Instability of the Hole Solution in the Complex Ginzburg-Landau Equation

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The complex Ginzburg-Landau equation has a hole solution as a localized structure. The hole solution shows two types of instability and bifurcates to more complicated solutions. The existence of those attractors makes the behaviors of the system complicated.

Oscillatory media near the Hopf bifurcation are generally described by the complex Ginzburg-Landau equation:

\[ \dot{W} = W - (1 + i c_2)|W|^2 W + (1 + i c_1) \partial^2 W, \]

where \( W = X + i Y \) is a complex variable. We consider hereafter only the one dimensional system and then \( \partial^2 W \) is replaced by \( \partial^2 W/\partial x^2 \). This equation has a family of plane wave solutions:

\[ W_0 = \sqrt{1 - Q^2} \exp(iQx - i\omega t), \]

where \( Q \) is the wavenumber and the frequency \( \omega \) is \( c_2 \cdot (1 - Q^2) + c_1 \cdot Q^2 \). A phase modulation \( \phi(x, t) \) on the plane wave solution satisfies the following phase equation:

\[ \dot{\phi} = \Omega_1 \frac{\partial \phi}{\partial x} + \Omega_2 \left( \frac{\partial \phi}{\partial x} \right)^2 + \Omega_3 \left( \frac{\partial^2 \phi}{\partial x^2} \right) + \Omega_4 \left( \frac{\partial^3 \phi}{\partial x^3} \right), \]

where \( \Omega_1 \sim \Omega_4 \) are parameters determined by \( c_1, c_2 \) and \( Q \). The stability of the plane wave solution is determined by the sign of the diffusion coefficient \( \Omega_2^{(1)} = 1 + c_1 c_2 - 2 Q^2 \cdot (1 + c_2^2)/(1 - Q^2) \).

When \( Q_2^{(1)} \) is negative but small enough, the plane wave solution becomes unstable but the phase modulation is small and it can be described by the phase equation (3). But when \( Q_2^{(1)} \) is negative and its absolute value becomes lar-
Fig. 2. Amplitude pattern of the final states starting from weakly perturbed plane wave solutions for $c_2=2.0$.
(a) Initial wavenumber $Q=24\pi/200$ and $c_1=0.32$.
(b) Initial wavenumber $Q=10\pi/200$ and $c_1=-0.4$.

We further study the hole solution to understand the difference of the final states by the value of $c_2$. A hole solution is an exact solution of the complex Ginzburg-Landau equation. Nozaki and Bekki obtained the solution by Hirota's bilinear method. The explicit form of the hole solution is given by

$$W_H = \sqrt{1-Q^2} \tanh(kx) \exp(i\theta(x) - i\omega t) ,$$

where the phase $\theta$ satisfies $\partial \theta / \partial x = -Q \cdot \tanh(kx)$, the frequency $\omega = c_2(1-Q^2) + c_1 Q^2$ and the wavenumber $Q$ is given by using $k$ as $Q = (2k^2 - 1)/(3kc_1)$. The width of the hole in the amplitude pattern is expressed by $k^{-1}$ and $k$ is given by a solution of the equation:

$$\{4(c_2-c_1)+18c_1(1+c_2)\}k^4 - \{4(c_2-c_1)+9c_1(1+c_2)\}k^2 + (c_2-c_1) = 0 .$$

The hole solution approaches a plane wave solution with the wavenumber $\mp Q$ as
We studied the stability of the hole solution numerically. We assumed the free boundary condition, i.e., $\partial W/\partial x = 0$ at the boundary $x = 0$ and $x = L$. We calculated a system whose size was 100, 200 or 400. Figure 3 is a phase diagram showing the stability of the hole solution. The hole solution is stable in the region between the line $c$ and $p$. The line $p$ indicates the phase instability line $\Omega^{(1)}(Q) = 0$ where the asymptotic plane wave with the wavenumber $Q$ becomes unstable. Figure 4 shows the behaviors of the system when $c_1$ is gradually decreased across the line for the constant $c_2 = 2.0$ and system size $L = 400$. The line $p$ is an instability line for an infinite size system $L = \infty$. In our finite system, the phase fluctuation grows but travels toward the boundaries and is absorbed at the boundaries. So the hole solution keeps its form even below the line $x \rightarrow \pm \infty$. The amplitude is zero at $x = 0$ or the hole solution is a solution which has a locally non-oscillating region in contrast to the locally oscillating solution found by Fauve and Thual.\(^7\) If we assume $c_1 = c_2 = 0$, we obtain a kink solution (domain wall) $W = \tanh \sqrt{\frac{1}{2}} x$ of the usual Ginzburg-Landau equation. If we assume $c_1 = c_2 \rightarrow \infty$, we obtain an oscillating kink solution $W = \tanh \sqrt{\frac{1}{2}} x \exp(-ic_2t)$, which is one of the kink solutions of the nonlinear Schrödinger equation:

\[
\dot{W} + c_1 W_{xx} - c_2 |W|^2 W = 0. \tag{6}
\]

Fig. 3. Phase diagram for the hole solution. The line $p$ indicates the phase instability, the line $c$ indicates the core instability and the line $b$ is the boundary line for the attractors with large amplitude fluctuation.

Fig. 4. Amplitude distributions for $c_1 = 0.35, 0.29, 0.23$ and 0.17. They are a result of a numerical simulation in which $c_1$ is gradually decreased from the stable region to the phase instability region for $c_2 = 2.0$. 

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But when $c_1$ is decreased further, the phase fluctuation grows and two holes are
created apart from the original hole. When $c_1$ is decreased still further, the phase
fluctuation with the shorter wavelength grows and another hole is created between
the previous two holes. When $c_1$ is larger than about 0.2, similar processes occur and the
hole number increases but each hole is almost stationary and stable. But when $c_1$ is
still smaller, the holes become non-stationary or the creation and annihilation proc­
esses of the holes occur, and the system becomes turbulent gradually.

We call the line $c$ a core instability line, where the amplitude distribution near the
hole becomes asymmetric and the hole begins to move. We determined the line $c$
umerically, i.e., we check up that the form (4) of the hole solution was not kept up
in the numerical time evolution of (1) above the line $c$, even if the initial condition was
assumed to be (4). In Fig. 5(a) we show a snap-shot pattern of the amplitude
distribution for $c_2=2.0$ and $c_1=0.67$. Note that the amplitude pattern is asymmetric
around the hole position and the asymptotic values of the amplitude far from the hole
position are different for the right and left. Figure 5(b) shows the trajectory of the
hole position which is defined as the minimum point of the amplitude. The hole
position begins to oscillate at the core instability. The oscillation grows and at last
the hole is absorbed at the left boundary $x=0$. This solution is similar to a propagat­
ing hole solution found by Bekki and Nozaki but their solution has a constant
velocity and does not exhibit the oscillation. The mechanism of the core
instability and the oscillation is left to the future study. The final state is the
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In conclusion we studied numerically the instability of the plane wave solution and the hole solution. In the numerical simulation starting from an unstable plane wave solution, a wavenumber changing process occurred and the unstable plane wave solution led to another stable plane wave solution for small $c_z$. On the other hand for large $c_z$ the unstable plane wave solution led to a hole solution or an amplitude-turbulence state. In the study of the hole solution we found that the hole solution and the attractors originated from the hole solution exist stably only for large $c_z$ (larger than about 1.1). This is closely related to the result of the first simulation that the final states starting from the unstable plane wave solutions are different by the value of $c_z$. Namely when $c_z$ is small, there are no attractors other than the family of plane wave solutions and so an unstable plane wave leads to another stable plane wave solution necessarily. On the other hand when $c_z$ is large, there exist the attractors born through the instability of the hole solution and therefore there is a possibility that the unstable plane wave state is attracted by the attractor originated from the hole solution. The result of the numerical simulation of Fig. 2 shows that this possibility was realized. It means that the unstable plane wave solution was in the basin of the attractor. We can say that the existence of the stable hole solution and the attractors born from the hole solution makes the behaviors of the complex Ginzburg-Landau equation complicated. We carried out a numerical simulation further to investigate the transition to the amplitude turbulence and the transition to the uniform oscillation across the line $b$. The result of the detailed simulation will be reported elsewhere.

4) H. Sakaguchi, Prog. Theor. Phys. 84 (1990), 792.