(2+1)-Dimensional Quantum Gravity

— Case of Torus Universe —

Akio HOSOYA and Ken-ichi NAKAO*

Department of Physics, Tokyo Institute of Technology, Oh-okayama, Tokyo 152
*KEK, National Laboratory for High Energy Physics, Tsukuba 305

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The (2+1)-dimensional pure Einstein gravity is studied in the canonical ADM formalism, assuming that the spatial surface is closed and compact. In the torus case for the spatial manifold we can do an explicit analysis. Owing to the constraints, the dynamical variables are reduced to the moduli parameters of the 2-surface. Upon quantization, the system becomes a quantum mechanics of moduli parameters in a curved space endowed with the Weil-Petersson metric. The superspace, on which the wave function of universe is defined, turns out to be the fundamental region in the moduli space. The solution of the Wheeler-DeWitt equation is explicitly given as the Maass form which is perfectly regular in the superspace.

§ 1. Introduction

Quantum gravity is still in its infancy. There remain many conceptual and technical difficulties to overcome. In this paper, we would like to concentrate on the clarification of the structure of superspace, the functional space of the spatial metric modulo diffeomorphism, on which the wave function of the universe is defined. We will not discuss the interpretation problem of the wave function. Let the spatial metric be \( h_{ij}(x) \) and let us remind the reader of the Hamiltonian (Wheeler-DeWitt) and the momentum constraints to the wave function \( \phi[h_{ij}] \):

\[
H(h_{ij}(x), \delta/\delta h_{ij}(x))\phi[h_{ij}]=0 ,
\]

\[
H^k(h_{ij}(x), \delta/\delta h_{ij}(x))\phi[h_{ij}]=0
\]

(1-1)

(1-2)*

with \( i, j, k \) being spatial indices. As is well known, the momentum constraint (1-2) implies that the wave function is invariant under the infinitesimal spatial coordinate transformation, since \( H^k \) is its generator. So it is natural to further demand that \( \phi[h_{ij}] \) is also invariant under the global diffeomorphism. Then the wave function of the universe is a functional defined on the superspace. The momentum constraint also guarantees the integrability of the Wheeler-DeWitt equation (1-1) which must hold at each spatial point, through the closed algebra between the constraints

\[
[H(x), H(y)]=\{H_i(x)+H_j(y)\}\delta^x_\phi\delta(x, y).
\]

(1-3)

So far is the standard but very formal line of arguments. Actually very little progress has been made for the analysis of the full Wheeler-DeWitt equation (1-1). People tend to look at the grossly simplified version of the superspace called mini-superspace, which may not even be a good approximation to the full version.

*) For precise forms of the constraints see e. g., Ref. 2).
In the present work, we study the (2+1)-dimensional pure Einstein gravity as a testing ground for the real (3+1)-dimensional gravity with matter degrees of freedom. We assume that the topology of the space-time is $\Sigma \times R^1$, where $\Sigma$ is a closed compact 2-surface and $R^1$ is time. In this simple model, we can make explicit the structure of the superspace, now the space of the 2-metric modulo diffeomorphism. This is nothing but the moduli space of the 2-surface in the theory of Riemann surfaces, if it is divided by the conformal group. In the case of torus for the 2-surface, we can explicitly solve the Wheeler-DeWitt equation.

In a sense, the (2+1)-dimensional pure Einstein gravity is an ideal toy model to see the global structure of space-time. The (2+1)-dimensional Einstein gravity contains no gravitational waves which are only local deformation modes in spacetime and therefore irrelevant to the more interesting global motion of the spatial manifold. In the model, only a finite number of degrees of freedom remain corresponding to the Teichmüller deformations of the spatial surface. Roughly speaking, the Teichmüller deformation describes the change of shape of the spatial manifold modulo local conformal expansion. In the case of torus for the 2-surface, for instance, the Teichmüller deformations induce a change from a fat torus to a slim torus and also a twist.

