonic reflectivity. For simplicity let us write $R = R_0$. The recursion providing $R_0$ is the following one:

$$R_N = r_N$$

$$R_{j-1} = \frac{r_{j-1} + z_j R_j}{1 + r_{j-1} z_j R_j} \quad (j = N, N - 1, \ldots, 1)$$

where the reflection coefficients as a function of the ray parameter $p$ are

$$r_j = \frac{\rho_{j+1} P_j(p) - \rho_j P_{j+1}(p)}{\rho_{j+1} P_j(p) + \rho_j P_{j+1}(p)} \quad (j = 0, 1, \ldots, N)$$

with

$$P_j(p) = \sqrt{p_j^2 - p^2} = \left\{ \begin{array}{ll}
|P_j(p)| & \text{if } 0 < p < p_j = 1/c_j \\
-i |P_j(p)| & \text{if } p > p_j
\end{array} \right. \quad (j = 0, 1, \ldots, N + 1).$$

Furthermore we have

$$z_j = \exp (-2i\omega r_j) \quad (j = 0, 1, \ldots, N)$$

where

$$r_j = \left\{ \begin{array}{ll}
hP(p) & \text{if } j = 0 \\
(H_j - H_{j-1})P_j(p) & \text{if } j = 1, 2, \ldots, N.
\end{array} \right.$$}

As is readily verifiable, the following properties hold for $j = 0, 1, \ldots, N$

(i) \hspace{1cm} $r_j = \left\{ \begin{array}{ll}
r_j & \text{if } p > \max(p_j, p_{j+1}) \\
1/r_j & \text{if } \min(p_j, p_{j+1}) < p < \max(p_j, p_{j+1})
\end{array} \right.$

(ii) \hspace{1cm} $z_j = \left\{ \begin{array}{ll}
z_j & \text{if } p > p_j \\
1/z_j & \text{if } 0 < p < p_j
\end{array} \right.$

and the following ones for $j = 1, 2, \ldots, N$

(iii) \hspace{1cm} $R_{j-1} \left( 1/r_{j-1}, 1/z_j, 1/r_j \right) = R_{j-1} \left( r_{j-1}, z_j, R_j \right)$
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(iv) \[ R_{j-1} \left( \frac{1}{r_{j-1}}, z_j, R_j \right) = R_{j-1} \left( \frac{1}{r_{j-1}}, \frac{1}{z_j}, 1/R_j \right) = 1/R_{j-1} \left( r_{j-1}, z_j, R_j \right). \]

The above four properties lead to the subsequent important result for \( j = 0, 1, \ldots, N \)

\[
\tilde{R}_j = \frac{R_j(p, \omega)}{1/R_j(p, \omega)} = \begin{cases} 
R_j(p, \omega) & \text{if } p > \max(p_j, p_{N+1}) \\
1/R_j(p, \omega) & \text{if } p_{N+1} < p < p_j 
\end{cases}
\]

which includes the statement that \( R_0 \) is real for \( p > \max(p_0, p_{N+1}) \).

We will now prove the correctness of the result (5) by induction. For \( j = N \) expression (5) is identical with expression (i). Now suppose that it is true for some value of \( j \) \((0 < j < N)\).

We wish to show that expression (5) is also true for \( j - 1 \).

Observe that

\[
\tilde{R}_{j-1} = \frac{\tilde{r}_{j-1} + \tilde{z}_j \tilde{R}_j}{1 + \tilde{r}_{j-1} \tilde{z}_j \tilde{R}_j}. \tag{6}
\]

Assuming \( p > \max(p_{j-1}, p_{N+1}) \), we may consider two cases, namely \( p > p_j \) and \( p < p_j \).

**Case 1:** \( p > p_j \)

Clearly \( \tilde{r}_{j-1} = 1/r_{j-1} \) and \( \tilde{z}_j = z_j \). Moreover, as \( p > \max(p_j, p_{N+1}) \) our induction hypothesis says that \( \tilde{R}_j = R_j \). It follows then from equation (6) that \( \tilde{R}_{j-1} = R_{j-1} \).

**Case 2:** \( p < p_j \)

Now clearly \( \tilde{r}_{j-1} = 1/r_{j-1} \) and \( \tilde{z}_j = 1/z_j \). Also as \( p_{N+1} < p < p_j \) we have by our induction hypothesis \( \tilde{R}_j = 1/R_j \). Hence we get from condition (6)

\[
\tilde{R}_{j-1} = R_{j-1} \left( \frac{1}{r_{j-1}}, \frac{1}{z_j}, 1/R_j \right) = R_{j-1}.
\]

We now proceed to prove the case \( p_{N+1} < p < p_{j-1} \).

We again consider two cases

**Case 1:** \( p > p_j \)

Then \( \tilde{r}_{j-1} = 1/r_{j-1} \), \( \tilde{z}_j = z_j \). As \( p > \max(p_j, p_{N+1}) \) it follows from our induction hypothesis that \( \tilde{R}_j = R_j \). Hence

\[
\tilde{R}_{j-1} = R_{j-1} \left( \frac{1}{r_{j-1}}, z_j, R_j \right) = 1/R_j.
\]

**Case 2:** \( p < p_j \)

Then \( \tilde{r}_{j-1} = r_{j-1}, \tilde{z}_j = 1/z_j \) and, as \( p_{N+1} < p < p_j \), \( \tilde{R}_j = 1/R_j \) by induction.

Hence

\[
\tilde{R}_{j-1} = R_{j-1} \left( r_{j-1}, 1/z_j, 1/R_j \right) = 1/R_{j-1}.
\]

This proves condition (5) in case \( p_{N+1} < p < p_{j-1} \) for \( j - 1 \), thus finishing the proof.
Conclusion

The reflectivity method is a well-known tool for constructing seismic responses for causal concentrated sources. Most of the attention, that has been given to the development of this method in the last decades has essentially been devoted to the stable and efficient computation of this time-harmonic reflectivity function \( R(p, \omega) \) (i.e. to the computation of the time harmonic plane wave response of a stack of uniform or piecewise smooth layers) and to solving problems for increasingly complex source–receiver configurations. To our knowledge no attempts have been made to transform in the indicated way the infinite solution integrals. The recipe by which the transformation can be performed is described. It may, so we believe, also be considered in connection with other solutions, which have so far been formulated with the help of the reflectivity method. Though the theory described in this note has not as yet been implemented, one can anticipate that the numerical computations will equal that of the traditional approach. One starts by evaluating (3a) at the same discrete frequencies one would consider for the usual evaluation of (1d). Thereafter one performs a discrete Hilbert transform of the discrete imaginary part of the spectrum in order to obtain the discrete real part of the spectrum \( \phi_{\text{ref}}(\omega) \) representing the desired causal reflection response. Instead of considering a numerical frequency integration one may also employ an analytic one. Further details on this are described by Tygel & Hubral (1985).

References