Sphaleron Transition of Reduced $O(3)$ Nonlinear Sigma Model

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By introducing an ansatz, infinite degrees of field freedom of the 1+1 dimensional $O(3)$ nonlinear sigma model are reduced to a single one along the noncontractible loop parameter. The mechanism is examined how the quantum-statistical transition rate of the reduced model by quantum tunneling at low temperatures turns smoothly into that by thermal activation at high temperatures. Such a transition pattern is considered to simulate that of the electroweak theory. At high temperatures, where the transition rate of the reduced model is able to be compared with the field theoretical zero-mode factor and entropy term are found to play a minor role in numerical results.

§ 1. Introduction

The sphaleron transition may be of particular significance in the sense that the well established electroweak theory is relevant to the baryon to photon ratio of our universe. If the sphaleron transition may wash out the primordial baryon number asymmetry after the electroweak phase transition, then any realistic model prior to the electroweak theory, whether the GUT type, the string type or the others, should be such that its baryon number asymmetry does survive the sphaleron transition.

The baryon number change due to the electroweak anomaly is induced by a transition between a pair of neighboring, topologically inequivalent vacua in field configuration space, which are separated by a barrier whose height is equal to the sphaleron energy $E_{sph} \sim O(10 \text{ TeV})$. At low temperatures, the transition takes place by quantum tunneling (the instanton transition in the zero temperature limit, known to be completely negligible). At high temperatures, the transition by thermal activation over the barrier is expected to occur. This transition over the wide temperature range is, however, too complicated to treat from first principles of quantum field theory. Therefore, when one examines the transition, he resorts to some toy models in 1+1 dimensions or manages to reduce infinite degrees of freedom of field to few ones by introducing some ansatz or other.

Among the toy models, the $O(3)$ nonlinear sigma model in 1+1 dimensions examined by Mottola and Wipf and also by the present authors is particularly interesting, since the model shares many features with the electroweak theory such as scale invariance, asymptotic freedom, topological aspects of winding number together with instanton, and chiral anomaly. It should be noted that the sphaleron of the model exists in the broken phase due to a symmetry breaking term while the instanton is defined in the symmetric phase, which is similar to the situation of the electroweak theory. On the other hand, a reduced action of the electroweak theory has been given by Aoyama, Goldberg and Ryzak, who regard the noncontractible loop parameter $\mu$
connecting the topologically inequivalent vacua as a single degree of freedom of coordinate, introducing at the same time a variational parameter $a$ to compensate possible crudeness of the ansatz.

In the present paper, we examine in detail the transition of the topological number in the 1+1 dimensional $O(3)$ nonlinear sigma model reduced á la Aoyama et al. At first sight, one might feel that such a reduced toy model might hardly serve to anything. However, this is not the case. First, the pattern of the reduced action of the model bears a close resemblance to that of the electroweak theory. Second, the quantum-statistical transition rate at finite temperature of 1 dimensional systems such as the reduced model has been basically formulated by Langer and Affleck. Third, the field theoretical transition rate of the $O(3)$ model, with which the reduced model should be compared, has been obtained at high temperatures. The last but not least, many of calculations are able to be done analytically once a tiny approximation is made, which helps to make the situations transparent.

In this way, we clarify the transition mechanism whose key is the bounce solution corresponding to quantum tunneling. It spans a gap between the sphaleron in the broken phase and the instanton in the symmetric phase, where the variational parameter plays a role to distinguish the two phases. We also find what corrections be necessary on the transition rate $\Gamma$ of a reduced model to obtain that of the original field theoretical model $\Gamma_{FT}$. We expect that the present approach would serve to economically estimate the transition rate of the electroweak theory over a wide range of temperature.

In § 2 the reduced action together with the reduction ansatz is presented. In § 3 the reduced classical action is given analytically in the tiny approximation. The pattern how the instanton limit of the action turns smoothly into the sphaleron action is investigated. The pattern simulates that of the reduced action of the electroweak theory, but is in sharp contrast to that of the reduced Abelian Higgs model in 1+1 dimensions, another toy model in which, however, both the instanton and the sphaleron coexist in the broken phase. In § 4 two subtle problems in applying the approximate formulae by Affleck are discussed. The first problem comes from the fact that the ground state energy of the “metastable” vacuum cannot always be neglected at extremely low temperatures. The second one is peculiar to the transition with the two degenerate vacua, which cause a new zero mode in the low temperature limit. In § 5 results of the transition rate $\Gamma$ obtained by numerical integration to avoid the problems are shown. The minimum of $\Gamma$ with respect to $a$ by quantum tunneling at low temperatures turns smoothly into that by thermal activation at high temperatures. In § 6 the field theoretical transition rate $\Gamma_{FT}$ is compared with $\Gamma$ of the reduced model at high temperatures. The field theoretical corrections of zero-mode factor and entropy term are found to play a minor role in numerical results.

§ 2. Reduction of field theory to quantum mechanics

The Lagrangian of the $O(3)$ nonlinear sigma model of three-component field
\( n(t, x) \) in 1+1 dimensions is

\[
\mathcal{L} = (1/g^2)[(1/2)\partial_\mu n \cdot \partial^\mu n - \omega^2 U(n)] \quad \text{with} \quad U(n) = 1 + n^2, \tag{2.1}
\]

where the potential \( U(n) \) is introduced by hand to break the \( O(3) \) symmetry to \( O(2) \).

A convenient parametrization of \( S^2 \) of \( n \cdot n = 1 \) is given by

\[
n = (-\sin \mu \sin f, -\cos \mu \sin (1 + \cos f), \sin^2 \mu \cos f - \cos^2 \mu) \tag{2.2}
\]

with \( \mu \in [0, \pi] \) and \( f \in [-\pi, \pi] \).

