Elementary Excitations in a Solid Nuclear Matter

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The effects of elementary excitations to the nature of dense solid nuclear matter are investigated using the Pandharipande-Smith (PS) potentials. A trial wave function embodying a simple cubic structure of nucleonic lattice leads to the Hamiltonian describing the dynamics of lattice vibrations (or phonons) spin-waves (or magnons) and isospin-waves (or ‘isomagnons’). Magnons and isomagnons are treated as the Holstein-Primakoff bosons. In the density range from around 0.3 to 1 nucleons/fm$^3$, the dynamics of phonons approximately reproduces the equations of state for PS’s potentials 2 and 3. At the lowest order of perturbations, due to the tensor interaction, magnon has an energy gap of the order of 100 MeV at infinite wavelength, while isomagnon is gapless. The energy is lowered by $O(100 \text{ MeV/nucleon})$ in virtue of the zero-point fluctuation of spins. A simple variational method indicates that the ground state is likely to be realized by the simultaneous condensation of $\pi^\pm$, $\pi^0$ and of isomagnons, which further lowers the energy by $O(100 \text{ MeV/nucleon})$.

§ 1. Introduction

A number of studies on the equation of state of cold and dense nuclear matter have been done based on various different methods including variational method, relativistic mean field theory, soliton description for nucleons and so on. Many of them are based on the assumption that the high density nuclear matter is in the liquid state of nucleons, hyperons, isobars and leptons as electrons and muons. Then, nucleons have been supposed to be in the state of superfluidity and superconductivity, which are expected to arise from attractive interactions in, e.g., $^1S_0$ and $^3P_2$ channels.

At the same time, there appeared another intriguing picture that the cold, dense nuclear matter is likely to solidify. This possibility arose by noting the presence of strong repulsive core in nuclear potential. The factor which acts against solidification is large zero-point lattice vibrations caused by localizations of nucleons. But, if the repulsive force at short distances as well as attractive force in intermediate distances is strong enough, it will overcome the zero-point energies to cage nucleons. Calculations based on $t$-matrix method were performed and showed that locally stable solidifications could occur.

In principle, one could judge which of liquid and solid is the true ground state by comparing energies of those two states. The problem was that calculations based on different schemes resulted in different conclusions. As long as the true ground state is concerned, there seem no definite theoretical consensus. The observations, on the other hand, provided some clue to this problem: The glitches observed in some pulsars are explained by the two components (crust and superfluid) model of neutron stars. This may be one of the reasons that the liquid picture is now widely accepted to be realized at least in some moderate density regions.

At very high density, however, the circumstances will be changed. Pandharipande
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and Smith (PS)\(^4\) pointed out a possibility that \(x^0\)-condensations (or possibly equivalently \(\Delta\)-isobars) will enhance the attractive tensor force to solidify the neutrons. To extract maximal attractive force, a special spin configuration was assumed: All spins are aligned to, say, positive \(z\)-direction in a single layer of the simple cubic lattice. In the neighboring layers, all spins are assumed to direct negative \(z\)-direction. (This spin configuration was called 'Alternating Layer Spin Structure' or [ALS]-structure in Ref. 28. We call it simply ALS in this paper.) Their constrained variational calculation showed that neutrons together with some fraction of \(\Delta\)-isobars are likely to solidify at densities higher than 0.4 nucleons fm\(^{-3}\).

For the same spin configuration, Takatsuka and Tamagaki (TT)\(^28\) and Takatsuka et al.\(^29\) adopted a different set of potentials and performed variational calculations to find that the solidification might be possible. Since their result showed the energy of the system had a broad minimum in the variation of lattice constant, they arrived at a different picture for dense nuclear matter: The solidification occurs in one dimension; in other two dimensions, nucleons behave as liquid.

The variational method usually starts by setting the forms of nuclear force as well as the form of the trial wave function of nuclear matter. It has been common that the trial wave functions are chosen so as to reflect the spatial correlations of nucleons. Other degrees of freedom, e.g., spins and isospins, are treated as classical variables. However, the treatment of this kind may be insufficient for solids because dynamics of spins are known to play essential roles to determine solid-state properties.

In solid-state physics, the dynamical nature of spin systems is typically described by the quantum Heisenberg model,\(^30\) whose Hamiltonian is given by

\[
H_{\text{spin}} = -\frac{1}{2} \sum_{(j,k)} J_{jk} \mathbf{S}_j \cdot \mathbf{S}_k, \quad S_j^2 = S(S+1),
\]

where \(\mathbf{S}_j\) is the spin operator at the \(j\)th lattice site. The system is (anti) ferromagnetic for \(J_{jk} > 0\) \((< 0)\). According to the quantum statistical mechanics, the system described by the Hamiltonian (1.1) is not labeled by the, e.g., \(z\)-component of each spin operator. In fact, none of the components of individual spin operator commute with the Hamiltonian. Instead of this, the system must be described in terms of spin-wave excitations or magnons, the quantum version of spin-waves. Then, it is known that thermodynamical quantities of the spin system are correctly reproduced by this picture.

Since the nuclear force has strong dependence on spin states, we can anticipate the same type of interaction to appear in solid state physics of nuclear matter and to play important roles in determining its physical properties. Of course, one may expect that the quantum model converges to the classical one when \(S\) gets larger. However, many previous works\(^1-5,9\) suggest that, in liquid nuclear matter, the lowest spin state, \(S=1/2\), will dominate over higher spin states called baryonic resonances in a wide range of density. Since it is very unlikely that this situation is drastically changed in solid phase, quantum mechanical studies of \(S=1/2\) spin-waves may be indispensable for obtaining knowledges about solid nuclear matter, if any.

The purpose of this paper is to analyze how the elementary excitations like
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magnon affect the property of dense nuclear matter when the solidification is supposed to occur. Nuclear forces are also dependent on the isospin state, and effective isospin-isospin interactions analogous to (1·1) will result in. They will be treated on the basis same to the spin-wave theory. Then, we will have to deal with the isospin-waves or their quantum version — in the present paper we call them 'isomagnons'. Also important are the lattice vibrations or phonons, for the nuclear force is sensitive to the spatial distance of two nucleons at high densities. The solid is assumed to be of three-dimensional structure, which enables us to escape from the Mermin-Wagner theorem\(^{31}\) which states that the one- and two-dimensional Heisenberg model with short-range interaction has no spin-orderings at nonzero temperatures.

The form of nucleon-nucleon interactions is specified in § 2. For the sake of comparison of our study with a previous work based on the well-known variational method, we adopt the potentials used by PS who reported the details of their analysis based on the constrained variational method.\(^{4}\) In § 3, we construct a trial state which embodies the solid structure of nucleon system. In § 4, the effective Hamiltonian and the equation of motion for nucleon wavefunction are obtained by the variational method in which spins and isospins are also treated as dynamical variables. In § 5, the contribution of phonon modes to system's energy is investigated in the lowest order of lattice vibrations, and compared with the PS-calculation. Spins and isospins are treated as classical variables which represent ALS. In § 6, we investigate the quantum mechanics of spin and isospin associated with nucleons. Their dynamics are described by Heisenberg-like Hamiltonian with nearest-neighbor interactions due to short-ranged nuclear potentials. Quantizations of spin- and isospin-waves are performed by Holstein and Primakoff (HP)'s method.\(^{32}\) In § 7, the coupled equations of pions and isomagnons are solved for a trial state describing their microscopic condensations. Section 8 is devoted to concluding remarks. HP transformation utilized in the spin-wave theory is elaborated in the Appendix.