In our previous work, we solved the classical (2+1)-dimensional Einstein equation employing York's time slice, the trace of the extrinsic curvature=const over the 2-manifold. It was shown there that the motion of the moduli parameters follow the geodesic curve defined by the Weil-Petersson metric in the moduli space. In the present work, we shall investigate the action in phase space rather than the equation of motion, which we studied in the previous work. This turns out to be more illuminating so that we can see the canonical structure and the Hamiltonian and therefore easily go over to quantum mechanics of the moduli parameters. We shall show that, in the case of torus for the 2-surface, the action reduces to

$$S = \int dt \left[ p_\alpha(\rho)^a \frac{\partial \rho^{(a)}}{\partial t} + \tau \frac{\partial \rho^{(a)}}{\partial t} - N(g^{(a)(b)} P_\alpha P_\beta - \tau^2 v^2) \right], \quad (1.4)$$

where $\rho^{(a)} (a=1, 2)$ are the moduli parameters of the torus and $v=\int d^2 x \sqrt{h}$ is the total volume of the 2-space. As they stand in the action (1.4), $P_\alpha$ and $\tau$ are conjugate momenta to $\rho^{(a)}$ and $v$, respectively, while $\tau$ originally has a geometrical meaning of the trace of the extrinsic curvature. $g^{(a)(b)}$ is the inverse of the Weil-Petersson metric:

$$ds^2 = \frac{1}{(\rho^{(a)})^2}((d\rho^{(1)})^2 + (d\rho^{(2)})^2) = g^{(a)(b)} d\rho^{(a)} d\rho^{(b)}, \quad (1.5)$$

*) Actually, the (2+1)-dimensional Einstein space is locally flat. This can be most easily seen by looking at the identity to the Riemann tensor:

$$R_{\mu\nu} = g_{\rho\sigma} R_{\rho\sigma} + g_{\nu\sigma} R_{\rho\sigma} - g_{\rho\sigma} R_{\nu\sigma} - g_{\nu\sigma} R_{\rho\sigma} + R g_{\rho\sigma}, \quad (1.6)$$

which holds only for the three dimensional space-time. The space-time is locally flat, because the Riemann tensor vanishes due to the vacuum Einstein equation. Therefore our 2-surface sweeps a part of the full Minkowski space. The motion of the 2-surface is not at all trivial, however, if its topology is nontrivial, i. e., its genus is non zero.
which coincides with the Poincaré metric in the Lobachevsky geometry. The Lagrange multiplier \( N \) is essentially a homogeneous component of the lapse function in the ADM formalism.\(^{10} \) The action (1·4) is analogous to the one for a relativistic point particle in a curved space-time with the metric:

\[
(g_{\mu\nu}) = \begin{pmatrix} -v^{-2} & 0 \\ 0 & g^{(a)(b)} \end{pmatrix}
\] (1·6)

Therefore we see that the trajectory of the moduli parameters is a geodesic, a semi-circle in the upper \((\rho^{(1)}, \rho^{(2)})\)-plane with its center on the \( \rho^{(1)} \)-axis.

The constraint equation, which is obtained from the variation of the action (1·4) with respect to \( N \),

\[
g^{(a)(b)} P_{(a)} P_{(b)} - \tau^2 v^2 \approx 0
\] (1·7)

is replaced by the differential equation via \( P_{(a)} \to -i \partial / \partial \rho^{(a)} \), \( \tau \to -i \partial / \partial v \), as

\[
\left[ -\frac{\partial^2}{\partial s^2} - (\rho^{(2)})^2 \left( \frac{\partial}{\partial \rho^{(1)}} \right)^2 + \left( \frac{\partial}{\partial \rho^{(2)}} \right)^2 \right] \phi(s, \rho^{(1)}, \rho^{(2)}) = 0
\] (1·8)

with \( s = \log v \).

