Cascade of Attractor-Merging Crises to the Critical Golden Torus and Universal Expansion-Rate Spectra

Takehiko HORITA, Hiroki HATA* and Hazime MORI**

Department of Physics, Kyushu University 33, Fukuoka 812
*Department of Physics, Kagoshima University, Kagoshima 890
**Faculty of Engineering, Kyushu Kyoritsu University, Kitakyushu 807

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Two chaotic attractors with rotation numbers $\rho_m = F_{m-1}/F_m$, $F_m$ being the $m$-th Fibonacci number, and $\rho_{m+1}$, just before their attractor-merging crisis converge to the critical golden torus as $m \to \infty$. A critical scaling is shown to hold for those chaotic attractors for large $m$, leading to a universal spectrum of the local expansion rates of nearby orbits.

The two-frequency dynamical systems exhibit a great variety of bifurcations in the two-dimensional space of parameters which control the ratio of the two frequencies and the strength of the nonlinearity, resulting in fascinating chaotic attractors. A typical example of the two-frequency systems is the sine-circle map, which is the map on a circle $[0, 1]$ onto itself expressed as

$$\theta_{t+1} = f(\theta_t) = \theta_t + \Omega - (K/2\pi)\sin(2\pi\theta_t), \mod 1$$

with $t = 0, 1, 2, \ldots$. The rotation number is the ratio of the two frequencies defined by $\rho(\theta_0) = \lim_{n \to \infty} [\theta_n - \theta_0]/n$ without taking modulo in (1) and plays an important role. In the $\Omega-K$ space, for a given rational number $p/q$, the region where the rotation number takes $p/q$ for some initial point $\theta_0$ is called the $p/q$-Arnold tongue. In a subregion of the $p/q$-Arnold tongue, there is a phase-locked chaotic attractor, i.e., a chaotic attractor with the rotation number $p/q$. When two Arnold tongues overlap, there is a point of attractor-merging crisis where two coexisting phase-locked chaotic attractors simultaneously merge into one phase-unlocked chaotic attractor by colliding with unstable periodic orbits. Such attractor-merging crises are ubiquitous in the two-frequency systems; indeed they are also observed in the dissipative standard map and the driven damped pendulum. In the previous paper, we have shown that the chaotic attractors just before and after the attractor-merging crisis are characterized by the expansion-rate spectra $\phi(\lambda)$. In the present paper, we shall discuss a cascade of such attractor-merging crises to the critical golden torus and a critical scaling law of their $\phi(\lambda)$ spectra.

The sine-circle map (1) is invertible for $K<1$ so that the map (1) with some $\Omega$ has a quasi-periodic orbit, i.e., an orbit with an irrational rotation number that is confined on a smooth invariant torus and has zero Liapunov exponent. For $K>1$, the map (1) is non-invertible and there no longer exist smooth invariant tori with zero Liapunov exponents. At $K=1$, the map (1) with some $\Omega$ has an invariant torus that just loses its smoothness, i.e., a critical torus.

Let us now consider a path in the $\Omega-K$ space along which a series of evolutions...
of chaotic attractors converging to a critical torus occurs as $K$ approaches $K_\infty = 1$. Figure 1 shows such a path for the critical torus with golden-mean rotation number $\rho_G = (\sqrt{5} - 1)/2$, i.e., the critical golden torus, where $A_{p/q}$ denotes the boundary of the $p/q$-Arnold tongue and $C_{p/q}$ denotes the line of crises. This path is composed of the subsets of $C_m$ with $l=1, 2, 3, \ldots$, where $\{\rho_l\}$ denotes the series of the best approximations of $\rho_G$ by rational numbers and is given by $\rho_l = F_{l-1}/F_l$ with the Fibonacci sequence $\{F_l\} = \{1, 1, 2, 3, 5, \ldots\}$ generated by $F_{l+1} = F_l + F_{l-1}$ with $F_0 = F_1 = 1$. On the path, there is a phase-locked chaotic attractor just colliding with an unstable crossing point of the two crisis lines, i.e., the attractor at the crisis which is composed of $F_l$ bands and has the rotation number $\rho_l$. Each turn on the path is denoted by $T_m$ ($m=1, 2, 3, \ldots$). It is the crossing point of two crisis lines, that is, $T_m = C_{\rho_m} \cap C_{\rho_{m+1}}$, and is the point of attractor-merging crises of two chaotic attractors with rotation numbers $\rho_m$ and $\rho_{m+1}$.