§ 2. Nucleon-nucleon interactions

Since we are interested in the nature of possible solid state, it is preferable to adopt a set of interactions which is likely to realize such structures. Here, we borrow those interactions from the work by PS.\(^{4}\) However, the main conclusion of our study reported below will not be affected strongly by the details of the forms of interaction, if solidification does occur.

The non-relativistic Hamiltonian for the nuclear matter is assumed to take the following form:

\[
H = \int dr \left[ -\frac{1}{2M} \frac{\partial^2 \phi^*(r)}{\partial r^2} \phi(r) - \frac{1}{2} \sum_i \phi^*_i(r) O_i \phi(r) \int d' r' V_i(r, r') \phi^*(r') O_i \phi(r') \right]. \tag{2·1a}
\]

\(\phi\) is the second quantized nucleon field which obeys the equal time commutation relation

\[\{\phi(r), \phi^*(r')\} = \delta(r - r'). \tag{2·1b}\]
The number of \( O_i \) which must be used for the accurate description of \( NN \) scattering is quite large, especially when we try to incorporate \( \Delta \)-isobars. The ones PS adopted are

\[
O_1 = 1, \quad O_2 = \sigma, \quad O_3 = \sigma \otimes \tau. \tag{2·1c}
\]

\( \sigma \) and \( \tau \) are Pauli's matrices for spin and isospin, respectively. \( V_i \) are potential functions relevant to \( i \)th type of operator. This representation of Hamiltonian is regarded as having been derived from more fundamental one describing interactions among nucleons and mesons. For example, the Hamiltonian for nucleon-pion system is given by

\[
H_{NN} = \frac{1}{2m} \partial^\dagger \partial \phi + \frac{1}{2} \left[ \phi^2 + \left( \partial \phi \right)^2 + \mu^2 \phi^2 \right] + \frac{f}{\mu} \phi^\dagger \sigma \otimes \tau \phi \partial \phi,
\]

\[
f^2/4\pi = 0.081, \tag{2·2b}
\]

where \( \phi \) is the isovector pion field. Eliminating pion field by functional integration for the corresponding Lagrangian, one has the interaction of the type 4 in (2·1) with

\[
V_i = \left( \frac{f}{\mu} \right)^2 \delta_{\alpha_1\alpha_2} \delta_{i_1i_2} D(r_1 - r_2)
\]

\[
= -\frac{1}{4\pi} \left( \frac{f}{\mu} \right)^2 \delta_{\alpha_1\alpha_2} \left[ (3\bar{n}_{i_1} \bar{n}_{i_2} - \delta_{i_1i_2}) \left( \frac{1}{2\mu^2 r^2} + \frac{\mu^2}{3} + \frac{\mu^2}{3} \right) \right] e^{-\mu r}/r,
\]

\[
\bar{n} = (r_1 - r_2)/r, \quad r = |r_1 - r_2|. \tag{2·3}
\]

Indices \( i \) and \( \alpha \) are for the three-dimensional Euclidean space and the isospin space, respectively. \( D(r) \) is the Green's function defined by

\[
D(r) = -\frac{1}{4\pi} \frac{e^{-\mu|r|}}{|r|}, \quad (\partial^2 - \mu^2)D(r) = \delta(r). \tag{2·4}
\]

The choice of \( O_i \) given in (2·1b) implies the following forms of potentials:

\[
V_1 = V_{C_1}(r_1 - r_2), \tag{2·5a}
\]

\[
V_{2_{i_1i_2}} = \delta_{i_1i_2} V_{C_2}(r_1 - r_2), \tag{2·5b}
\]

\[
V_{3_{i_1i_2i_3}} = \delta_{i_1i_3} \delta_{i_2i_3} V_{C_3}(r_1 - r_2), \tag{2·5c}
\]

\[
V_{4_{i_1i_2i_3i_4}} = \delta_{i_1i_4} (3\bar{n}_{i_1} \bar{n}_{i_2} - \delta_{i_1i_2}) V_{T}(r_1 - r_2). \tag{2·5d}
\]

Contributions from other mesons than pions (as rho and omega) together with higher order effects will eventually yield several types of effective potentials, among of which PS chose the set of potentials which are the generalization of Green-Haapakoski potentials for the description of \( ^1S_0 \) phase shifts in nucleon-nucleon interactions. The details of the forms of \( V_{C_1} - V_T \) will be given in § 5.

PS have exclusively taken the effect of \( \Delta \) into their calculations as the corrections of the strengths of the tensor interaction whose strength is allowed to vary up to a value about four times larger than the one given by (2·2b) in accordance with the change in concentration of \( \Delta \). Here, we follow their method, too.
As stressed by PS, the neutral pion field in general acquires nonzero classical value. This situation can be taken into account in our formulation by splitting the pion field into the classical and quantum parts:

\[ \phi = \phi_c + \phi_q, \quad \phi_c = (0, 0, \phi_0), \]  

(2.6)

and by performing the functional integration over \( \phi_q \) in the original expression of the Lagrangian. Here, the classical field satisfies the equation of motion

\[ \ddot{\phi}_c - \partial^2 \phi_0 + \mu^2 \phi_0 = -\frac{f}{\mu} \partial_\alpha S_{\alpha\mu}(r), \]  

(2.7a)

\[ S_{\alpha\mu}(r) = \langle \phi^\dagger(r) \tau_\alpha \sigma_\mu(\mathbf{r}) \rangle. \]  

(2.7b)

The resultant Hamiltonian is (2.1) added by

\[ \frac{1}{2} \int dr [\phi_0^2 + (\partial \phi_0)^2 + \mu^2 \phi_0^2] - \frac{1}{2} \int dr dr' S_{\alpha\mu}(r) S_{\alpha\mu}(r') V_{4; \alpha, \mu, \alpha, \mu}. \]  

(2.8)

This contribution, although identically vanishes in case \( \phi_0 \) is time-independent as in § 5 and in Refs. 4), 28) and 29), will have to be reserved for the general case in which the trial state takes into account the spin-wave excitations in a general form.

In PS's approach, all the effects of meson fields are supposed to be incorporated in the form of nuclear potentials. Especially, the classical pion field can be regarded to contribute in enhancing the concentration of \( \Delta \)-isobar, resulting in the increase of the effective mass of nucleon. Later, we will compare it with (the first integration of) (2.8) to see one aspect of the correspondence between the potential method and meson field method for the description of nucleon interactions.

### § 3. Trial wave functions

We seek the local minimum of the expectation value of Hamiltonian (2.1) in terms of trial functions which represent a crystalline structure of nuclear matter. As a structure of solid, we assume the simple cubic lattice. Its state shall be determined by a set of parameters \( \{ R_j, r_j, s_j, t_j \} \), each of which denotes the spatial position of \( j \)th lattice site, spatial position of the nucleon around the equilibrium position at \( j \)th lattice site, the nucleon's spin state and isospin state, respectively. For a simple cubic lattice, there are three fundamental lattice vectors \( (a_1, a_2, a_3) \) which are orthogonal to each other. Then, any vector \( R_j \) is represented by a set of integers \( (l_j, m_j, n_j) \) as

\[ R_j = l_j a_1 + m_j a_2 + n_j a_3. \]  

(3.1)

The most general trial state will be constructed from the direct product of the spatial wave function and the spinors of spin and isospin as

\[ |(s_j, t_j)\rangle = \int (\Pi dr_j) (\Pi \phi^\dagger(r_j) \chi(s_j) \zeta(t_j)) \Phi(r_1, r_2, \ldots, r_N) |0\rangle, \]  