The wave function of the torus universe \( \phi(s, \rho^{(1)}, \rho^{(2)}) \) is manifestly invariant under the infinitesimal diffeomorphism. As is well known, however, there remains the global diffeomorphism corresponding to the \( SL(2, \mathbb{Z}) \) modular transformation.\(^5 \)

\[
\rho \to \rho' = \frac{a \rho + b}{c \rho + d},
\]

\[
a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1
\] (1·9)

with \( \rho = \rho^{(1)} + i \rho^{(2)} \). Therefore the true superspace, \( \{ h_0 \}/\text{Diff} = \{ \rho^{(1)}, \rho^{(2)} \}/SL(2, \mathbb{Z}) \), is the fundamental region \( F \) in the complex \( \rho \)-plane as depicted in Fig. 1.

For \( \phi \) to be well-defined in the superspace, it has to be invariant under the \( SL(2, \mathbb{Z}) \) transformation,

\[
\phi(s, \rho) = \phi\left(s, \frac{a \rho + b}{c \rho + d}\right)
\] (1·10)

and it must vanish at \( \rho^{(2)} \to \infty \) (cusp):

\[
\phi(s, \rho^{(2)} \to \infty) = 0,
\] (1·11)

so that it is squarely integrable. Such functions called cusp forms are now well understood in mathematics.\(^{11} \) Therefore we understand the functional space of the wave function of the torus universe.

The organization of this paper is as
follows. We recapitulate the ADM canonical formalism of gravity in the case of the 
(2+1)-dimensional pure Einstein gravity in § 2. In § 3, we shall demonstrate the 
reduction of the dynamical variables to moduli parameters. In § 4, we will discuss 
the quantum mechanics of the moduli of the torus in greater detail. Section 5 is 
devoted to a summary and discussion.

§ 2. ADM formalism

In this section we recapitulate the ADM canonical formalism\textsuperscript{10} of the 
(2+1)-dimensional Einstein gravity. The (2+1) decomposition of the metric reads

$$ds^2 = -(Ndt)^2 + (dx^i + N^i dt)(dx^j + N^j dt)h_{ij},$$

where $N$, $N^i$ and $h_{ij}$ are the lapse, shift functions and the spatial metric. The spatial
suffix $i$ runs from 1 to 2.

We obtain the decomposition of the Einstein-Hilbert action\textsuperscript{*1} as

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R^{(3)} - 8\pi G \right)$$

$$= \frac{1}{16\pi G} \int d^3x N\sqrt{h}(K_{ij}K^{ij} - K^2 + R^{(2)}) + \text{surface term},$$

where $K_{ij} = \frac{1}{2N}(h_{ij,0} - N_{i,0} - N_{j,0})$ is the extrinsic curvature and $K = h_{ij}K^{ij}$ is its
trace. $R^{(3)}$ and $R^{(2)}$ denote the three and two dimensional scalar curvatures, re­
spectively. Hereafter we will take the unit $16\pi G = 1$. The stroke indicates the
covariant derivative defined by the spatial metric $h_{ij}$. The canonical conjugate
momentum $\pi^i$ to $h_{ij}$ is given by

$$\pi^i = \sqrt{h} \left( K_{ij} - h_{ij}K \right).$$

It is straightforward to obtain the action in the phase space,

$$S = \int d^3x \left[ \pi^i \frac{\partial h_{ij}}{\partial t} - NH - N_i H^i \right],$$

$$H = \frac{1}{\sqrt{h}}(\pi^i \pi_{ij} - \pi^2) - \sqrt{h} R^{(2)},$$

$$H^i = -2\pi^{ik} h_{kj},$$

with $\pi = \pi^i = -\sqrt{h} K^i, K^i$. Note that the form of the super Hamiltonian (2·5) is slightly
different from the four dimensional gravity ($\pi^2$ instead of $(1/2)\pi^3$). In what follows,
it is more convenient to use the traceless part of the extrinsic curvature $\tilde{K}_{ij} \equiv K_{ij} - (1/2)h_{ij}K_m^m$ and its trace $\tau = -K^i_k$ instead of the canonical momenta $\pi^i$. The
action becomes

$$S = \int d^3x \left[ \tilde{R}_{ij} \frac{\partial \tilde{h}_{ij}}{\partial t} + \tau \frac{\partial \sqrt{h}}{\partial t} - \sqrt{h} N \left( \tilde{K}_{ij} \tilde{K}^{ij} - \frac{\tau^2}{2} - R^{(2)} \right) + 2\sqrt{h} N_i \tilde{K}_{ij} \right].$$