We reduce infinite degrees of freedom of the fields \( \mu(t, x) \) and \( f(t, x) \) to one by the following ansatz:

\[
\mu(t, x) \rightarrow \mu(t), \quad f(t, x) \rightarrow f(x) = 2 \arcsin(\tanh(\omega x / a)), \tag{2.3}
\]

where \( a \) is the variational parameter. At \( \mu = \pi/2 \) and \( a = 1 \), \( n \) coincides with the \( O(3) \) sphaleron \( n = (-\sin f, 0, \cos f) \) with the energy \( E_{\text{sph}} \) as \(^4\), \(^5\)

\[
f = f_{\text{sph}}(x) = 2 \arcsin(\tanh(\omega x)) \quad \text{with} \quad E_{\text{sph}} = 8 \omega / g^2. \tag{2.4}
\]

On the other hand, \( \mu = 0 \) and \( \pi \) give the topologically inequivalent vacua \( n = (0, 0, -1) \) separated by the sphaleron barrier. Hereafter the variable \( \mu(t) \), which parametrizes a noncontractible loop in field configuration space, is regarded as coordinate of a quantum mechanical system.*

The reduced Euclidean action from \( \mathcal{L} \) based on the ansatz (2.3) is

\[
S_E[\mu] = \frac{1}{g^2} \int_0^{\beta} dt \int_{-\infty}^{\infty} d\xi \left[ \frac{1}{2} (\dot{\mu})^2 + \frac{1}{2} (\dot{\mu})^2 + (1 + n^2) \right]
\]

\[
= \frac{1}{g^2} \int_0^{\beta} dt \left[ \frac{1}{2} M(\mu, a) \dot{\mu}^2 + V(\mu, a) \right], \tag{2.5}
\]

with \( \dot{\mu} = d\mu/dt, \dot{n} = d\mu/d\xi \) and \( \mu = d\mu/d\tau, \xi = \omega x, \tau = \omega t, \beta = \omega \beta \) and

\[
M(\mu, a) = 8a(1 - \sin^2 \mu / 3),
\]

\[
V(\mu, a) = V_0(a) \sin^2 \mu \tag{2.6}
\]

are dimensionless mass and potential respectively. The periodic boundary condition is imposed on \( \mu(\tau) \) at \( \tau = 0 \) and \( \beta \). Thus, \( S_E[\mu] \) describes a fictitious point particle with the effective mass \( M(\mu, a) \) moving in the effective attractive potential \(- V(\mu, a)\) with the period

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*) Since \( \mu(t) \) is originally an angle variable, we must be careful to treat it in quantization. (The eigenvalues of its momentum and the Hamiltonian would be discrete.) We regard it as a usual coordinate whose spectrum is unbounded in the semiclassical treatment that follows.
\[ \beta = \omega \beta, \] where \( 1/\beta \) denotes the temperature. The Minkowskian version of the potential, \( + V(\mu, a) \), has a peak at \( \mu = \pi/2 \) as Fig. 1. In order for the ansatz to be physically reasonable, the particle in one of the valleys, \( \mu = 0 \), is hoped at high temperatures \( 1/\beta \sim E_{\text{sph}} \) to pass over this barrier at the variational parameter \( a = 1 \) reproducing the sphaleron, which is actually the case as will be shown.

A dimensional consideration shows that the mass \( M(\mu, a) \) coming from \( (\dot{n})^2 \) is proportional to \( a \), while the first term of the potential \( V_0(a) \) coming from \( (\dot{n}')^2 \) is proportional to \( 1/a \) and the second one from \( U(n) \) to \( a \). Since the sphaleron is a static solution \( (\dot{n} = 0) \), it should sit at the minimum of \( V_0(a) \), i.e., \( a = 1 \). Since the instanton of the \( O(3) \) nonlinear sigma model is defined in the symmetric phase \( (U(n) = 0) \), it should be at \( a = 0 \).

\section*{§ 3. Reduced action of \( O(3) \) model and its comparison with Abelian Higgs model}

The equation of motion from the reduced action (2.5) is

\[ M(\mu, a) \mu^2 + (1/2) \mu^2 \partial M(\mu, a) / \partial \mu - \partial V(\mu, a) / \partial \mu = 0. \] (3·1)

\subsection*{3.1. Classical solutions and their actions}

As analyzed in I, we have three types of classical solutions to (3·1).

(i) Vacuum solutions \( \mu_0 = 0, \pi \) at the minima of \( V(\mu, a) \) with \( S_0[\mu_0] = S_0 = 0 \).

(ii) Sphaleron solution \( \mu_{\text{sph}} \) at the top of \( V(\mu, a) \):

\[ \mu_{\text{sph}} = \pi/2 \quad \text{with} \quad S_0[\mu_{\text{sph}}] = S_{\text{sph}}(a) = (4 \beta/g^2)(1/a + a), \] (3·2)

which is minimized at \( a = 1 \) and, as expected,

\[ S_{\text{sph}}(a = 1) = \beta E_{\text{sph}}. \] (3·3)

(iii) Bounce solution \( \mu_b \): In view of the weak \( \mu \) dependence of \( M(\mu, a) \), we make an approximation in a self-consistent way, which enables an analytic treatment. That is, we first put \( \sin^2 \mu(\tau) \) in \( M(\mu, a) \) in (2.6) to be a constant, solve (3·1) by replacing \( M(\mu, a) \rightarrow M_0(a) \), evaluate the average \( \langle \sin^2 \mu \rangle \) from the solution, and put it back to \( M(\mu, a) \). Namely, we start from the energy conservation law by (3·1) neglecting the second term:

\[ (1/2)M_0(a) \mu^2 - V_0(a) \sin^2 \mu = -E. \] (3·4)