(3.2)

where \( \chi \) and \( \zeta \) are two-component column vectors for spin and isospin, respectively. \( \Phi = \Phi(r_1, r_2, \ldots, r_N) \) is the spatial wave function of \( N \) nucleons localized at the lattice...
sites represented by (3.1). \( |0\rangle \) is the perturbative vacuum annihilated by the nucleon field \( \phi \). \( j \) ranges over integers from 1 to \( N \). These functions are presumed to be normalized and orthogonal:

\[
\chi'(s)\chi(s') = \delta_{s,s'}, \quad \xi'(t)\xi(t') = \delta_{t,t'}, \quad (3.3a)
\]

\[
\int(\Pi dr_j)\Phi^*\Phi = 1, \quad \int(\Pi dr_j)\Phi^*\Phi_{[j,k]} = 0 \quad \text{for} \quad j \neq k. \quad (3.3b)
\]

In (3.3b), \( \Phi_{[j,k]} \) denotes that the coordinates of \( j \)th and \( k \)th nucleons have been interchanged in \( \Phi \):

\[
\Phi_{[j,k]} = \Phi(\cdots, r_j \rightarrow r_k, \cdots, r_k \rightarrow r_j, \cdots). \quad (3.4)
\]

With these conditions, the states \( |{s_j, t_j}\rangle \) are also assured to satisfy the orthogonality condition:

\[
\langle{s_j, t_j}|{s'_j, t'_j}\rangle = \prod \delta_{s_j,s'_j}\delta_{t_j,t'_j}. \quad (3.5)
\]

The expectation value of Hamiltonian is

\[
\langle H \rangle = \langle{s_j, t_j}|H|{s_j, t_j}\rangle. \quad (3.6)
\]

In the subsequent sections, the local minimum of \( \langle H \rangle \) is sought not only by the variation of \( \Phi \) but also by the variation of spin- and isospin-state, under the constraint of normalization condition imposed on \( \Phi \).

§ 4. The equation of motion

We calculate \( \langle H \rangle \) by using the commutation relation (2.1b). To this end, we need to know the effect of the operation of \( \phi \) on \( |{s_j, t_j}\rangle \). It reads

\[
\psi_{\sigma}(r)|{s_j, t_j}\rangle = \sum (\pm)\chi_{\sigma}(s_j)\xi_{\sigma}(t_j)\int(\Pi dr)|{\phi^*(r)\chi(s_j)\xi(t_j)}\rangle\Phi(\cdots, r_j = r, \cdots)|0\rangle. \quad (4.1)
\]

The signs \( (\pm) \), coming from Fermi statistics of \( \phi \), mean it takes values +1 or −1 in accordance with the position of the \( j \)th operator \( \phi^*(r) \) in \( |{s_j, t_j}\rangle \). In the products over \( l \) in (4.1), \( l = j \) is omitted.

Let us take a matrix element of Hamiltonian (2.1a) between two states \( |{s_j, t_j}\rangle \) and \( |{s'_j, t'_j}\rangle \), which have in general different sets of spin and isospin states. Field operators in the Hamiltonian together with the operators in the trial states will meet and annihilate each other with various combinations. Some examples are depicted in Fig. 1. Figure 1(a) represents an example of direct diagrams, in which the first field operators in the bra and ket state are contracted respectively with \( \phi^*(r) \) and \( \phi(r) \) of the same spatial coordinate \( r \) in the Hamiltonian. Similarly, the second operators in the trial states are contracted with the operators having spatial coordinate \( r' \) in the Hamiltonian. Other operators in the ket vector are contracted directly with the corresponding operators in the bra vector. As a result, the operators \( O_i \) which describe spin and isospin dependences of nuclear force have the matrix elements for
In Fig. 1(b) is presented another type of diagram. There, \( \phi^+(r) \) and \( \phi(r) \) having the same coordinate in the Hamiltonian are contracted with operators of different nucleons in the bra vector. The operators of corresponding nucleons in the ket vector are contracted with other field operators in the Hamiltonian. Then, the operators \( O_i \) acquire their matrix elements between spin and isospin states of different nucleons. Another important effect is that the spatial coordinates in \( \Phi \) are also exchanged. If, for instance, \( \Phi \) describes the localized nucleons by Gaussian forms for the spatial coordinates, then the exchange of \( j \)th and \( k \)th coordinate will bring about a suppression factor like \( \exp \left[ - \frac{\nu (R_j - R_k)^2}{2} \right] \) for the wave function. Other nucleons in the trial state are contracted directly with the same nucleons.

We see that, due to the localization, the exchange diagrams will yield one suppression factor per exchange of two nucleons, which gets rapidly smaller when the spatial positions of exchanged nucleons get farther. Therefore, for the exchange diagrams, we will keep only the ones in which only single pairs of nearest-neighbors are exchanged. The result is

\[
\langle \{ s_j, t_j \} | H | \{ s'_j, t'_j \} \rangle = \langle \{ s_j, t_j \} | H | \{ s'_j, t'_j \} \rangle_D + \langle \{ s_j, t_j \} | H | \{ s'_j, t'_j \} \rangle_E ,
\]

\[
\langle \{ s_j, t_j \} | H | \{ s'_j, t'_j \} \rangle_D = \int (dr_j) \sum_{j, \delta, \bar{\delta}} \frac{1}{2M} \partial_j \phi^* \partial_j \phi \delta_{(st)} \]

\[
+ \sum_{j, \delta, \bar{\delta}, p, q} \phi^* \phi \langle j | O_{i, p} | k' \rangle \langle k | O_{i, q} | j' \rangle V_{i, p, q}(r_j, r_k) \delta_{(st)},
\]

\[
\langle \{ s_j, t_j \} | H | \{ s'_j, t'_j \} \rangle_E = \int (dr_j) \sum_{j, \delta, \bar{\delta}} \delta_{(st)} \left[ \frac{1}{2M} \partial_j \phi^* \partial_j \phi_{\{ jk \}} \langle j' | k' \rangle \langle k | j' \rangle \right]
\]

\[
+ \sum_{j, \delta, \bar{\delta}, p, q} \phi^* \phi_{\{ jk \}} \langle j | O_{i, p} | k' \rangle \langle k | O_{i, q} | j' \rangle V_{i, p, q}(r_j, r_k) \delta_{(st)} .
\]

\( \langle H \rangle_D \) and \( \langle H \rangle_E \) are the contributions from direct and exchange interaction, respectively. Their diagrammatic representations were given in Fig. 1. In (4·2), bra and ket vectors on the r.h.s. represent states of spin and isospin. For example, \( |j'\rangle \), an 'initial' spin- and isospin-state of the nucleon at \( R_j \) is a simplified notation of \( \chi_{s_j}(s_j) \xi_{t_j}(t_j) \). \( \delta_{(st)} \) and \( \delta_{(st)} \) are Kronecker's deltas concerning spin and isospin states defined by
Subscripts \( p \) and \( q \) denote the set of space indices concerning the operator \( O_i \). For example, for the operator \( \sigma \), they represent the suffix \( i (=1,2,3) \) which denotes the dimensional indices of \( O(3) \) space. For the operator \( \sigma \otimes \tau \), \( p \) and \( q \) represent sets of \( O(3) \) index \( i \) and \( SU(2) \) index \( \alpha (=1,2,3) \). \( \Phi_{[jk]} \) in the above expression has been defined by (3·4).

In usual variational methods, the diagonal elements of the Hamiltonian \( \langle H \rangle \) are minimized. In our method, however, we keep the off-diagonal matrix elements too to take the dynamics of spin and isospin into account. Seeking the (local) minima of the Hamiltonian by the variations of spin and isospin states reduces to the problem of diagonalizing \( \langle H \rangle \).

