How Many “Times” Do We Have in Quantum Gravity?
—— A Path-Integral Approach ——

Akio HOSOYA and Jiro SODA*

Department of Physics, Tokyo Institute of Technology, Oh-okayama, Tokyo 152
*Department of Physics, Hiroshima University, Hiroshima 730
and
Uji Research Center, Yukawa Institute for Theoretical Physics, Kyoto University, Uji 611

(Received June 2, 1990)

Apparently there are an infinite number of time-like variables in the Wheeler-DeWitt equation in quantum gravity. This gives rise to an obvious conceptual difficulty and further becomes an obstacle if one wants to canonically third quantize the universe. In this paper, adopting York’s gauge in the path-integral approach, we formulate quantum geometrodynamics so that it contains only a single time-like variable corresponding to the total volume of the universe.

§ 1. Introduction

The most fundamental equation in quantum geometrodynamics is the so-called Wheeler-DeWitt equation,  
\[ G_{ijkl} \left( \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} + R(3) \sqrt{h} - T_{o0} \sqrt{h} \right) \Psi = 0 , \]

\[ G_{ijkl} = (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}) / \sqrt{h} . \]  

(1)

Here \( h_{ij} \) is the spatial metric tensor, \( R(3) \) is the spatial scalar curvature and \( T_{o0} \) is the energy density of matter. The wave functional of the universe \( \Psi \) is primarily a functional of the spatial metric \( h_{ij} \) and other matter degrees of freedom, if any. The Wheeler-DeWitt equation (1) is the second order functional partial differential equation and should hold at each spatial point \( x \). The supermetric \( G_{ijkl} \) in Eq. (1) is symmetric under the exchange: \( i \leftrightarrow j \) and \( k \leftrightarrow l \) as well as \( (i, j) \leftrightarrow (k, l) \). The supermetric is essentially a \( 6 \times 6 \) matrix with a signature \((-+ + + + +)\). The minus signature corresponds to the volume element \( \sqrt{h} \), as one can easily verify. One might naturally identify this volume element to be the time-like variable and consider that the Wheeler-DeWitt equation is analogous to the Klein-Gordon equation.

However, the volume element \( \sqrt{h} \) depends on the spatial coordinates. So, it appears that there are continuously an infinite number of time-like variables corresponding to spatial points. This is not what we want, because, if so, we cannot do the causal description of the wave function of the universe \( \Psi \). Further, the canonical third quantization seems very difficult, if there were time-like variables more than two.

We have already encountered a similar case in ordinary quantum field theory, viz., the Tomonaga-Schwinger equation:
How Many "Times" Do We Have in Quantum Gravity?

\[
i\frac{\delta}{\delta \sigma(x)} \Psi = H(x) \Psi,
\]

where \(\delta \sigma(x)\) is the volume element for the infinitesimal deformation of the space-like hypersurface \(\Sigma\) at \(x\) and \(H(x)\) is the Hamiltonian density. As is well known, one can freely choose the most convenient slice of the Minkowski space-time, viz., \(x^0=\text{constant}\) on \(\Sigma\). This clearly suggests that we can overcome the difficulty of the infinite number of time-like variables in the Wheeler-DeWitt equation, if we choose a suitable gauge, or time-slice. In this paper, we are going to show that this is indeed the case, by using York's slice, i.e., mean curvature=constant on the hypersurface \(\Sigma\). In the York gauge, we shall see that we have only a single time-like variable and the rest are gravitational wave and matter degrees of freedom.

§ 2. ADM formalism

To simplify the presentation, let us forget matter degrees of freedom and concentrate on the pure gravity.

The action for the Einstein gravity, in the unit \(16\pi G=1\), is cast in a form,

\[
S = \int d^4x \sqrt{-g} R^{(4)}
\]

\[
= \int dt d^3x [\sqrt{\hbar}(K^{ab} - K\hbar^{ab})\partial h_{ab}/\partial t - N\sqrt{\hbar}(K^{ab}K_{ab} - K^2 - R^{(3)}) + 2N_a\nabla_b (\sqrt{\hbar}(K^{ab} - K\hbar^{ab}))].
\]