For example, $T_1 = (\Omega_1, K_1) = (0.7861195\ldots, 3.4057350\ldots)$. At this point two phase-locked chaotic attractors with rotation numbers $1/1$ and $1/2$ merge into one phase-unlocked chaotic attractor. Several values of $T_m = (\Omega_m, K_m)$ are listed in Table I.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\Omega_m$</th>
<th>$K_m$</th>
<th>$F_{m+1}A_m(\theta_{m+1})$</th>
<th>$F_mA_m(\theta_m)$</th>
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<tr>
<td>1</td>
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<td>3.405735089365</td>
<td>1.413443</td>
<td>1.418130</td>
</tr>
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<td>1.436573</td>
<td>1.497271</td>
</tr>
<tr>
<td>4</td>
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<td>1.268475456003</td>
<td>1.445470</td>
<td>1.499052</td>
</tr>
<tr>
<td>5</td>
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<td>1.156702126804</td>
<td>1.442263</td>
<td>1.504854</td>
</tr>
<tr>
<td>6</td>
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<td>1.444407</td>
<td>1.504013</td>
</tr>
<tr>
<td>7</td>
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</tr>
<tr>
<td>8</td>
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<td>1.032051470891</td>
<td>1.443994</td>
<td>1.504835</td>
</tr>
<tr>
<td>9</td>
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<td>1.443806</td>
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<td>1.443811</td>
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</tr>
<tr>
<td>$\vdots$</td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
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</tr>
<tr>
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<td>1.0</td>
<td>1.443813</td>
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$K_m$ converges to $K_\infty = 1$ and $\Omega_m$ converges to $\Omega_\infty = 0.60666106347011201\cdots$ as $m \to \infty$, and $\rho_m$ and $\rho_{m+1}$ converge to $\rho_C$ so that this is a cascade of attractor-merging crises towards the critical golden torus. Its renormalization group analysis predicts that $K_m$ converges to $K_\infty$ geometrically as $K_m - K_\infty \propto (a_{cm} - m)^{-\mu}$ for large $m$, where $a_{cm} = 1.66043\cdots$ is the eigenvalue of the fixed point of the renormalization transformation along the unstable direction associated with the change of the nonlinearity of the map.\(^\text{1,2}\) $\Omega_m$ also obeys a scaling law. The fixed point of the renormalization transformation has two unstable directions, one corresponding to the change of $K$ with the eigenvalue $a_{cm}$ and the other corresponding to the change of $\Omega$ with the eigenvalue $\delta = -2.83361\cdots$. Now consider the invariant manifold with codimension one, in the space of maps, which is transversal to the unstable direction with the eigenvalue $\delta$, and let us denote the intersection of the sine-circle map with this manifold by $\Omega = \overline{\Omega}(K)$, where $\overline{\Omega}(K_\infty = 1) = \Omega_\infty$. $\overline{\Omega}$ has been determined by MacKay and Tresser\(^\text{3}\) as $\overline{\Omega} = \Omega_\infty - 0.017482e^{-0.0005e^2 + \cdots}$ for $e = K - 1 > 0$. The corresponding intersection transversal to the eigendirection with $a_{cm}$ is $K = 1$. Using $\overline{\Omega}$, we obtain $\Omega_m - \overline{\Omega}(K_m) \propto \delta^{-m}$ for large $m$. Figure 2 shows a plot of $\varepsilon_m = K_m - K_\infty$ versus $\mu_m = |\Omega_m - \overline{\Omega}(K_m)|$ for $1 \leq m \leq 15$, which ensures the scaling for $K_m$ and $\Omega_m$.