We first take the variation of \( \langle H \rangle \) in terms of \( \Phi \) under the condition (3·3b). This is achieved by adding to the Hamiltonian the following terms:

\[
-\lambda \int (\Pi dr_j) |\Phi|^2 - \sum_{j,k} \lambda_{jk} \int (\Pi dr_j) \Phi^{*} \Phi_{[jk]} .
\]

Before obtaining the equation of motion for \( \Phi \), it is convenient to express various matrix elements appeared in the Hamiltonian (4·2) in terms of Pauli’s matrices of spin and isospin, \( \sigma \) and \( \tau \) as well as the unit matrix. Note that \( \langle j|\cdots|j' \rangle \) is regarded as the matrix element of an operator pertinent to the nucleon at the \( j \)th lattice site. It can be expressed with a suffix \( j \) in matrix representation. We introduce the following notations,

\[
\langle j|j' \rangle = 1, \quad \langle j|\sigma|j' \rangle = \sigma_j, \quad \langle j|\tau|j' \rangle = \tau_j .
\]

\( \langle j|\cdots|k' \rangle \) is the matrix element between nucleons at different lattice sites. They can be expressed by using Heisenberg and Bartlett exchange operators: \( P_{\sigma;jk} = (1+\sigma_j \cdot \sigma_k)/2 \) and \( P_{\tau;jk} = (1+\tau_j \cdot \tau_k)/2 \). For example,

\[
\begin{align*}
\langle j|k' \rangle \cdot \langle k|j' \rangle &= P_{\sigma;jk} P_{\sigma;jk} , \\
\langle j|\sigma|k' \rangle \cdot \langle k|\sigma|j' \rangle &= \sigma_j \cdot \sigma_k P_{\sigma;jk} P_{\sigma;jk} , \\
\langle j|\tau|k' \rangle \cdot \langle k|\tau|j' \rangle &= \tau_j \cdot \tau_k P_{\tau;jk} P_{\tau;jk} .
\end{align*}
\]

Using these representations, the equation of motion for \( \Phi \) obtained by the variation of the Hamiltonian takes on the form

\[
-\frac{\lambda}{2M} \partial_j^2 \Phi + \frac{1}{2M} \partial_k^2 \Phi_{[jk]} P_{\sigma;jk} P_{\tau;jk} + \sum_{j,k} [\Phi - \Phi_{[jk]} P_{\sigma;jk} P_{\tau;jk}] V(r_j, r_k) \\
= \lambda \Phi + \sum_{j,k} \lambda_{jk} \Phi_{[jk]} ,
\]

\[
V(r_j, r_k) = V_{C1}(r_j, r_k) + \sigma_j \cdot \sigma_k V_{C2}(r_j, r_k) + \sigma_j \cdot \sigma_k \tau_j \cdot \tau_k V_{T}(r_j, r_k) ,
\]

\[
S_{j,k} = 3 \sigma_j \cdot \tau_j \sigma_k \cdot \tau_k - \sigma_j \cdot \sigma_k .
\]
In (4.7a), \( \lambda \)-term comes from the normalization condition for \( \Phi \), while \( \lambda_{jk} \)-terms come from the orthogonality condition. We may be able to write (4.7a) into more compact form

\[
H \Phi = \lambda \Phi .
\]

(4.8)

Then, we see that the problem of obtaining a stationary solution for \( \Phi \) reduces to the problem to diagonalize \( H \). This problem is not still so easy for not only the Hamiltonian \( H \) but also the 'eigenvalue' \( \lambda \) include the exchange operations of spatial coordinates as well as of spins and isospins. In dealing with this problem, we can envision the following steps: First, we solve (4.7a) by keeping \( \lambda \)-term only and obtain the wave function \( \Phi^{(0)} \) and the corresponding eigenvalue \( \lambda^{(0)} \). \( \Phi^{(0)} \) will be normalizable but will not satisfy the orthogonality condition. However, if \( \Phi^{(0)} \) describes the system of sharply localized nucleons, and the overlappings among nucleon's wave functions are small, then we can expect any exchange integrations appeared in (4.2b) and (4.2c) to be also small. In this case, we will be able to proceed to the second step: Let \( \delta \Phi \) be the correction to \( \Phi^{(0)} \) and insert \( \Phi^{(1)} = \Phi^{(0)} + \delta \Phi \) into (4.7a), keeping the \( \lambda_{jk} \)-term, too. Then, linearizing the equation, \( \delta \Phi \) will be obtained as a function of \( \lambda_{jk} \) and \( \delta \lambda \), the correction to \( \lambda^{(0)} \). The normalization and orthogonality conditions yield \( 1 + N(N-1)/2 \) equations for \( \Phi^{(1)} \), while we have \( 1 + N(N-1)/2 \) unknown parameters \( \delta \lambda \) and \( \lambda_{jk} \). Then, we will be able to get the set of eigenvalues to this order. We can repeat these steps to any orders. We assume that this procedure gives rise to a convergent result, and in the present study, we will restrict our arguments to the first step of this procedure only. Nonetheless, in a certain region of density, the result of our calculation will show a nice agreement with the previous work by PS based on the constrained variational method. We will discuss this in the next section.

\section{Phonon and stability of lattice}

First, by harmonic approximations, we analyze the eigenmodes of phonon to investigate the stability of simple cubic lattice, keeping spin and isospin as classical variables. We start with the classical ALS configuration of spins and isospins which were adopted by previous authors.\(^4,28,29,35\)

Now, we expand \( H \) appeared in (4.8) in terms of the power series of the deviation of spatial coordinate from the equilibrium positions. We redefine the spatial variables by shifting them: \( r_j \to r_j + R_j \). Similarly, we write the position of \( x \)th nearest-neighbor of \( j \)th nucleon as \( r_{jk} + R_{jk} \). Their relative deviation is then \( u_{jk} = r_{jk} - r_j \). Assuming that \( |u_{jk} + a_x| = a \) (\( a_x = R_{jk} - R_j \)) is small, the rotationally invariant potential functions \( V_i(u_{jk} + a_x), (i = C1, C2, C3, T) \) appeared on the r.h.s. of (2.5) are expanded up to second order of \( u_{jk} \) as

\[
V_i(u_{jk} + a_x) = V_i^{(0)} - \frac{V_i^{(1)}}{2} \left[ 2a_x \cdot u_{jk} + u_{jk}^2 + w_i(\vec{a}_x \cdot u_{jk})^2 \right],
\]

(5.1a)

\[
V_i^{(0)} = V_i(a), \quad V_i^{(1)} = -\frac{1}{a} \frac{dV_i(a)}{da}, \quad w_i = -\frac{1}{V_i^{(1)}} \frac{d^2V_i(a)}{da^2} - 2,
\]

(5.1b)
where $V_i$ are assumed to be functions of the relative distance only. $\vec{a}_k$ is the unit vector to the direction of $a_k$.