Here \(K_{ab}\) is the extrinsic curvature, \(K_{ab} = (\partial h_{ab}/\partial t - N_{a;b} - N_{a;b})/(2N)\) and \(K\) is its trace, \(K^{ab} h_{ab}\). As can be easily read off, the canonically conjugate momentum of \(h_{ab}\) is given by \(\pi^{ab} = \sqrt{\hbar}(K^{ab} - K\hbar^{ab})\). The Lagrange multipliers \(N \approx \sqrt{-g_{00}}\) and \(N_a \approx g_{0a}\) are the so-called lapse and shift functions. Upon the variation with respect to the lapse and shift functions, we obtain the Hamiltonian and the momentum constraints,

\[
K^{ab}K_{ab} - K^2 - R^{(3)} = 0, \tag{3}
\]

\[
\nabla_a (K^{ab} - K\hbar^{ab}) = 0, \tag{4}
\]

respectively. The standard quantization method by Dirac\(^4\) is to first write everything in terms of the canonical variables, \(h_{ab}\) and \(\pi^{ab}\), replace the conjugate-momentum \(\pi^{ab}\) by the differential operator \(-i(\delta/\delta h_{ab})\) and finally reinterpret the constraint as the conditions to the state \(\Psi\). Corresponding to the Hamiltonian constraint we have the Wheeler-DeWitt equation (1) shown in the Introduction, while the momentum constraint reads

\[
\nabla_a \frac{\delta}{\delta h_{ab}} \Psi = 0. \tag{5}
\]

The momentum constraint (5) implies the diffeomorphism invariance of the wave functional of the universe \(\Psi\).

It is frequently said that the Hamiltonian constraint has a dynamical content,
whereas the momentum constraint is a kinematical one. As we shall see, this is not quite precise. Only the homogeneous component of the Hamiltonian is dynamical, that is, generates the time translation and the remaining inhomogeneous ones are not dynamical which have to be treated on the same footing as the momentum constraint. Namely, we have to impose the appropriate gauge conditions to fix the gauge freedom corresponding to the inhomogeneous part of the Hamiltonian constraint and the momentum constraint.

§ 3. York’s time slice

Let us split the extrinsic curvature into the trace $K$ and the traceless part $\Sigma_{ab}$

$$ K_{ab} = \Sigma_{ab} + \frac{1}{3} K h_{ab}, $$

$$ h_{ab} \Sigma_{ab} = 0. $$

The action can be rewritten as

$$ \int dt d^3 x \left[ \sqrt{h} \Sigma_{ab} \frac{\partial h_{ab}}{\partial t} + \frac{3}{4} K \partial \sqrt{h} / \partial t - N \sqrt{h} \left( \Sigma_{ab} \Sigma_{ab} - \frac{2}{3} K^2 - R^{(3)} \right) 
+ 2 N \partial_b \left( \Sigma_{ab} - \frac{2}{3} K h_{ab} \right) \right]. $$

Following York, we take a special time slice: the mean curvature $K =$ constant on the hypersurface. We can easily recognize that the action is considerably simplified in this gauge and that the momentum constraint implies the transversality of the traceless part of the extrinsic curvature,

$$ \nabla_b \Sigma_{ab} = 0. $$

York’s crucial observation is that the conformally transformed quantity

$$ \tilde{\Sigma}_{ab} = \Omega^{1/2} \Sigma_{ab} $$

is traceless and transverse with respect to the conformally transformed metric

$$ \tilde{h}_{ab} = \Omega^{-1} h_{ab}. $$

This conformal freedom enables us to solve the Hamiltonian constraint while keeping the momentum constraint and is the essence of York’s formulation of the initial value problem. In terms of new variables with hats, the Hamiltonian constraint reads

$$ \Omega^{-3} \tilde{\Sigma}_{ab} \tilde{\Sigma}_{ab} - \frac{2}{3} K^2 - \Omega^{-1} \left[ \tilde{R} - 2 \frac{\tilde{\Delta} \Omega}{\Omega} + \frac{3}{2} \left( \tilde{\partial} \Omega / \partial \Omega \right)^2 \right] = 0. $$

It is known that a unique solution of this equation exists if the spatial manifold is closed and compact provided that the metric $\tilde{h}_{ab}$ and the traceless and transverse part of the extrinsic curvature $\tilde{\Sigma}_{ab}$ are given on a spatial hypersurface with a mean curvature $K$. By conformally transforming back, we can obtain the metric $h_{ab}$ and the extrinsic curvature $K_{ab}$ which satisfy both the momentum and Hamiltonian...
constraints. So far is the standard prescription for the initial value problem.

In the following section we will use a similar conformal technique.

§ 4. The path integral

In the case of the asymptotically flat space-time, the dynamical content is contained in the surface term. On the contrary, in the case of the compact space on which we are concentrating, only the homogeneous component of the Hamiltonian is dynamical. Therefore, it is convenient to decompose the Hamiltonian constraint to the homogeneous part and the inhomogeneous ones such that

\[
\prod_{i,x} \delta(H) = \int \prod_{i,x} dE(t) \delta \left( \frac{\int d^3x H}{\sqrt{\frac{\hbar}{\mathcal{E}(t)}}} \right) \prod_{i,x} \delta(H - \sqrt{\hbar} E(t)),
\]

where \( H \) is the Hamiltonian constraint. We shall demonstrate how to entangle the inhomogeneous parts of the conformal factor. Now, we shall start from the following path integral expression:

\[
\int \prod_{i,x} d\pi^{ab} \prod_{i,x} dh_{ab} \prod_{i,x} \delta \left( \frac{1}{\sqrt{\hbar}} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - \sqrt{\hbar} R^{(3)} \right) \prod_{i,x} \delta(\mathcal{V}_b \pi^{ab}) \\
\times \prod_{i,x} \delta \left( \frac{2\pi}{3\sqrt{\hbar}} - \tau(t) \right) \prod_{i,x} \left| \text{det} \left( \frac{2\pi}{3\sqrt{\hbar}} , H \right) \right| \exp \left( i \int dt d^3x \left[ \pi^{ab} \partial h_{ab} / \partial t \right] \right).
\]