By parametrizing the binding energy as

\[ E = V_0(a)(1 - \kappa^2), \quad 0 \leq \kappa \leq 1 \] (3·5)

we obtain the bounce solution:

\[ \mu_b(\tau) = \arccos(-\kappa \text{sn}(b(\kappa, a) \tau; \kappa)) \] (3·6)

with \( b(\kappa, a) = \sqrt{2V_0(a)/M_0(a)} \), where \( \text{sn}(x; \kappa) \) is the elliptic function. Because of time translation invariance, the bounce solution has one zero mode. Once the temperature \( 1/\beta \) is given, the parameter \( \kappa \) is determined from the periodic boundary condition as
The integer \( n \) corresponds to the number of node of \( \mu_b \), and we take \( n=1 \) to have the minimal bounce action. The average of \( \sin^2 \mu \) from (3-6) is

\[
\langle \sin^2 \mu_b \rangle = \frac{1 + \kappa \sqrt{1 - \kappa^2}}{\arcsin \kappa} / 2 ,
\]

so that the approximate bounce solution is (3-6) with

\[
M_0(\kappa, a) = \frac{4}{3} a \left( 5 - \frac{\kappa \sqrt{1 - \kappa^2}}{\arcsin \kappa} \right) = \begin{cases} 
(16/3)a & \text{for } \kappa = 0 , \\
(20/3)a & \text{for } \kappa = 1 ,
\end{cases}
\]

where \( M_0(\kappa, a) \) is monotonically increasing with respect to \( \kappa \). Note the weak \( \kappa \) dependence of \( M_0 \) of 20%.

From (3-6) and (3-9), we have the bounce action:

\[
S_\kappa[\mu_b] = S_b(\kappa, a) = (4g^2)^{1/2} \sqrt{M_0(\kappa, a)} V_0(a) / 2 \left[ 2E(\kappa) - (1 - \kappa^2) K(\kappa) \right] \\
= (1/g^2) [8\sqrt{M_0(\kappa, a)} V_0(a) / 2 \{ E(\kappa) - (1 - \kappa^2) K(\kappa) \} + (1 - \kappa^2) V_0(a) \bar{\beta}] ,
\]

where

\[
E(\kappa) = \int_0^{\pi/2} d\theta (1 - \kappa^2 \sin^2 \theta)^{1/2} .
\]

At \( \kappa = 0 \), the bounce action coincides with the sphaleron action:

\[
S_b(\kappa = 0, a) = (1/g^2) V_0(a) \bar{\beta} = S_{\text{sphe}}(a) .
\]

At \( \kappa = 1 \), the bounce action,

\[
S_b(\kappa = 1, a) = (16/g^2) \sqrt{10(1 + a^2)}/3 ,
\]

is minimized at \( a = 0 \) as

\[
S_b(\kappa = 1, a = 0) = 16\sqrt{10}/3 / g^2 = 1.16(8\pi/g^2) = 1.16(2S_{\text{inst}}) ,
\]

where \( S_{\text{inst}} = 4\pi/g^2 \) is the \( O(3) \) instanton action. The bounce motion going to and fro between the two vacua with the zero binding energy \( E = 0 \) (\( \kappa = 1 \)) implies that the particle passes the Minkowskian sphaleron peak twice in a single period. On the other hand, the \( O(3) \) instanton defined in the symmetric phase requires \( a = 0 \) as mentioned before. Thus the result (3-14) is perfectly reasonable and would mean that the possible errors due to the ansatz (2-3) and the approximation (3-8) would amount only to 16%.

The last expression in (3-10) is rewritten as (Legendre transform)

\[
S_\kappa[\mu_b] = (1/g^2) [W(E) + ET(E)] = S_b(E) .
\]

Here \( E \) is the binding energy in (3-5),
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\[ T(E) = 2 \int_{\mu(E)}^{s(E)} d\mu \sqrt{M_0(\kappa, a)/2(V(\mu) - E)} \]
\[ = 4\sqrt{M_0(\kappa, a)/2 V_0(a) K(\kappa)} \]  \hspace{1cm} (3.16)

is the period of the bounce motion, and

\[ W(E) = 2 \int_{\mu(E)}^{s(E)} d\mu \sqrt{2M_0(\kappa, a)(V(\mu) - E)} \]
\[ = 4\sqrt{2M_0(\kappa, a) V_0(a)} [E(\kappa) - (1 - \kappa^2) K(\kappa)] \]  \hspace{1cm} (3.17)

is the "volume" of the sphaleron barrier in the WKB approximation, where the $\mu_i$'s are determined from $V(\mu_i) = E(0 \leq \mu_1 \leq \mu_2 \leq \pi)$ as Fig. 1.

3.2. Reduced action of the $O(3)$ nonlinear sigma model

First, some kinematical remarks from (3.7) with $n=1$ are in order. The low temperature limit $1/\beta = 0$ corresponds to $\kappa = 1$. As $1/\beta$ increases, $\kappa$ decreases to 0, where the temperature arrives at

\[ 1/\beta = b(0, a)/4K(0) = \sqrt{(3/2)(1 + 1/a^2)}/2\pi = t_b(a). \]  \hspace{1cm} (3.18)

Given $a$, $0 \leq 1/\beta \leq t_b(a)$ is the region where exists the bounce solution, which we call bounce region. In other words, we have from (3.7) for $\kappa = 0$

\[ a = 1/\sqrt{8\pi^2/3 \beta^2 - 1} = a_b(\beta). \]  \hspace{1cm} (3.19)

Given $1/\beta > t_b(\infty) = \sqrt{3/2}/2\pi$, $a \leq a_b(\beta)$ is the bounce region. We call the region $a \geq a_b(\beta)$ sphaleron region, since, at $a = a_b(\beta)$, the bounce action continues smoothly to the sphaleron action of (3.12). On the other hand, because $b(\kappa, a) \to \infty$ as $a \to 0$, the $a=0$ limit requires $\kappa = 1$ irrespective of the temperature. We call this point instanton limit because of (3.14).