The tensor factor $S_{j,k} = S_{j,k}$, which was defined by (4.7c), is also subject to this expansion. It reads

$$S_{j,k} = 3\sigma_j \cdot \vec{a}_k \sigma_k \cdot \vec{a}_k - \sigma_j \cdot \sigma_k \cdot \vec{a}_k \cdot \vec{a}_k - \frac{3}{2} \left( \frac{2}{a} \frac{\vec{a}_k \cdot \vec{u}_{jk}}{a^2} u_{jk}^2 \right) \vec{a}_k \sigma_j \sigma_k \cdot \vec{u}_{jk}$$

$$+ \frac{3}{a} \left( 1 + \frac{2}{a} \frac{\vec{a}_k \cdot \vec{u}_{jk}}{a^2} \right) \vec{a}_k \cdot \vec{a}_k \sigma_j \sigma_k \cdot \vec{u}_{jk}$$

$$+ \frac{3}{a^2} \sigma_j \cdot \vec{u}_{jk} \sigma_k \cdot \vec{u}_{jk} \cdot \vec{u}_{jk}. \quad (5.2)$$

From (5.1) and (5.2), the tensor interaction part can be written as

$$\sum_{k=1}^{3} \tau_j \cdot \vec{u}_{jk} S_{j,k} V_{T}(\vec{u}_{jk} + a_k) = -4V_{T}(0) + \frac{V_{T}(1)}{2}(u_{j1}^2 + w_T(\vec{a}_1 \cdot \vec{u}_{j1})^2)$$

$$+ \frac{3V_{T}(0)}{a^2}(\vec{a}_3 \cdot \vec{u}_{j1})^2 + \frac{V_{T}(1)}{2}(u_{j2}^2 + w_T(\vec{a}_2 \cdot \vec{u}_{j1})^2)$$

$$+ \frac{3V_{T}(0)}{a^2}(\vec{a}_3 \cdot \vec{u}_{j2})^2 + \left( \frac{3V_{T}(0)}{a^2} + V_{T}(1) \right) u_{j3}^2$$

$$- 18 \frac{V_{T}(0)}{a^2} w_T(12V_{T}(1))(\vec{a}_3 \cdot \vec{u}_{j3})^2. \quad (5.3)$$

Similar expansions are used for other potentials.

Now, there are many possible sets of potentials which are compatible with the data of low energy nucleon-nucleon scattering. PS gave a precise result of their investigation about the sets they called potentials 2 and 3. In this paper, we also adopt the same sets of potentials for the sake of comparison of our calculation to other method. In units of MeV, they are given by

$$V_{c1} = 1.3A \frac{e^{-x_w}}{x_w}, \quad (5.4a)$$

$$V_{c2} = \frac{43}{3} \frac{e^{-x_2}}{x_2}, \quad (5.4b)$$

$$V_{c3} = 3.5 \frac{e^{-x}}{x} + 0.1A \frac{e^{-x_w}}{x_w}, \quad (5.4c)$$

$$V_{T} = \left\{ 3.5(1 - b^2)^2 + 4 \times \frac{2}{3} \times 7.55(1 - b^2)^{3/2} b + 4 \times \frac{4}{9} \times 16.2(1 - b^2) b^2 \right\}$$

$$\times \frac{e^{-x}}{x} \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \left( 1 - e^{-x} \right)^{2}, \quad (5.4d)$$

$$x = 0.7r, \ x_2 = 2.75r, \ x_w = 3.9r, \quad (5.4e)$$

$$A_2 = 5000 \text{ MeV}, \ B_2 = 1.8 \text{ fm}^{-2}; \ A_3 = 10000 \text{ MeV}, \ B_3 = 2.6 \text{ fm}^{-2}. \quad (5.4f)$$
\( r \) is the distance between two nucleons measured in units of fm. \( 1 - b^2 \) and \( b^2 \) represent the numbers of neutrons and \( \Delta \)-isobars per nucleon, respectively. PS obtained the expression for \( V_T \) by replacing the effect of pion condensation by the appearance of \( \Delta \)-isobars assuming \( SU(3) \) symmetry for meson-baryon interactions.

Before we go into the quantum mechanics of lattice vibrations, we need to contrive to handle the exchanged wave function \( \Phi_{(jk)} \). If it were not for lattice vibrations, \( \Phi_{(jk)} \) would be given by the value of \( j \)th nucleon wave function at the position of \( k \)th lattice site. Therefore, it will be appropriate to expand \( \Phi_{(jk)} \) in terms of \( u_{jk} \). At the order of our approximation in which the power series expansion in terms of spatial deviation is truncated at the quadratic terms, the single nucleon wave function will have a Gaussian form:

\[
\Phi_j(r_j + R_j) \approx \exp \left( -\frac{1}{2} \nu r_j^2 \right). \tag{5.5}
\]

Strictly speaking, \( \nu \) will have a dependence on the direction of \( r_j \). But we neglect it for simplicity. Then, interchanging the coordinates of two nucleons will result in the change in the form of wave function into

\[
\Phi_j(r_k + R_k) \Phi_k(r_j + R_j) \approx \exp \left[ -\frac{1}{2} \nu ((r_k + R_k - R_j)^2 + (r_j + R_j - R_k)^2) \right], \tag{5.6a}
\]

\( (k=j) \)

or

\[
\Phi_{(jk)} \approx \exp \left( -\nu a^2 - \nu a \cdot u_{jk} \right) \Phi \approx \exp \left( -\nu a^2 \right) \left[ 1 - \nu a \cdot u_{jk} + \frac{\nu^2}{2} (a \cdot u_{jk})^2 \right] \Phi. \tag{5.6b}
\]

Similarly, the kinetic part will be written as

\[
\partial_j^2 \Phi_{(jk)} \approx \exp \left( -\nu a^2 \right) \left[ \nu^2 a^2 + \partial_j^2 - 2\nu a \cdot u_{jk} a \cdot \partial_j + \frac{\nu^2}{2} (a \cdot u_{jk})^2 \right] \Phi. \tag{5.7}
\]

Putting these all together, we have the eigenvalue equation (4.8) with the harmonic approximation for potentials.

In order to solve this equation, it is convenient to transfer to another representation of coordinates which preserves the algebra of the operators \( \partial_j \) and \( r_j \):

\[
r_j = \frac{1}{\sqrt{N}} \sum_k \frac{1}{\sqrt{2\omega}} \left[ c(k) e^{i k \cdot r_j} + c^\dagger(k) e^{-i k \cdot r_j} \right], \tag{5.8a}
\]

\[
\partial_j = \frac{1}{\sqrt{N}} \sum_k \sqrt{\frac{\omega}{2}} \left[ c(k) e^{i k \cdot r_j} - c^\dagger(k) e^{-i k \cdot r_j} \right], \tag{5.8b}
\]

\[
[c_i(k), c^\dagger_j(k')] = \delta_{ij}, \delta_{kk'}, \quad [c_i(k), c_r(k')] = [c_i^\dagger(k), c_r^\dagger(k')] = 0, \tag{5.8c}
\]

where \( c_i(k) \) is the \( i \)th component of \( c(k) \).

\( \omega \) is a parameter which is to be determined by the equation of motion for \( c \) and \( c^\dagger \). Then, the Hamiltonian \( H^{ph} \) in the eigenvalue equation \( H^{ph} \Phi = \lambda \Phi \) takes the form
\[H^{(0)} = \frac{1}{2M} \sum_k \omega \left[ 2c^\dagger(k)c(k) - c(k)c(-k) - c^\dagger(k)c^\dagger(-k) + 1 \right] \]
\[+ \frac{V_T^{(0)}}{\alpha^2} \sum_k \Gamma_i \left[ 2c^\dagger(k)c(k) + c(k)c(-k) + c^\dagger(k)c^\dagger(-k) + 1 \right] \]
\[+ NV^{(0)}, \quad (5.9)\]

where \(M = M(1 - 2\eta)\), \(\eta = \exp(-\nu a^2)\), and

\[\Gamma_i = \frac{\alpha_0}{\sqrt{\nu}} \gamma_\alpha, \quad \gamma_\alpha(k) = 1 - \cos(k \cdot \alpha), \quad \Gamma_{\alpha} = \Gamma_{\alpha}(\alpha) = \eta \Gamma_{\alpha}(\alpha), \quad (5.10a)\]