For a chaotic orbit $\{\theta_t\}$ of (1), let $\lambda_i(\theta_m)$ be the local expansion rate of nearby orbits at $\theta_m$, i.e., $\lambda_i(\theta_m) = \log |f'(\theta_m)|$ and define the coarse-grained local expansion rate\(^\text{4}\)

$$\Lambda_n(\theta_0) = S_n(\theta_0)/n \quad \text{with} \quad S_n(\theta_0) = \sum_{m=0}^{n-1} \lambda_i(\theta_m).$$

(2)

$\Lambda_n(\theta_0)$ converges to a positive Liapunov exponent $\Lambda^\infty$ in the limit $n \to \infty$. The probability density for $\Lambda_n(\theta_0)$ to take a value around $\Lambda$ for given $n$ is given by

$$P(\Lambda; n) = \langle \delta(\Lambda_n(\theta_0) - \Lambda) \rangle = \lim_{N \to \infty} (1/N) \sum_{t=0}^{N-1} \delta(\Lambda_n(\theta_t) - \Lambda),$$

(3)

where $\delta(g)$ denotes the $\delta$-function of $g$. For chaotic attractors, we expect that $P(\Lambda; n)$ decays with $n$ for given $\Lambda$ as\(^\text{5}\)

$$P(\Lambda; n) \approx \exp(-n\phi(\Lambda))P(\Lambda^\infty; n)$$

(4)

for $n \to \infty$, where the spectrum $\phi(\Lambda)$ is a concave function of $\Lambda$ independent of $n$ with $\phi(\Lambda) \geq \phi(\Lambda^\infty) = 0$.

At $T_m$, there are two chaotic attractors just before the attractor-merging crisis; one is denoted by $a_m^\infty$, which has the rotation number $\rho_m$ and is composed of $M_m^\infty \equiv F_m$.
bands, and the other is denoted by \( a_{m}^{+} \), which has the rotation number \( \rho_{m+1} \) and is composed of \( M_{m}^{+} \equiv F_{m+1} \) bands. Since \( F_{m} = \frac{(\rho - (m+1) - (-\rho)(m+1))}{(\rho - 1 + \rho)} \sim \rho^{-m} \), the numbers \( M_{m}^{+} \) of bands of the attractors \( a_{m}^{\pm} \) grow as \( M_{m}^{\pm} \sim e^{-\kappa} \) for \( \varepsilon = K - K_{m} \rightarrow 0 \), where \( \kappa = \log \rho^{-1}/\log a_{m} = 0.9489 \cdots \). Similarly to the case of the cascade of band-splitting bifurcations to the critical 2\(^{m}\) attractor in the logistic map, the \( \phi(A) \) spectra may obey a critical scaling law for the cascade of attractor-merging crises to the critical golden torus, since there is a similarity of the map between two points \( T_{m-1} \) and \( T_{m} \), which is implied by the existence of the fixed point of the renormalization transformation. Indeed, the Liapunov exponents obey a scaling law

\[
\lambda_{m}^{\pm} = c_{m}^{\pm}/M_{m}^{\pm} \quad \text{for} \quad m \gg 1,
\]

where \( c_{m}^{+} \approx 0.693 \) and \( c_{m}^{-} \approx 0.692 \). Let \( \{\theta_{m,i}^{\pm}\} \), \( (i = 0, 1, 2, \cdots, M_{m}^{\pm} - 1) \) be the unstable periodic orbit lying at the edge of the attractor \( a_{m}^{\pm} \) just collided with the attractor. The expansion rates \( \lambda_{m}(\theta_{m,i}^{\pm}) \) of those periodic orbits also obey the scaling law

\[
\lambda_{m}(\theta_{m,i}^{\pm}) \sim z_{m}^{\pm}/M_{m}^{\pm} \quad \text{for} \quad m \gg 1,
\]

as shown in Table I, where \( z_{m}^{+} \approx 1.443813 \) and \( z_{m}^{-} \approx 1.505033 \). Moreover the \( \phi(A) \) spectrum obeys a scaling law. Let \( \phi_{m}^{\pm}(A) \) be the expansion-rate spectra of the attractors \( a_{m}^{\pm} \). Then

\[
M_{m}^{\pm} \phi_{m}^{\pm}(z/M_{m}^{\pm}) \rightarrow \phi^{\pm}(z) \quad \text{for} \quad m \rightarrow \infty.
\]