The reduced action $S_\Sigma[\mu]$ versus the variational parameter $a$ is given in Fig. 2. At low temperatures $1/\beta < t_b(\infty)$, the reduced action is the bounce action $S_b(a)$ for all $a$, whose minimum $S_b(0)$ is the instanton limit. As $1/\beta$ exceeds $t_b(\infty)$, the sphaleron action $S_\text{sph}(a)$ appears for large $a$ and continues at $a_b(\beta)$ to $S_b(a)$ for small $a$. $S_b(0) = S_b(1)$ at
which gives the measure that the local minimum of $S_E[\mu]$ at $a=1$ becomes lower than the instanton limit. As $1/\beta$ further increases, $a_s(\beta)$ decreases, and, at $1/\beta > \sqrt{3}/2\pi$ where $a_s(\beta) < 1$, $S_{\text{ sph}}(1)$ reproducing the sphaleron configuration by (3.3) is the true minimum of $S_E[\mu]$. Figure 2 clearly shows how the minimum point of $S_E[\mu]$ changes with $1/\beta$. We stress that this pattern in Fig. 2 is completely the same with that of the reduced action of the electroweak theory (Fig. 1 of Ref. 6), and that the pattern of the reduced action of the Abelian Higgs model discussed below forms a striking contrast to these.

3.3. Reduced action of the Abelian Higgs model

The Lagrangian of the Abelian Higgs model in 1+1 dimensions is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu \phi)^\dagger D_\mu \phi - \lambda (\phi \phi^\dagger - v^2)^2$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \phi = (\partial_\mu - ieA_\mu) \phi$. By putting $\phi = v h(x) \exp(i\theta(x))$, we have the static sphaleron solution in the $\phi=0$ gauge:

$$h = h_{\text{sph}}(x) = \tanh(\sqrt{\lambda} v x) \quad \text{with} \quad E_{\text{sph}} = (8/3)\sqrt{\lambda} v^3.$$  

From the equation of motion the gauge field is given by

$$A_1(x) = (1/e) \partial_1 \theta(x),$$

from which the Chern-Simons number is defined to be

$$N_{\text{cs}} = \frac{e}{2\pi} \int_0^\infty dx A_1(x) = \frac{1}{2\pi} [\theta(\infty) - \theta(-\infty)].$$

In order to have the reduced action, the noncontractible loop parameter $\mu(t) \in [0, \pi]$ and the variational parameter $a$ are introduced by the following ansatz:*)

$$\phi = v \exp(i\mu(t) h(x)) [\cos(\mu(t)) - isin(\mu(t)) h(x)],$$

$$A_1(x) = (1/e) \mu(t) \partial_1 h(x),$$

$$h(x) = \tanh(\sqrt{\lambda} v x/a) \quad \text{with} \quad h = h_{\text{sph}} \quad \text{at} \quad a=1.$$  

At $\mu=0$ and $\pi$, $N_{\text{cs}}=0$ and 1 respectively. At $\mu=\pi/2$, $N_{\text{cs}}=1/2$ corresponds to the winding number of the sphaleron. The reduced action is

$$S_E[\mu] = v^2 \int_0^\beta d\tau \left[ \frac{1}{2} M(\mu, a) \dot{\mu}^2 + V(\mu, a) \right],$$

where $\tau = ev\tau$, $\beta = ev\beta$ and

$$M(\mu, a) = (8e/3\sqrt{\lambda})[(\lambda/2e^2)/a + asin^2\mu],$$

$$V(\mu, a) = (4\sqrt{\lambda}/3e)[1/a + asin^2\mu]sin^2\mu.$$  

*) The ansatz for $\phi$ is different from that of Ref. 10) in a finite spatial domain in the $x$ dependence of the phase.
The first (second) term of $M$ comes from $F_{
u}^2(|D_0\phi|^2)$ while the first (second) term of $V$ from $|D_1\phi|^2$ (the Higgs potential), as understood from the dimensional consideration. As before, we apply the average approximation (3·8) to $\sin^2\mu$ in the parentheses of both $M$ and $V$: $M_0(\kappa, a) = (8e/3\sqrt{\lambda})[(\lambda/2e^2)/a + a\langle\sin^2\mu\rangle]$ and $V_0(\kappa, a) = (4\sqrt{\lambda}/3e)[1/a + a\langle\sin^2\mu\rangle]$. Then we again have the same three types of classical solutions as the $O(3)$ model:

(i) Vacuum solution $\mu_0 = 0$, $\pi$ with $S_\mu[\mu_0] = 0$.

(ii) Sphaleron solution:

$$\mu_{sph} = \pi/2 \quad \text{with} \quad S_\mu[\mu_{sph}] = S_{sph}(a) = \beta(4\sqrt{\lambda} v^2/3e)(1/a + a).$$  \hspace{1cm} (3·28)

(iii) Approximate bounce solution:

$$\mu_b(\tau) = \arccos(-\kappa \sin(b(\kappa, a) \tau; \kappa))$$  \hspace{1cm} (3·29)

with $b(\kappa, a) = \sqrt{2} V_0(\kappa, a)/M_0(\kappa, a)$, whose action is of the form similar to (3·10):

$$S_\mu[\mu_b] = S_b(\kappa, a) = 4\sqrt{\lambda} v^2 M_0(\kappa, a) V_0(\kappa, a)/2[2 E(\kappa) - (1 - \kappa^2) K(\kappa)].$$  \hspace{1cm} (3·30)

Here also, $S_b(\kappa=0, a) = S_{sph}(a)$ as (3·12), which is minimized at $a=1$ as (3·3):

$$S_{sph}(a=1) = \beta E_{sph} \quad \text{with} \quad E_{sph} = 8\sqrt{\lambda} v^3/3e^{10}.$$

The reduced action of the Abelian Higgs model is shown in Fig. 3 in the case of

![Figure 3](https://example.com/fig3.png)