\[\Gamma_{11} = \frac{\alpha^2}{2 V_T^{(0)}} \left[ V_T^{(1)}(1 + w_T) - V_{\alpha}^{(1)}(1 + w_{c1}) - V_{\beta}^{(1)}(1 + w_{c2}) - V_{\gamma}^{(1)}(1 + w_{c3}) \right], \quad (5.10b)\]

\[\Gamma_{22} = \frac{\alpha^2}{2 V_T^{(0)}} \left[ V_{\alpha}^{(1)} - V_{\beta}^{(1)} - V_{\gamma}^{(1)} \right], \quad (5.10c)\]

\[\Gamma_{33} = \frac{\alpha^2}{2 V_T^{(0)}} \left[ 2 V_T^{(1)} + V_{\alpha}^{(1)} + V_{\beta}^{(1)} + V_{\gamma}^{(1)} \right] + 3, \quad (5.10d)\]

\[\Gamma_{22} = \Gamma_{11}, \quad \Gamma_{22} = \Gamma_{33} = \Gamma_{22}, \quad (5.10e)\]

\[\Gamma_{33} = \Gamma_{33} = \frac{\alpha^2}{2 V_T^{(0)}} \left[ V_T^{(1)} - V_{\alpha}^{(1)} - V_{\beta}^{(1)} - V_{\gamma}^{(1)} \right] + 3, \quad (5.10f)\]

\[\Gamma_{33} = \frac{\alpha^2}{2 V_T^{(0)}} \left[ V_T^{(1)}(2 W_T + 26) + V_{\alpha}^{(1)}(1 + w_{c1}) + V_{\beta}^{(1)}(1 + w_{c2}) + V_{\gamma}^{(1)}(1 + w_{c3}) \right] - 15, \quad (5.10g)\]

\[\Gamma_{11} = \frac{\alpha^2}{V_T^{(0)}} \left[ - \frac{V_T^{(1)}}{2}(1 + w_T) + V_{\alpha}^{(1)}(1 + w_{c1}) + V_{\beta}^{(1)}(1 + w_{c2}) + V_{\gamma}^{(1)}(1 + w_{c3}) \right] \]
\[+ \nu^2 a^4 - \frac{\nu^2 \alpha^4}{V_T^{(0)}} \left( V_{\alpha}^{(0)} + V_{\beta}^{(0)} + V_{\gamma}^{(0)} \right), \quad (5.10h)\]

\[\Gamma_{22} = \frac{\alpha^2}{V_T^{(0)}} \left[ - \frac{V_T^{(1)}}{2} + V_{\alpha}^{(1)} + V_{\beta}^{(1)} + V_{\gamma}^{(1)} \right], \quad (5.10i)\]

\[\Gamma_{33} = \frac{\alpha^2}{V_T^{(0)}} \left[ - V_T^{(1)} + 2 V_{\alpha}^{(1)} + 2 V_{\beta}^{(1)} \right] - 3, \quad (5.10j)\]

\[\Gamma_{11} = \Gamma_{11}, \quad \Gamma_{22} = \Gamma_{22}, \quad \Gamma_{33} = \Gamma_{33}, \quad (5.10k)\]

\[\Gamma_{33} = \frac{\alpha^2}{V_T^{(0)}} \left[ - V_T^{(1)}(w_T + 13) + 2 V_{\alpha}^{(1)}(1 + w_{c1}) + 2 V_{\beta}^{(1)}(1 + w_{c2}) \right] \]
\[+ 15 + 2\nu^2 a^4 - \frac{\nu^2 \alpha^4}{V_T^{(0)}} \left( V_{\alpha}^{(0)} + V_{\beta}^{(0)} \right), \quad (5.10l)\]
Then, the Hamiltonian (5.9) is diagonalized as

\begin{align}
\mathbb{H}^{ph}_{\text{diag}} &= \sum_{\mathbf{k}, \mathbf{i}} \omega(\mathbf{k}, i) \left[ c_i^\dagger(\mathbf{k}) c_i(\mathbf{k}) + \frac{1}{2} \right] + NV^{(0)}, \\
\omega(\mathbf{k}, i) &= \left( \frac{4 V_T^{(0)} \Gamma_i}{M a^2} \right)^{1/2}, \\
\bar{c}_i(\mathbf{k}) &= \alpha(\mathbf{k}, i) c_i(\mathbf{k}) - \beta(\mathbf{k}, i) c_i^\dagger(-\mathbf{k}), \\
\alpha(\mathbf{k}, i) &= \frac{1}{\sqrt{2}} \left( \frac{1}{\omega(\mathbf{k}, i)} \left( \frac{\tilde{\omega}}{2M} + \frac{2 V_T^{(0)} \Gamma_i}{\tilde{\omega} a^2} \right) + 1 \right)^{1/2}, \\
\beta(\mathbf{k}, i) &= \frac{1}{\sqrt{2}} \left( \frac{1}{\omega(\mathbf{k}, i)} \left( \frac{\tilde{\omega}}{2M} + \frac{2 V_T^{(0)} \Gamma_i}{\tilde{\omega} a^2} \right) - 1 \right)^{1/2}.
\end{align}

We require that \( r_j \) and its conjugate momentum \( i M [\mathbb{H}^{ph}, r_j] \) also satisfy the canonical commutation relation. Then, we have \( \tilde{\omega} = \omega(\mathbf{k}, i) = M\omega(\mathbf{k}, i) \). \( \bar{c}_i(\mathbf{k}) \) and \( \bar{c}_i^\dagger(\mathbf{k}) \) are the annihilation and creation operator of a phonon of wave vector \( \mathbf{k} \). In order for a stable solution to exist, none of \( \Gamma_i \) can be negative. The ground state \(|0\rangle\) for phonon is defined by

\begin{equation}
\bar{c}_i(\mathbf{k}) |0\rangle = 0
\end{equation}

or

\begin{equation}
|0\rangle = A \exp \left[ -\sum_{\mathbf{k}} \frac{\beta(\mathbf{k}, i)}{\alpha(\mathbf{k}, i)} c_i^\dagger(\mathbf{k}) c_i^\dagger(-\mathbf{k}) \right] |0\rangle,
\end{equation}

where \(|0\rangle\) is the vacuum for the old operators \( c_i(\mathbf{k}) \), and \( A \) is the normalization factor.

Using this solution, we can calculate the mean square of the spatial deviation of a nucleon from the equilibrium point.

\begin{equation}
\langle r_i^2 \rangle = \langle 0 | r_i^2 | 0 \rangle = \frac{1}{4N} \sum_{\mathbf{k}} \frac{a}{\sqrt{MV_T^{(0)} \Gamma_i}}.
\end{equation}

On the other hand, from our ansatz for the functional form of the wave function, \( \langle r_i^2 \rangle \) must be equal to 1/2\( \nu \). As a matter of fact, \( \langle r_i^2 \rangle \) has a direction dependence because of the anisotropy of the nuclear force in the solid. Therefore, we simply require that 1/2\( \nu \) must be equal to the average of \( \langle r_i^2 \rangle \):

\begin{equation}
\frac{1}{2\nu} = \frac{1}{3} \sum_{z=1}^{3} \langle r_i^2 \rangle.
\end{equation}

Solving this self-consistency condition, we get the energy of the system for any nucleon density. Since the presence of \( \Delta \) isobars changes the form of kinetic term, in the actual calculations, the neutron mass \( M \) was replaced by the effective mass \( M_{\text{eff}} \) which is given by

\begin{equation}
\frac{1}{M_{\text{eff}}} = \frac{1}{M} + \frac{b^2}{M_\Delta}.
\end{equation}

For a given \( \nu \), (5.15) gives a continuous set of lattice constant \( a \) and \( \Delta \)-concentration
Fig. 2. Energy per nucleon (the neutron rest mass is subtracted). \( \triangle \) (potential 2) and \( \square \) (potential 3) are the present calculations. Statistical errors, which are due to Monte Carlo integration in the three dimensional wavevector space, are smaller than 5\%. \( \triangle \) (potential 2) and \( \square \) (potential 3) are of PS's calculations. Some values of \( b^2 \) are given in parentheses.