\textsuperscript{*1} We follow the conventions in the textbook by Wald.\textsuperscript{12}
with $\bar{h}_{ij} = h_{ij} / \sqrt{h}$. Here we have employed the time slice $\tau = \text{const}$ over the spatial surface following York. Variations with respect to the Lagrange multipliers $N$ and $N_i$ would give the Hamiltonian and the momentum constraints

$\bar{K}_{ij} - \frac{\epsilon^2}{2} - \bar{R} = 0$, \hspace{1cm} (2.8)

$\bar{K}^{ij}_i = 0$. \hspace{1cm} (2.9)

§ 3. Reduction to moduli parameters

We restrict our phase space so that the transversality holds for the traceless part of the extrinsic curvature,

$\bar{K}^{ij}_i = 0$.

It is well known that there exist $6g - 6$ independent traceless transverse tensors (holomorphic quadratic differentials) on the closed Riemann surface of genus $g \geq 2$, while the number becomes zero for the sphere and 2 for the torus ($g = 1$). \cite{8} Hereafter we assume $g \geq 1$. Therefore we can expand $\bar{K}^{ij}$ in terms of the basis $\{\phi^{(a)ij}\}$ of the quadratic differentials

$\bar{K}^{ij} = \sum_a P_{(a)ij} \phi^{(a)ij} / 2v$ \hspace{1cm} (3.1)

with $v = \int d^2 x \sqrt{h}$. The denominator $2v$ is introduced just for convenience. The tensor $\bar{h}_{ij}$ is arranged so that it is invariant under the Weyl transformation $h_{ij} \rightarrow \Omega h_{ij}$. The deformation of $\bar{h}_{ij}$ is therefore so-called Teichmüller deformations modulo diffeomorphism,

$\frac{\partial \bar{h}_{ij}(x)}{\partial t} = \sum_a \frac{\partial \rho^{(a)}}{\partial t} \mu_{(a)ij}(x) \bar{h}_{ij}(x) \text{+ diffeo}$. \hspace{1cm} (3.2)

Equation (3.2) defines the Teichmüller parameters $\rho^{(a)}$ and the corresponding Beltrami differentials $\mu_{(a)ij}$. The term "+ diffeo" indicates the redundancy in the form $(\nabla_i \xi_j + \nabla_j \xi_i - h_{ij} \nabla_m \xi^m) / \sqrt{h}$ which does not contribute to the action because of the transversality and the tracelessness of $\bar{K}^{ij}$. With $\{\mu_{(a)ij}\}$ defined by Eq. (3.2), we can arrange the linear combinations of the holomorphic quadratic differentials $\{\phi^{(a)ij}\}$ in a standard way,

$\langle \mu_{(a)}, \phi^{(b)} \rangle = \int d^2 x \sqrt{h} \mu_{(a)ij} \phi^{(b)ij} / 2v = \delta_a^b$, \hspace{1cm} (3.3)

i.e., $\{\phi^{(a)}\}$ is dual to $\{\mu_{(a)}\}$ with respect to the Petersson inner product.

We can introduce a natural metric $g^{(a)(b)}$ called the Weil-Petersson metric in the Teichmüller parameter space \cite{8} as

$g^{(a)(b)} = \int d^2 x \sqrt{h} \phi^{(a)ij} \phi^{(b)ij} h_{ik} h_{jl} / 2v$, \hspace{1cm} (3.4)

which is a function of $\rho$'s and $v$, in general. Some of the properties of the Weil-
Petersson metric have been known to mathematicians. For example, the metric (3·4) is Kählerian and the Ricci tensor composed of it is negative definite and the Weil-Petersson metric is incomplete in the moduli space, namely the moduli space is geodesically incomplete.