**Fig. 3.** (a) Reduced action of the Abelian Higgs model $S_\mu[\mu]/v^2$ at various $1/\beta$ versus $a$ for $\lambda/e^3 = 0.5$, where $\langle\sin^2\mu\rangle = 1$ for simplicity. (b) The minimum of the action $S_\mu[\mu_{\text{min}}]/v^2$ at $a_{\text{min}}$ versus $1/\beta$, to be compared with the dash-dotted curves in Fig. 5.
\( \lambda/e^2 = 0.5 \). For \( 2 > \lambda/e^2 \), for example, the reduced action at low temperatures \( 1/\beta < (\sqrt{\lambda}/e)/2\pi \) is the bounce action for all \( a \), which is minimized at \( a = (\lambda/2e^2 \sin^2 \mu)/\beta^{14} \neq 0 \) in the broken phase while diverges at the symmetric phase point \( a = 0 \). At high temperatures \( 1/\beta > 1/\sqrt{2\pi} \), the reduced action is completely given by the sphaleron action, whose minimum at \( a = 1 \) is also in the broken phase. Only in the narrow intermediate temperature region the bounce action at small \( a \) continues to the sphaleron action at large \( a \) at \( a = a_b(\beta) \). As is well known, the instanton of the Abelian Higgs model in \( 1+1 \) dimensions exists in the broken phase \( (a \neq 0) \), unlike the \( O(3) \) nonlinear sigma model and \( 3+1 \) dimensional gauge theories. Namely, all the dynamics is realized in the broken phase. This brings about a peculiar pattern to Fig. 3, and the pattern would remain the same even if we take another reduction ansatz. Presumably the Abelian Higgs model\(^{10} \) would not imitate the transition of the electroweak theory.

§ 4. Formulation of transition rate by reduced action

**Imagining** that \( V(\mu, a = 0) < V(\mu = 0, a) \) in Fig. 1, we treat the transition rate of the “metastable” vacuum at \( \mu = 0 \) to the true vacuum at \( \mu = \pi \) of the reduced \( O(3) \) model in the Minkowskian version (the particle under the potential \(+ V\)). Although the basic formalism of the quantum-statistical transition at finite temperature has already been given by Langer\(^8 \) and Affleck,\(^9 \) we need to discuss some subtle problems. To avoid unessential complexity we often replace in this section \( M(\mu, a) \rightarrow M_0 \) and \( V(\mu, a) \rightarrow V(\mu) = V_0 \sin^2 \mu \) and regard \( M_0 \) and \( V_0 \) to be constants.

The transition rate at the temperature \( 1/\beta \) is defined to be the Boltzmann average of the probability flow \( \rho(\epsilon) \Gamma(\epsilon) \) through the potential barrier: \(^9 \)

\[
\Gamma = Z_0^{-1} \int_{\epsilon_0}^{\infty} d\epsilon \exp(-\beta \epsilon) \rho(\epsilon) \Gamma(\epsilon).
\]  

(4·1)

Here \( \epsilon = (\omega/g^2) E \) is the dimensional energy,\(^* \) \( \rho(\epsilon) = 1/2\pi \hbar \) is the incident flux per unit energy,\(^** \) \( \epsilon_0 = (1/2)\omega_0 \) with \( \omega_0/\omega = \Omega_0 = \sqrt{V''(0)/M_0} = \sqrt{2V_0/M_0} \) is the ground state energy of the metastable vacuum, which we do not put to be zero. The harmonic oscillation there gives rise to the positive-mode factor of the vacuum:

\[
Z_0 = [2\sinh(\beta \omega_0/2)]^{-1}.
\]  

(4·2)

In the WKB approximation \( \Gamma(\epsilon) \) is given by the “volume” \( W(\epsilon) \) of the barrier, and (4·1) is rewritten as

\[
\Gamma = \frac{\omega}{g^2} \frac{Z_0^{-1}}{2\pi} \int_{\epsilon_0}^{\infty} d\epsilon \exp(-1/g^2) \{ \beta E + W(E) \},
\]  

(4·3)

where \( E_0 = (\omega^2/\omega) \epsilon_0 = g^2 \Omega_0/2 \) is the dimensionless ground state energy and \( W(\epsilon) \) is given in (3·17). The negative mode frequency of the sphaleron barrier is given by \( \omega_- = \omega = \sqrt{\langle V''(\pi/2)/2 \rangle}/M_0 = \sqrt{2V_0/M_0} \).

\(^* \) The dimensional mass and potential, \( m = (1/\omega g^2) M_0 \) and \( v(\mu) = (\omega g^2) V(\mu) \), might be helpful as well to derive formulae that follow.

\(^** \) It can be shown that \( \hbar \) always appears in the combination \( g^2 \hbar \), so that we put \( \hbar = 1 \) hereafter.
4.1. Affleck formulae

We summarize approximate formulae of $\Gamma$ derived by Affleck.\cite{Affleck}

(i) High temperature region: $1/\beta \gg \Omega_\gamma /2\pi \equiv t_0(a)$ (more correctly, the sphaleron region in Fig. 2). Since the integral in (4·3) is dominated by $E \approx V_0$, the sphaleron barrier approximated by the parabolic form leads to

$$
\Gamma \approx (\omega_\gamma /4\pi) Z_0^{-1} \exp\left(-\bar{\beta} V_0 / g^2 / \sin(\beta \omega_\gamma / 2)\right). \tag{4.4}
$$

Note that $\bar{\beta} V_0(a) / g^2 = \beta E_{sph}^2$ at $a=1$ as remarked before.