Since \( a_{m}^{+} \) and \( a_{m}^{-} \) are the attractors at the crisis, each \( \phi(A) \) of them has linear slopes \( s_{m}^{\pm} = -1 \) and \( s_{m}^{\mp} \). Then (5) and (6) lead to

\[
\varphi^{\pm}(z) = \begin{cases} 
(1/2)z_{m}^{\pm}/(z_{m}^{\pm} - c^{\pm}) \quad & \text{for} \quad z < c^{\pm} \\
\infty \quad & \text{for} \quad z_{m}^{\pm} < z,
\end{cases}
\]

where \( s_{m}^{\pm} \) are universal \( s_{m} \) slopes given by \( s_{m}^{\pm} = (1/2)z_{m}^{\pm}/(z_{m}^{\pm} - c^{\pm}) \) with \( z_{m}^{+} \approx 0.961 \) and \( z_{m}^{-} \approx 0.926 \). In Fig. 3, the scaled spectra \( M_{m}^{\pm} \phi_{m}^{\pm}(z/M_{m}^{\pm}) \) obtained by numerical experiments with \( n = 20 \times M_{m}^{\pm} \) are shown for \( m = 5 \) and 6, assuring (7) and (8).

For a chaotic attractor with the rotation number \( \rho_{m} = F_{m-1}/F_{m} \) existing between the successive band-merging crises \( T_{m-1} \) and \( T_{m} \), the scaling (7) may be extended to

\[
\phi(A) = (1/F_{m})\varphi_{k}(F_{m}A),
\]

where \( k = (K_{m-1} - K)/(K_{m-1} - K_{m}) \), and \( \varphi_{k}(z) \) with \( 0 \leq k \leq 1 \) is a function of \( z \) independent of \( K \) and \( \rho_{m} \) but dependent on \( K \) between \( K_{m-1} \) and \( K_{m} \) with \( \varphi_{0} = \varphi^{+} \) and \( \varphi_{1} = \varphi^{-} \).

A similar cascade of attractor-merging crises to a different critical torus with an irrational rotation number \( \rho \) exists and a similar critical scaling of \( \phi(A) \) would hold if the rotation number
The envelope grows as $\log n$ in the critical regime $n < M_{10}^+ = 144$, where the variance consists of blocks whose width is $\Delta \log n \approx \log \rho_G^{-1}$.

Figure 4 shows the variance for the attractor $a_{10}^+$ with $A^* = 0.00481$. It has an envelope that grows with $n$ as $\log n$ in the critical regime $n < M_{10}^+ = 144$ and nearly constant in the chaotic regime $n > M_{10}^+$. For the critical golden torus in the limit $m \to \infty$ (i.e., $M_m^* \to \infty$), this $\log n$ dependence is achieved in the whole time scale and suggests an algebraic spectrum $\psi_p(\beta)$ of local expansion rates.\(^5\)\(^,\)\(^6\)\(^,\)\(^7\)\(^,\)\(^8\) The critical regime of Fig. 4 also indicates that the variance of the critical golden torus consists of blocks with width $\Delta \log n \approx \log \rho_G^{-1}$ which develop a self-similar inversely-nested structure as time $n$ proceeds. This is a remarkable memory phenomenon without mixing. This temporal structure may be constructed by a rule associated with the Fibonacci tree composed of 0 and 1, in contrast to the critical $2^\omega$ attractor with the binary tree.\(^7\)\(^,\)\(^8\) These will be reported on future occasion.

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