Here, we have taken York's time slice. At this stage, we make the following orthogonal decomposition:

\[
\pi^{ab} = \sigma^{ab} + \sqrt{\hbar} (LW)^{ab} + \frac{1}{3} h^{ab} \pi,
\]

\[
h_{ab} = \Omega h_{ab}
\]

with \( \pi = \pi^{ab} h_{ab} \), where \((LW)^{ab} = \mathcal{V}^a W^b + \mathcal{V}^b W^a - 2h^{ab} \mathcal{V}^c W^c / 3\) and the measure of the metric part takes the form

\[
dh_{ab} = d\tilde{h}_{ab} d((LW)^{ab}) d\Omega,
\]

where \( v^a \) represents the longitudinal part of the metric and henceforth hereafter \( d\tilde{h}_{ab} \) denotes the integration measure of physical degrees of freedom. Using this decomposition, we obtain the following expression:

\[
\int \prod_{i,x} d\sigma^{ab} \prod_{i,x} d(\sqrt{\hbar} (LW)^{ab}) \prod_{i,x} d\tau \prod_{i,x} d\tilde{h}_{ab} dv^a d\Omega \left| \text{det} \left( L' L \right) \right|^{1/2} \\
\times \prod_{i,x} \delta \left( \frac{1}{\sqrt{\hbar}} \left( \sigma^{ab} \sigma_{ab} + 2\sqrt{\hbar} \sigma^{ab} (LW)^{ab} + h(LW)^{ab} (LW)_{ab} - \frac{3}{8} \hbar \pi^2 \right) - \sqrt{\hbar} R^{(3)} \right) \\
\times \prod_{i,x} \delta(\sqrt{\hbar} \mathcal{V}_b (LW)^{ab}) \prod_{i,x} \delta \left( \frac{2\pi}{3\sqrt{\hbar}} - \tau(t) \right) \prod_{i,x} \left| \text{det} \left( \frac{2\pi}{3\sqrt{\hbar}} , H \right) \right| \\
\times \exp \left( i \int dt d^3x \left[ \sigma^{ab} \partial h_{ab} / \partial t + \sqrt{\hbar} (LW)^{ab} \partial h_{ab} / \partial t + \tau \partial \sqrt{\hbar} / \partial t \right] \right),
\]
where $L^a L^a = \nabla_b (L^a)^b_{ab}$. Here we used the delta functional in the integrand. As the integrand is diffeomorphism invariant, we can factor out the integration measure of the longitudinal part of the metric. After performing the integration of the longitudinal part of the extrinsic curvature, the path integral becomes

$$
\int \prod_{t,x} d\hat{\sigma}^{ab} \prod_{t,x} d(\sqrt{h}) \prod_{t,x} d\hat{h}_{ab} d\Omega \prod_{t,x} \delta(H^a) \times \prod_{t,x} \left| \det(\tau, H^a) \right| \exp i \int dt d^3 x \left[ \hat{\sigma}^{ab} \hat{\sigma}_{ab} / \partial t + \tau \partial(\sqrt{h} \Omega^{3/2}) / \partial t \right],
$$

(15)

where

$$
H^a = \frac{1}{\sqrt{h} \Omega^{3/2}} \left( \hat{\sigma}^{ab} \hat{\sigma}_{ab} - \frac{3}{8} \Omega^3 \hat{h} \tau^2 \right) - \sqrt{h} \Omega^{1/2} \left[ R - 2 \frac{\hat{A}_a}{\Omega} + \frac{3}{2} \left( \frac{\partial \Omega}{\Omega} \right)^2 \right].
$$

(16)

Using Eq. (11) and inserting the identity

$$
1 = \int \prod_{t,x} dv \delta \left( v - \int d^3 x \sqrt{h} \Omega^{3/2} \right),
$$

(17)

we obtain

$$
\int \prod_{t,x} d\hat{\sigma}^{ab} \prod_{t,x} d\hat{h}_{ab} \prod_{t,x} d\tau \prod_{t,x} dv \delta \left( v - \int d^3 x \sqrt{h} \Omega^{3/2} \right) \prod_{t} dv \times \prod_{t,x} \left| \det(\tau, H^a) \right| \times \exp i \int dt \int d^3 x \left[ \hat{\sigma}^{ab} \hat{\sigma}_{ab} / \partial t + \tau \partial(\sqrt{h} \Omega^{3/2}) / \partial t - n(t) H^a \right].
$$