For potential 2, however, the agreement with PS's result may be fictitious because \( b \)-values, and in turn, the forms of potentials are different. We have not been able to find the origin of this discrepancy.

The functional form of neutron wave function determined above can be used to obtain the information about the neutral pion field. In the approximation used in this section, the neutral pion source function is expressed as

\[
S(r) = \langle \psi^*(r) \sigma \psi(r) \rangle = \sum_j S_j \rho(r-R_j),
\]

\[
S_j = (0,0,\pm 1), \quad \rho(r-R_j) = \left( \frac{\nu}{\pi} \right)^{3/2} e^{-\nu(r-R_j)^2}.
\]

In (5·17b), \( \pm 1 \) means that it takes the value +1 or −1 in accordance with the classical value of the \( z \)-component of \( j \)th neutron's spin (+1/2 or −1/2). Then, the classical pion field is expressed as

\[
\phi_{1,2} = 0, \quad \phi_3(r) = -\frac{f}{\mu} \int dr' D(r-r') \sigma \cdot S(r'),
\]

where \( D(r-r') \) is the Green's function (2·4). The energy \( E_\pi \) of pion field is given by the spatial integration of the second term in (2·2). The result is

\[
E_\pi = \frac{N}{2} \left( \frac{f}{\mu} \right)^2 \left[ \frac{1}{3} \left( \frac{\nu}{\pi} \right)^{3/2} [1 + O(\mu^2/\nu)] + O(\nu \alpha^2 \rho(\alpha_x) + \mu^2 D(\alpha_x)] \right].
\]

The energy per nucleon \( E_\pi/N \) given by the leading term in (5·19) together with the increment \( \Delta M = M_{\text{eff}} - M \) of the effective neutron mass is also presented in Table I.
Table I. Density dependences of $\pi^0$ energy and $\Delta M$ for potential 3. Both quantities are shown in units of fm$^{-1}$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\Delta M$</th>
<th>$E_{\pi}/N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.266</td>
<td>0.17</td>
<td>0.097</td>
</tr>
<tr>
<td>0.364</td>
<td>0.17</td>
<td>0.19</td>
</tr>
<tr>
<td>0.466</td>
<td>0.22</td>
<td>0.30</td>
</tr>
<tr>
<td>0.547</td>
<td>0.40</td>
<td>0.43</td>
</tr>
</tbody>
</table>

The increment of $E_{\pi}/N$ parallels the increment of $\Delta M$. Thus, we may confirm that the effect of classical pion field is qualitatively incorporated as a parameter describing the potentials as long as the energy of pion is concerned.

Our treatment of potential model will fail to yield a correct answer as the density gets larger, that is, in such a region of density, the harmonic oscillator approximation of nucleon interaction is absolutely invalid as expected. Beyond the transcendental density, we have to deal with the repulsive core of nuclear potential explicitly, which is out of scope of this paper. What we learned in this section is that (1) PS-potentials allow solid phase of nuclear matter which is at least locally stable against lattice distortions. (2) For a certain class of PS-potentials, there should be density regions in which the results of calculations based on PS's constrained variational method can be understood from the viewpoint of the dynamics of phonon. Then, (3) the microscopic description of nuclear solid in terms of other elementary excitations will be indispensable if the solid state exists. In the following sections, we investigate other excitation modes which are expected to be important in the solid nuclear matter.

§ 6. Spin- and isospin-waves

The spin-wave theory of magnetic system shows that the dynamical systems of quantum spins manifest some peculiarities which classical systems do not possess. Furthermore, in systems with lower spins, quantum effects can be regarded as becoming larger because fluctuations of spin state will be larger. The nature of solid is thus sometimes strongly affected by the quantum spin dynamics.

Since the nuclear force is strongly dependent to spin and isospin states of nucleons, the above situation is also expected to be the case for the solid nuclear matter. This is manifested in (4·7) and (4·8). There, nearest-neighbor interactions among quantum spins and quantum isospins appear explicitly. Apart from spin-isospin coupling terms, it is nothing but the $S=1/2$ quantum Heisenberg model which is familiar in solid state physics.\cite{30}

In this section, we study the effect of the spin dynamics by shedding a light on its quantum nature from the viewpoint of the phenomenological spin-wave theory. Except for the one-dimensional spin chain, the dynamical nature of the quantum Heisenberg model has not been satisfactorily clarified yet. However, the spin-wave theory describes well the physics of spin systems when the number of elementary excitations is not large. Since ferromagnetic system is easier to handle, we begin our discussion for the case of isospin-wave.

Isospin-waves

We assume that all nucleons are at their equilibrium positions and that the exact classical ALS is formed. Then, apart from an additional constant term, the
Hamiltonian for the isospin extracted from (4·7) and (4·8) takes on the form

$$H^r = \sum_\alpha \sum_\beta J_{\alpha \beta}^{(r)} \tau_\alpha \cdot \tau_\beta - E_0^{r,\text{cl}},$$  \quad (6·1)$$

where the energy $E_0^{r,\text{cl}}$, due to the classical configuration of spins and isospins, must be subtracted since they have already been included in the energy calculated in § 5. Coupling constants are given by

$$J_1^{(0)} = J_2^{(0)} = V_T^{(0)} - V_{\text{ex}}^{(0)} + \frac{e^{-1/a}}{2M} \left( V_{T1}^{(0)} + V_{T2}^{(0)} \right),$$  \quad (6·2a)$$

$$J_3^{(0)} = 2V_T^{(0)} + V_{\text{ex}}^{(0)}. $$  \quad (6·2b)$$

In deriving (6·1), we used the identity

$$P_{\tau_1 \tau_2} \tau_1 \cdot \tau_2 = \frac{1}{2} (3 - \tau_1 \cdot \tau_2).$$  \quad (6·3)$$

Exchange diagrams do not contribute to $J_3^{(0)}$ to this order because the classical value of the exchange operator $P_{\tau_1 \tau_2}$ for $k = j_2$ vanishes for ALS. The values of $J_\alpha^{(0)}$, together with $J_\alpha$, the values corrected by zero-point lattice vibrations are given in Table II. The rate of the contribution of the exchange part is less than 1% and was neglected.

The dynamics of isospin is thus described by the unisotropic $S=1/2$ 'ferromagnetic' Heisenberg Hamiltonian. Its quantum nature is conventionally described by the quantum spin-wave theory. Our Hamiltonian can be transformed to the convenient form by invoking the Holstein-Primakoff (HP) transformation.\(^{32,30}\) Assuming that nuclear matter at high density is dominated by neutrons ($\tau_3 = -1$), HP transformation for isospin variables is defined by

$$\tau_1^+ = \tau_{1,1} + i\tau_{1,2} = 2d_j^1 (1 - d_j^1 d_j^1)^{1/2}, \quad \tau_1^- = \tau_{1,1} - i\tau_{1,2} = 2(1 - d_j^1 d_j^1)^{1/2} d_j,$$

$$\tau_3 = -1 + 2d_j^1 d_j. $$  \quad (6·4)$$

This representation fulfills $SU(2)$ algebra when $d_j$ and $d_j^1$ obey the Bose commutation relation. See the Appendix for details.