In the case of torus universe for $\Sigma^2$ we can choose the gauge such that $N(t, x) = N(t)$, i.e., the lapse function is constant on $\Sigma^2$. In § 5, we will show the incompatibility of this choice with York's time slice, $\tau = \text{const}$ on $\Sigma^2$ for higher genus case. Therefore we confine ourselves to the torus case in what follows. Substitute the expansions (3·1) and (3·2) for $\tilde{K}^{ij}$ and $\partial \tilde{h}_{ij}/\partial t$ into the action (2·7) in the phase space. Due to the special gauge $N = N(t)$, the integration over the spatial coordinates can be explicitly carried out. We obtain for the torus universe, with the help of Eqs. (3·3) and (3·4),

$$ S = \int dt \left[ \sum P_{(a)} \partial \rho^{(a)}/\partial t + \tau \partial v/\partial t - N' \sum P_{(a)} P_{(b)} g^{(a)(b)} - v^2 \tau^2 \right], \quad (3·5) $$

where

$$ v = \int d^2 x \sqrt{h} \quad (2\text{-volume}), \quad (3·6) $$

$$ N' = N/2v. $$

It is clear from the expression for the action that $(P_{(a)}, \tau)$ are conjugate momenta to $(\rho^{(a)}, v)$. $N'$ plays the role of the Lagrange multiplier, which produces upon variation the constraint equation,

$$ \sum g^{(a)(b)} P_{(a)} P_{(b)} - v^2 \tau^2 \approx 0. \quad (3·7) $$

As Eqs. (3·5) and (3·7) reveal, the dynamics of the Riemann surface $\Sigma^2$ is very similar to the one for a relativistic point particle in a curved space-time. The "background metric" $g^{(a)(b)}$ is now given by the Weil-Petersson metric (3·4). The total 2-volume $v$ of $\Sigma^2$ plays the role of time. As we alluded to in the Introduction, the classical trajectory of the moduli parameter $\rho^{(a)}$ is a geodesic curve given by the "space-time" metric

$$ \begin{pmatrix} -v^{-2} & 0 \\ 0 & g^{(a)(b)} \end{pmatrix}. \quad (3·8) $$

We have two Teichmüller parameters $\rho^{(a)} (a = 1, 2)$ with the Poincaré metric:

$$ ds^2 = \frac{1}{(\rho^{(a)})^2} (d\rho^{(1)x} + d\rho^{(2)y}). \quad (3·9) $$

In order to simplify the notation we write $\rho^{(1)} = x, \rho^{(2)} = y$ with which we may not confuse the real coordinates on $\Sigma^2$. Then the constraint (3·7) becomes

$$ y^2(P_x^2 + P_y^2) - v^2 P_v^2 = 0, \quad (3·10) $$
where we write $P_v=r$ to make it explicit as a conjugate momentum to the volume $v$. The geodesic of the Lobachevski geometry is a semi-circle in the upper $x$-$y$ plane with its center on the $x$-axis, as is well known.

§ 4. Quantum theory of the torus universe

We follow Dirac's prescription for the constraint equations in quantization. The momenta are replaced by differential operators,

$$P_{(a)} \rightarrow -i \frac{\partial}{\partial \rho_{(a)}},$$

$$r \rightarrow -i \frac{\partial}{\partial v}.$$  \hspace{1cm} (4.1)

The constraint is interpreted as a wave equation to the state vector. Let us analyse the torus case in greater detail. Equation (3.9) is translated as

$$\left[ \frac{\partial^2}{\partial s^2} - v^2 (\partial_x^2 + \partial_y^2) \right] \phi(x, y, s) = 0$$  \hspace{1cm} (4.2)

with $s = \log v$ and $(x, y)$ defined above Eq. (3.10). As briefly discussed in the Introduction, there remains a global diffeomorphism $SL(2, \mathbb{Z})$

$$z \rightarrow z' = \frac{az + b}{cz + d},$$

$$ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}$$  \hspace{1cm} (4.3)

with $z \equiv x + iy$. The operator ordering of the laplacian in (4.2) is fixed so that it is $SL(2, \mathbb{Z})$ invariant. The wave function $\phi(x, y, s)$ has also to be $SL(2, \mathbb{Z})$ invariant in order to be well defined in the superspace

$$\frac{\{h_{ij}\}}{(\text{Diff} \times \text{Conf})} = \frac{\{h_{ij}\}}{(\text{Diff}_0 \times \text{Conf})}$$

$$= \frac{\text{SL}(2, \mathbb{Z})}{\text{Teichm"uller}}.$$  \hspace{1cm} (4.4)