In the standard Euclidean path integral formalism, the imaginary part of the free energy, $F = -(1/\beta) \ln Z$, gives the transition rate. Here the partition function $Z$ is dominated by the three classical actions obtained in § 3 as $Z = Z_0 + Z_{sph} + Z_b$, where each $Z$ is calculated by ($A$ standing for 0, sph, b)

$$
Z_a = N \exp\left(-S_a + \text{Det}(\delta^2 S_a / \partial \mu^2)\right)^{-1/2}, \tag{4.5}
$$

$N$ being a normalization constant. $Z_0$ from the vacuum action coincides with the positive-mode factor (4·2). $Z_{sph} = -(i/2)N \exp\left(-\bar{\beta} V_0 / g^2 / 2 \sin(\beta \Omega_\gamma)\right)$ reproduces (4·4) as

$$
\Gamma \approx (\omega_\gamma / \beta / \pi) \text{Im} F, \tag{4.6}
$$

where the approximation $\ln Z \approx Z_{sph} / Z_0$ is made. This is the formula on which is based the field theoretical calculation of the sphaleron transition rate. Thus there is no problem in the sphaleron region.

(ii) Low temperature region: $1/\beta \ll \Omega_\gamma /2\pi$ (more correctly, the bounce region in Fig. 2). The integral is dominated by a stationary point $E_0$ obtained from $\bar{\beta} + W'(E_0) = \beta - T(E_0) = 0$, around which we have the expansion

$$
\bar{\beta}E + W(E) = \bar{\beta}E_0 + W(E_0) + (1/2)T'(E_0)(E - E_0)^2 + \cdots. \tag{4.7}
$$

Note that $T'(E) < 0$, since $T(E)$ is originally the period of the bounce motion with the binding energy $E$ in § 3.1.*

In so far as the stationary point $E_0$ is safely within the integral region, we can resort to the saddle-point method using the Gaussian approximation of the integral ($\int_{E_0} f^{E_\infty}$), which results in

$$
\Gamma \approx (\omega / g) Z_0^{-1} \left(2\pi |T'(E_0)|\right)^{-1/2} \exp\left(-S_\phi(E_0)\right), \tag{4.7}
$$

where $S_\phi(E)$ is given in (3·15). Formally, this is expressed as

$$
\Gamma \approx 2\text{Im} F \tag{4.8}
$$

in the approximation $\ln Z \approx Z_{sph} / Z_0$. The concrete expression of $|T'(E_0)|$ which causes a trouble is given later.

(iii) Intermediate temperature region: $1/\beta \approx \Omega_\gamma /2\pi$ (more correctly, around the boundary $a_\beta$ in Fig. 2). Although the classical action is continuous at $a=a_\beta$, the transition rates (4·4) and (4·7) which include the leading quantum corrections are not. An approximate formula connecting the high and low temperature regions is

\footnote{In contrast to the $O(3)$ model and the electroweak theory, $T(E) < 0$ for most values of $\bar{\beta}$ of the Abelian Higgs model, which is related to the peculiar pattern of Fig. 3.}
\[ \Gamma \approx (\omega/g)Z_0^{-1}[2\pi T'(V_0)]^{-1/2}\exp\left[-(1/g^2)(\bar{\beta}V_0-(\bar{\beta}-2\pi|\Omega_0|^2/2)T'(V_0))\right] \]
\[ \cdot \Phi((\bar{\beta}-2\pi|\Omega_0|)/g\sqrt{T'(V_0)}) \, , \tag{4.9} \]

where \( \Phi(x) \) is the error function defined by \( \Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-t^2/2} \, dt \).

### 4.2. Problems in extremely low temperature region

Once the temperature is given, the stationary point \( E_0 \) is given by \( \kappa \) determined from

\[ \bar{\beta} = T(E_0) = 4\sqrt{M_0(x, a)/2V_0(a)}K(\kappa) \quad \text{with} \quad E_0 = V_0(a)(1-\kappa^2) \, . \tag{4.10} \]

In a region near the instanton limit \((\kappa \sim 1, a \sim 0)\), the low temperature formula \((4.7)\) is not applicable, which is now discussed.

1. **Problem of integral region** At extremely low temperatures, the stationary point \( E_0 \) obtained above is too small to be within the integral region of \((4.3)\), so that the saddle-point method leading to \((4.7)\) is no longer applicable. For illustration, the limiting formula for low temperature retaining \( E_c \neq 0 \) in \((4.3)\) may be as follows:

\[ \Gamma \approx \frac{\omega}{g^2}Z_0^{-1}\int_{E_c}^{V_0} dE \exp\left[-(1/g^2)(\bar{\beta}E + W(E))\right] = \frac{\omega}{g^2}Z_0^{-1}\exp(-\beta\omega_0) \]
\[ \cdot V_0 \int_{E_c/V_0}^{1} dx \exp\left[-(1/g^2)(\bar{\beta}V_0(1-x) + W(x-E_c/V_0))\right] \, , \tag{4.11} \]

where \( x = 1 - E/V_0 \). In the limit \( \bar{\beta} \to \infty \), the last integral is dominated by \( x \approx 1 \) while \( Z_0^{-1}\exp(-\beta\omega_0) \to 1 \), so that

\[ \Gamma \sim (\omega/2g^2)\sqrt{V_0}\exp\left[-(1/g^2)W(1-E_c/V_0)\right] \, . \tag{4.12} \]

Furthermore, the exponential factor at \( a = 0 \) in the classical limit \((E_c \to 0)\) actually coincides with the instanton limit: \( \exp(-W(1)/g^2) \to \exp(-S_b(\kappa=1, a=0)) \) with \( S_b(\kappa=1, a=0) = 1.16(2S_{\text{inst}}) \). On the other hand, when we consider the zero temperature and classical limit from the first without using the bounce action, the result from \( \Gamma \sim 2\text{Im}E \) \((F = E \text{ at } 1/\beta = 0)\) is\(^{11}\)

\[ \Gamma \sim (\omega_0/g\pi)\sqrt{V_0}\exp(-S_b(\kappa=1, a=0)) \, . \tag{4.13} \]

As seen later, prefactors to the exponential one play a minor role in numerical results.