(18)

The following part of measure:

$$
\int \prod_{t,x} d\Omega \prod_{t,x} \delta(\hat{H}^a - \sqrt{h} \Omega^{3/2} E) \prod_{t,x} \left| \det(\tau, H^a) \right|
$$

becomes the identity and the $\Omega$ is solved as a functional of $\hat{\sigma}^{ab}$, $\hat{h}_{ab}$, $\tau$ and $E(t)$. Furthermore,

$$
\int \prod_{t} dE \delta \left( v - \int d^3 x \sqrt{h} \Omega^{3/2} \right)
$$

(19)

eliminate $E$ from the action and $\Omega$ is now the functional of $\hat{\sigma}^{ab}$, $\hat{h}_{ab}$, $\tau$ and $v$. Thus the final form of the path integral becomes

$$
\int \prod_{t,x} d\hat{\sigma}^{ab} \prod_{t,x} d\hat{h}_{ab} \prod_{t,x} d\tau \prod_{t} dv \prod_{t} dn \prod_{t} \left| \det(\tau, H^a) \right| \times \exp i \int dt \left[ \int d^3 x \hat{\sigma}^{ab} \hat{\sigma}_{ab} / \partial t + \tau \partial - n(t) \right] \int d^3 x H^a.
$$

(20)

This equation (20) takes the desired form, i.e., there exists only a single time-like variable $v$. The total Hamiltonian $\int d^3 x H^a$ is a functional of $\hat{\sigma}^{ab}$ and $\hat{h}_{ab}$ as well as $\tau$ and $v$. 
§ 5. Conclusion

In this paper, we have demonstrated that quantum geometrodynamics contains only a single time-like variable corresponding to the total volume of the universe. This is requisite for canonically formulating the gravity as a quantum field theory on the superspace.

We have started our argument with the phase space path integral on the basis of Faddeev's prescription for general constrained systems and chosen York's gauge, the mean curvature = constant on the spatial hypersurface. There the inhomogeneous components of the Hamiltonian constraint as well as the momentum constraint were treated as kinematical, i.e., constraints to be solved completely. Only the physical modes corresponding to gravitational waves remain after the longitudinal and conformal modes are eliminated by solving the constraints. On the other hand, the homogeneous part of the Hamiltonian constraint has a whole dynamical content and plays a role of the time (= the volume of the universe) translation generator. This reduction is our main result.

In our procedure, York's time slice has been technically crucial. Its shortcoming may be that the time slicing is legitimate only in a finite range of time interval and cannot be defined globally in space-time. It remains to be seen whether our method works in other gauges than York's gauge.

In our manipulation of changing the variables, we solved the equation for the conformal factor $\Omega$,

$$H^0 = \sqrt{\hbar} \, \Omega^{3/2} E.$$  

This equation is known to have a unique solution. We also traded $E$ for $v$ by solving

$$v = \int d^3 x \sqrt{\hbar} \, \Omega^{3/2}.$$  

At this stage we may encounter branches of solutions more than two.

Although it is not easy to explicitly solve the Hamiltonian constraint in the physical (3+1)-dimensional space-time, we can solve it completely in the case of toroidal universe in (2+1)-dimensional gravity. It was previously shown\(^6\),\(^7\) in that case that the geometrodynamics reduces to the dynamics of the relativistic point particle in the Teichmüller space with the Poincaré metric. So the third quantization of the torus universe is straightforward to work out.

We hope that our present investigation will provide an insight into the topology changing phenomena in quantum gravity from the third quantization point of view. The recently proposed theory of vanishing cosmological constant\(^8\) is related to this problem, where the branching-off process of baby universes plays an essential role. To this end we need a method to extract physical information from the obtained reduced Hamiltonian which is unfortunately rather implicit in its form. It is amusing to point out that in string theories one needs to choose a specific gauge, e.g., the light-cone gauge, if one wants to second quantize the string field and that only the
homogeneous part of the Virasoro generator plays a role of Hamiltonian.\(^9\)

**Acknowledgements**

We thank Professor M. Sasaki for useful discussions. The authors are grateful to the Japan Society for the Promotion of Science for the partial support under the Grant-in-Aid for Scientific Research for the Ministry of Education, Science and Culture No. 02640232 and No. 02952045. One of the authors (A.H.) is also grateful to Professor Ni for his hospitality at the Hsinchu School on Gravitation, 1989. This work was partly done at that occasion.

**References**

6) A. Hosoya and K. Nakao, Class. Quantum Grav. **7** (1990), 163.