Namely, the superspace is the functional space of the 2-metric $h_{ij}$ modulo full diffeomorphism Diff and the conformal transformation Conf. The full diffeomorphism consists of the diffeomorphism connected to the identity transformation $\text{Diff}_0$ (infinitesimal coordinate transformation) and the global modular transformation $SL(2, \mathbb{Z})$. The Teichm"uller space is defined as the metric space modulo $\text{Diff}_0$. Therefore our superspace is the quotient space $\{\text{Teichm"uller}\}/SL(2, \mathbb{Z})$ which is called the moduli space.

As is well known in the theory of automorphic functions, the quotient space $\{\text{upper half plane}\}/SL(2, \mathbb{Z})$ is the fundamental region. Therefore, for the torus universe, the superspace is nothing but the fundamental region $F$. The region $y \rightarrow \infty$ ("cusp") corresponds to the limit of the fattest torus and also of the thinnest one. Actually they are identified by an element of $SL(2, \mathbb{Z})$, the inside-out operation of the torus.
Separating variables, we can easily find a solution of Eq. (4·2) as

$$u^{(n)}(s, x, y) = \sqrt{y} K_{\nu}(2\pi|n|y) e^{2\pi i n x} e^{-iEs}, \quad (n=\text{integer})$$

with

$$\nu = \sqrt{E^2 - \frac{1}{4}}. \quad (4·5)$$

Here $K$ is the so-called modified $K$ Bessel function which approaches zero exponentially when its argument goes to infinity. The number $n$ has to be an integer due to the periodicity $x \rightarrow x + 1 \in \text{SL}(2, \mathbb{Z})$. Of course, (4·5) itself is not invariant under the full $\text{SL}(2, \mathbb{Z})$. We have to superpose (4·5) so that it satisfies the $\text{SL}(2, \mathbb{Z})$ invariance,

$$U_v(s, x, y) = \sum_{n=0}^{\infty} \rho_v(n) \sqrt{y} K_{\nu}(2\pi|n|y) e^{2\pi i n x} e^{-iEs}. \quad (4·6)$$

The coefficients $\rho_v(n)$ have not been analytically given and the discrete eigenvalues $\nu$ are known only numerically. (The smallest one is 13, 7797513⋯). However, their properties are fairly well studied by mathematicians in number theory. The function (4·6) is called the Maass form. Note that we have excluded $n=0$ mode in the sum (4·5), because we wanted the boundary condition $u_v(s, x, y) \rightarrow 0$ as $y \rightarrow \infty$. That is, we demand the singular universe has no chance to appear. The idea behind this boundary condition is similar to Hartle and Hawking's. In a sense the singularity of the space-time is circumvented in quantum cosmology of the torus universe. Our wave function of the universe is perfectly regular everywhere in the superspace. Although it is not unique, its variety is only discrete rather than continuous.

It is amusing to point out that the system of the differential equation (4·2) defined in the fundamental region is one of the examples of quantum chaos. This resembles the mixmaster model or the Bianchi type IX model of homogeneous and anisotropic universe.

§ 5. Summary and discussion

Assuming that the topology of space-time is $\Sigma^2$ (closed compact) $\times \mathbb{R}^1$, we have studied the $(2+1)$-dimensional pure Einstein gravity on the basis of the canonical ADM formalism. The dynamical variables reduce to the moduli parameters and the system becomes analogous to the one for the relativistic point particle in a curved space-time endowed with the Weil-Petersson metric. The logarithm of the 2-volume and the Teichmüller parameters play roles of the time and spatial coordinates, respectively. Going over to quantum gravity is straightforward and we find that the Wheeler-DeWitt equation becomes analogous to the Klein-Gordon equation. The operator ordering problem is at least partially solved by imposing the remaining discrete symmetry to the Laplacian. In the case of a torus universe, the operator ordering is essentially unique. The solution to the Wheeler-DeWitt equation is explicitly given as the Maass form which is modular invariant and perfectly regular in the fundamental region of the Teichmüller space, i.e., in the superspace. Therefore we have found an example of the wave function of universe in which the singularity
is avoided. We have heavily relied on the special time slicing, \( \tau = \text{const} \). It is an open question whether we can choose more general time slicing.