2. **Problem of divergence in the instanton limit** From \((4.10)\) the explicit form of \( T'(E_0) \) is calculated:

\[ -T'(E_0) = \Delta T_E(\kappa) + \Delta T_M(\kappa) > 0 \, , \]

\[ \Delta T_E(\kappa) = \sqrt{M_0(x, a)/2V_0(a)^2}[E(\kappa) - (1-\kappa^2)K(\kappa)]/[\kappa^2(1-\kappa^2)] \, , \]

\[ \Delta T_M(\kappa) = (4/3)\sqrt{1/2M_0(x, a)V_0(a)^3K(\kappa)}/[\kappa^2(1-\kappa^2)\arcsin \kappa] \]
\[ \cdot [\kappa(1-2\kappa^2)/\sqrt{1-\kappa^2} - \kappa^2(1-\kappa^2)/\arcsin \kappa]a \, , \tag{4.14} \]

where \( \Delta T_E \) comes from the derivative with fixed \( M_0 \), and \( \Delta T_M \) comes from the weak
The same divergent result is also derived in the standard Euclidean path integral formalism for the bounce action. We have from (4·5)

$$Z_b = N [\text{Det}([1/g^2](-M_0(\kappa, a)d^2/d\tau^2 + V'(\mu_b))]^{1/2}\exp(-S_b(a))$$

with $$V'(\mu_b) = 2V_0(a)(1-2dn(b(\kappa, a)\tau; K)).$$

The lowest discrete eigenvalues $$\varepsilon$$ and the eigenfunctions $$\upsilon$$ of the operator in the determinant have been obtained in I:

$$\upsilon^0 \propto \text{cn}(b(\kappa, a)\tau; K)$$ with $$\varepsilon^0 = 0,$$

$$\upsilon^- \propto \text{dn}(b(\kappa, a)\tau; K)$$ with $$\varepsilon^- = -(1-\kappa^2),$$

$$\upsilon^+ \propto \text{sn}(b(\kappa, a)\tau; K)$$ with $$\varepsilon^+ = \kappa^2.$$ (4·16)

Note that in the limit $$\kappa \rightarrow 1$$ (limit of the infinite spatial domain in I), the second negative mode turns into one more zero mode. The calculation of the determinant after eliminating the proper zero mode $$\varepsilon^0$$ according to standard literatures is found to lead to $$Z_b = -iN(\beta\omega/2g)[2\pi T'(E_0)]^{1/2}\exp(-S_b(a))$$ with $$T'(E_0) = -\Delta T\kappa(\Delta T\kappa = 0$$ by ignoring the weak $$\kappa$$ dependence of $$M_0$$). Here the first factor $$-i$$ comes from the negative sign of $$\varepsilon^-$$, and the denominator $$\kappa^2(1-\kappa^2)$$ does from the eigenvalues $$\varepsilon^+$$ and $$|\varepsilon^-|$$. The physical reason of the divergence is now clear. The bounce solution with the zero binding energy is regarded as a pair of instanton and anti-instanton as remarked before. In this limit $$\kappa = 1$$, where the bounce motion requires an infinite period $$\beta$$, the instanton and the anti-instanton are infinitely separated from each other and would move independently. Such a system has two zero modes. Even if one of them, due to time translation invariance of the center of mass coordinate of the pair, has been eliminated as the proper zero mode, the other one due to the relative coordinate is left uneliminated and causes the divergence.

In conclusion, while both the problems are specific to extremely low temperatures, the first one may be a general one of the Affleck approximate formulae, but the second one is specific to the transition via quantum tunneling between two degenerate vacua.

§ 5. Numerical results

5.1. Transition rate by the reduced model

Two types of numerical results on the transition rate in the form

$$r = -\ln\Gamma/\omega$$

are presented in Fig. 4. One is by direct numerical integration of (4·3) to avoid the subtle problems, where the upper bound of the integral is cut at $$V_0(a)$$, the top of the potential barrier. (This is justified for low temperatures.) The other is by the

\(^{(*)}\) In the definition of $$\Gamma$$ in (4·1), the boundary condition which corresponds to the decaying state is included, so that no problem arises concerning the degenerate vacua.
Fig. 4. Numerical result of \( r = -\ln(\Gamma/\omega) \) versus \( a \) for various temperatures \( 1/\beta \). The solid curve is by the direct numerical integration. The dashed curve is the Affleck formula of the low (bounce) and the high temperature (sphaleron) regions, the dotted curve around the boundary \( a_0 \) (denoted by the vertical line) being the intermediate formula. Every local minimum of the numerical integration curves is indicated by the arrow: (a) \( g^2 = 1.0 \), (b) \( g^2 = 0.1 \).

Affleck formulae (4.4), (4.7) with (4.14), and (4.9) for comparison. The effect of the divergence of \( r \) by the Affleck formulae at \( a = 0 \) is not restricted to the vicinity of the instanton limit, but extends to a rather wide region around it. Except the divergence, all the numerical results rapidly converge as \( g^2 \) decreases (the semiclassical limit).*)

Figure 5 shows the temperature dependence of the transition rate, \( r_{\text{min}} \) by the numerical integration at its minimum with respect to the variational parameter \( a \), which is compared with the minimum of the classical action \( S_\mu / g^2 \). The former basically follows the latter, that is, the thermal transition at high temperatures “transits” around \( 1/\beta_c \) in (3.20) to the quantum tunneling transition governed by the bounce action, whose zero temperature limit is the instanton transition. But the transition rate \( r_{\text{min}} \) behaves more smoothly than \( S_\mu / g^2 \).

\[ \Gamma_{\text{min}} \text{largely changes with the temperature:} \]

\[ \Gamma_{\text{min}} \sim \omega e^{-30/g^2} \text{ at } 1/\beta \sim 0 \]

\[ \rightarrow \Gamma_{\text{min}} \sim \omega e^{-10/g^2} \text{ at } 1/\beta \sim \omega, \]

where \( \omega \) is the unique energy scale of the \( O(3) \) model.

5.2. Comparison with field theoretical transition rate

*) See the second footnote on p. 1204.
Sphaleron Transition of Reduced \(O(3)\) Nonlinear Sigma Model

Fig. 5. Temperature dependence of \(r_{\text{min}}\), the minimum of the transition rate \(r = -\ln \Gamma/\omega\) by the direct numerical integration (the solid curve), which is compared with the minimum of the classical action \(S_F[\mu]_{\text{min}}/g^2\) (the dash-dotted curve). The dashed curve is \(r_{\text{min}}\) by the Affleck formulae: (a) \(g^2=1.0\), (b) \(g^2=0.1\).

Fortunately, the field theoretical transition rate \(\Gamma_{\text{FT}}\) of the \(O(3)\) nonlinear sigma model in 1+1 dimensions is completely known at high temperatures:\(^{4,5}\)

\[
\Gamma_{\text{FT}} = (\omega \cdot 4\pi \sin(\beta \omega \cdot 2)) F_{\text{zm}} \exp[-\beta E_{\text{sph}} - h(\beta)]
\]  

(5.3)

with

\[
F_{\text{zm}} = \sqrt{E_{\text{sph}}/2\pi \beta \sqrt{3}} \beta \omega / g,
\]

\[
h(\beta) = -(4 \beta/\pi) \int_0^{\infty} dx [1/(x^2 + \beta^2) + 1/(x^2 + 4 \beta^2)] \ln[1 - \exp(-\sqrt{x^2 + \beta^2})]
\]

\[
\rightarrow -\ln 3 + (4/\pi)(\arctan \beta + \arctan 2\beta/2 - 3\beta) \ln \beta - C + O(\beta)
\]

(5.4)

for \(1/\beta \ll 1\) with \(C=2.2515852\). Two zero modes of fluctuation operators around the sphaleron contribute to the zero mode factor \(F_{\text{zm}}\). The first is due to translation invariance of the sphaleron solution in (2.4) and the second to rotation invariance of \(\mathbf{n}\) around the \(n_3\) axis in (2.2),\(^{4,14}\) \(h(\beta)\) is the entropy factor obtained from the phase shift of the Rosen-Morse potential \((V'\) with \(\varepsilon=0\) in (4.5)),\(^4\) so that (6.1) is valid only at high temperatures.

Comparing (6.1) with (4.4), we see that \(F_{\text{zm}}\) and \(h(\beta)\) originating from the field degrees of freedom duly come in, while \(Z_0^{-1}\), the positive-mode factor of the vacuum
Fig. 6. $r_{\text{min}}$ by the numerical integration (the solid curve) and the field theoretical one $r_{\text{FT}}$ at high temperatures (the dash-dotted curve). The result where the zero-mode factor $F_{zm}$ and the entropy term $h(\beta)$ are omitted from the latter is shown for comparison (the dashed curve): (a) $g^2=1.0$, (b) $g^2=0.1$.

of the reduced action is dropped out and is implicitly contained in $h(\beta)$. These are all the corrections we need to obtain $\Gamma_{\text{FT}}$ from $\Gamma$ of the reduced model. $r_{\text{min}}$ of the reduced model and $r_{\text{FT}} = -\ln \Gamma_{\text{FT}}/\omega$ are shown in Fig. 6. Of course, the zero mode correction $F_{zm} \sim O(1)$ at high temperatures scarcely affects the magnitude of $\Gamma$. The correction by the entropy term $h(\beta)$ is also almost negligible as understood from Fig. 6.*

At low temperatures, unfortunately, the field theoretical analysis is not complete enough. There are six zero modes around the one instanton solution related to translation, dilatation and rotation.\(^{16}\) Their corrections on the reduced transition rate may not modify the magnitude of it. The entropy correction would not presumably be so serious.

\section*{§ 6. Conclusions}

We have investigated, in a wide range of temperatures, the transition rate of topological number of $1+1$ dimensional nonlinear sigma model, employing an ansatz which reduces the infinite degrees of freedom of field theory to one along the non-contractible loop with a variational parameter. A similar analysis \textit{at classical level} of four-dimensional $SU(2)$ gauge-Higgs system has been performed by Aoyama,\(^{15}\)

\* The entropy suppression in the electroweak sphaleron transition at high temperatures also plays a minor role.\(^{15}\)
et al. who started from the instanton solution, while we started from the sphaleron and studied the leading quantum corrections. The variational parameter, which is the scale of the static solution, while that of the instanton in the work by Aoyama, et al., is found by dimensional argument to play the role to distinguish the symmetric theory and the symmetry-broken one. Such an interpretation does not apply to the case of the 1+1 dimensional Abelian Higgs model where both instanton and sphaleron coexist in the broken phase.

We have also pointed out some problems in applying the approximate formula by Affleck\(^9\) to calculate the transition rate at extremely low temperatures. To avoid the problems, we numerically integrated the defining equation of the quantum-statistical transition rate and obtained it in a rather wide range of temperatures. The results show that the bounce solution mediates between the instanton transition and the sphaleron transition, that the sharp change of the rate is somewhat tamed by quantum corrections, and that the approximate formula by Affleck which is used for field-theoretic calculations is valid numerically at high temperatures. Comparing the rate with that obtained by the field-theoretic treatment at high temperatures, we found that the corrections due to the zero-mode factor and entropy term play a minor role.

For the four-dimensional \(SU(2)\) gauge-Higgs model, similar results are expected to hold. For that model, we should incorporate the effect of temperature dependence of the sphaleron mass, which would raise the transition rate at high temperatures since the height of the barrier decreases as temperature increases. Although we have not been equipped with reliable analytical methods, what is more interesting would be the transition above the critical temperature of the gauge-symmetry restoration.

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**References**