Some years ago, Martinec\(^20\) studied the torus universe in the same model of \((2+1)\)-dimensional quantum gravity. It seems to us, however, that the reduction procedure there is not very clear to us. Recently Moncrief\(^21\) investigated the higher genus case in the same York slice as ours. However, he completely solved the Hamiltonian constraint in contrast to our formulation. His reduced "Hamiltonian" is a very implicit function of moduli parameters. So it is not easy to compare our work with his result. We believe, however, that we can also formulate the higher genus case by using our reduction method but with very formal expression for the supermetric. The supermetric will not be so explicitly given as the Weil-Petersson metric.

Now we would like to discuss the relation of the lapse function and York's gauge as we promised before. As a matter of fact, we can always choose the lapse function \( N \) to be independent of spatial coordinates \( x^i \) by making a suitable transformation of the time coordinate: \( t \rightarrow t' = t'(t, x) \). What remains to be shown is its consistency with York's time slice \( \tau = \text{const} \) on \( \Sigma^2 \). Here we are going to see that the lapse function \( N \) cannot be a function of time \( t \) only. So our formulation actually works only for torus. Consider the evolution equation for \( \tau \):

\[
\frac{\partial \tau}{\partial t} = -\Delta N + (\frac{2}{2} R + \tau^2) N .
\]

(5.1)

In our case \( \frac{2}{2} R \) cannot be taken as a function of time only. This can be most easily seen from the Hamiltonian constraint equation,

\[
\bar{K}_{ij} \bar{R}^{ij} - \frac{\tau^2}{2} - (\frac{2}{2} R) = 0 .
\]

(5.2)

As is well known, \( \bar{K} \) has \( 4g - 4 \) zeroes and therefore cannot be a constant. From the equation above, this implies that \( \frac{2}{2} R \) cannot be a constant, neither. It is now clear from (5.1) that the lapse function should depend on the spatial coordinates as a consequence of York's gauge.

Some extensions of our pure Einstein gravity are easy to conceive. Introduction of the cosmological constant seems straightforward. Extension to the 2-manifold with boundaries will also be straightforward if one adds the surface term in the action. On the other hand, it is not so straightforward to include matter degrees of freedom. The system will become a mixture of field theory and "particle mechanics". We are planning to study the Einstein-Maxwell system in the \( \Sigma^2 \times R^1 \) space-time. Even in the fixed \( \Sigma^2 \times R^1 \) space-time, the Maxwell system exhibits an interesting feature.\(^16\)

We have obtained an explicit form of the wave function of universe. But can one say anything about physics? How is it related to the classical cosmology? At the moment we do not know the answer, since these problems are inherently connected with the interpretation problem of the wave function of universe. One of the possibilities may be the third quantization.\(^17\)

Recently, Witten\(^18\) found that the Einstein action can be written as a Chern-
Simons form and claimed that the wave function can be obtained by the conformal bloc. At the moment, we do not know the precise relationship between our method and his approach.

From the experience of the (2+1)-dimensional gravity, we convince ourselves that the mathematical knowledge of the 3-dimensional manifold is essential to understand the superspace. As is well known, it is still far-reaching.

Finally, can we say anything about topology change? The answer is probably yes. We suggest a possible topology changing process: $\text{torus} + \text{torus} \rightarrow g=2$ surface (virtual state) $\rightarrow \text{torus} + \text{torus}$. We are also planning to study this collision-scattering process in future, since we may get a better understanding of the so-called wormhole physics, which has been hotly disputed.

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