The physically meaningful states have the eigenvalues 0 or 1 for the number operators $d_j^1 d_j$. Therefore, the following isomorphism holds:\(^{36,37}\)

$$\tau_3^+ \approx 2d_j^1 (1 - d_j^1 d_j), \quad \tau_3^- \approx 2(1 - d_j^1 d_j) d_j,$$

\quad (6·5a)$$

Table II. Isospin-isospin couplings $(J^{(0)}, J)$ and spin-spin couplings $(K^{(0)}, K)$ for potential 3. The unit is MeV. Superscript \(^{(0)}\) denotes the values before the mean-field corrections. The unit is MeV/nucleon.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$J_1^{(0)}$</th>
<th>$J_3^{(0)}$</th>
<th>$J_{1,2}$</th>
<th>$J_5$</th>
<th>$K_1^{(0)}$</th>
<th>$K_2^{(0)}$</th>
<th>$K_3$</th>
<th>$K_4$</th>
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<td>58.5</td>
<td>52.9</td>
<td>52.3</td>
</tr>
<tr>
<td>0.304</td>
<td>31.0</td>
<td>67.3</td>
<td>31.3</td>
<td>21.2</td>
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<td>97.5</td>
<td>69.8</td>
<td>89.6</td>
</tr>
<tr>
<td>0.466</td>
<td>43.6</td>
<td>95.2</td>
<td>44.4</td>
<td>41.1</td>
<td>42.8</td>
<td>138</td>
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<tr>
<td>0.547</td>
<td>61.9</td>
<td>134</td>
<td>61.0</td>
<td>65.0</td>
<td>60.9</td>
<td>196</td>
<td>122</td>
<td>184</td>
</tr>
</tbody>
</table>

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Elementary Excitations in a Solid Nuclear Matter

For the representation (6·5b), the Hamiltonian expressed in terms of the Fourier components of $d_j$ and $d_j^*$ (i.e., creation and annihilation operators of isomagnon at $j$th lattice site) is written as

$$H_0 = \sum_k e^{(0)}(k)n_d(k) \frac{4}{N_{\text{site}}} \sum_{k',k''} \epsilon^{(0)}(k) \cos((\mathbf{k} - \mathbf{k}'') \cdot \mathbf{a}_x) - \cos((\mathbf{k} + \mathbf{k}' - \mathbf{k}'') \cdot \mathbf{a}_x)$$

$$\times d_j^*(k) d_j^*(k') d(k'') d(k + k' - k''),$$

$$n_d(k) = d_j^*(k) d(k),$$

$$\epsilon^{(0)}(k) = 4 \sum_{\xi} J^{(0)}(1 - \cos(k \cdot \mathbf{a}_x)).$$

The summation in $\kappa$ is taken over three non-parallel vectors connecting nearest-neighbors. We have the gapless dispersion relation familiar in the ferromagnetic Heisenberg model. The dispersion relation of one isomagnon is exactly given by (6·6c).

Spin-waves

In a similar way, the Hamiltonian for magnons which do not interact with phonons and isomagnons is given by

$$H_0^\sigma = -K_1^{(0)} \sum_j \sum_{\xi} \sigma_j \cdot \sigma_{j+\xi} + K_2^{(0)} \sum_j (\sigma_{j,x} \sigma_{j+1,x} + \sigma_{j,y} \sigma_{j+1,y} + \sigma_{j,z} \sigma_{j+1,z}) - E_0^{\sigma,cl},$$

where, the energy $E_0^{\sigma,cl}$ due to the classical spins and isospins has been subtracted. $K_1^{(0)}$ and $K_2^{(0)}$ are given by

$$K_1^{(0)} = V_T^{(0)} - V_{\xi_2}^{(0)} - V_{\xi_3}^{(0)} - \frac{\varepsilon^{(0)}}{2} \left( \frac{v^2 a^2}{2M} + 2V_T^{(0)} + V_{\xi_2}^{(0)} + V_{\xi_3}^{(0)} \right),$$

$$K_2^{(0)} = 3(1 - e^{-\nu^2}) V_T^{(0)}.$$
romagnetic system. We introduce Fourier coefficients by
\[ a_j = \frac{2}{N} \sum_{k} a(k) e^{i k \cdot R_j}, \quad b_j = \frac{2}{N} \sum_{k} b(k) e^{i k \cdot R_j}, \]  
(6·10)
where the \( k \)'s are in the first Brillouin zone of the reciprocal lattice of each sublattice. Then, we have the lowest order Hamiltonian
\[ H_{0}\sigma = \sum_{k} \left[ F(k) (a^\dagger(k)a(k) + b^\dagger(k)b(k)) + \frac{G(k)}{2} (a(k)a(-k) + a^\dagger(k)a^\dagger(-k) + b(k)b(-k) + b^\dagger(k)b^\dagger(-k)) \right. \]
\[ \left. - H(k)(a(k)b(-k) + a^\dagger(k)b^\dagger(-k)) \right], \]
(6·11a)
where
\[ F(k) = \sum_{x=1}^{2} [4K_{1}(1 - \cos(k \cdot a_{x}) + 2K_{2}(1 + \cos(k \cdot a_{x})) - 4K_{1}], \]
(6·11b)
\[ G(k) = 2K_{2}(\cos(k \cdot a_{1}) - \cos(k \cdot a_{3})) , \]
(6·11c)
\[ H(k) = 4K_{1}\cos(k \cdot a_{3}). \]
(6·11d)
To diagonalize (6·11a), we first introduce operators \( a(k) \), \( \beta(k) \) and their hermitian conjugates defined by
\[ a(k) = ua(k) + va^\dagger(-k), \quad b(k) = u\beta(k) + v\beta^\dagger(-k), \]
(6·12a)
\[ u = \frac{1}{\sqrt{2}} \frac{(F(k) + \sqrt{F(k)^2 - G(k)^2})^{1/2}}{G(k)}, \]
(6·12b)
\[ v = \frac{1}{\sqrt{2}} \frac{G(k)}{(F(k)^2 - G(k)^2)^{1/2}(F(k) + \sqrt{F(k)^2 - G(k)^2})^{1/2}}. \]
(6·12c)
This transformation diagonalizes the part of the Hamiltonian involving \( F(k) \) and \( G(k) \) term. Using \( a(k) \), \( \beta(k) \), etc., the diagonalized form of \( H_{0}\sigma \) reads
\[ H_{0}\sigma = \sum_{k} \left[ \varepsilon_{i}(k) \bar{a}^\dagger(k) \bar{a}(k) + \varepsilon_{2}(k) \bar{b}^\dagger(k) \bar{b}(k) \right] - \varepsilon_{0 ms}, \]
(6·13a)
\[ \varepsilon_{0 ms} = \sum_{k} \left\{ F(k) - \frac{1}{2}(\varepsilon_{1}(k) + \varepsilon_{2}(k)) \right\}, \]
(6·13b)
\[ \varepsilon_{1}(k) = \sqrt{F(k)^2 - (G(k) - H(k))^2}, \]
(6·13c)
\[ \varepsilon_{2}(k) = \sqrt{F(k)^2 - (G(k) + H(k))^2}, \]
(6·13c)
\[ a(k) = p\bar{a}(k) + q\bar{a}^\dagger(-k) + r\bar{b}(k) + s\bar{b}^\dagger(-k), \]
(6·13d)
\[ \beta(k) = -p\bar{a}(k) - q\bar{a}^\dagger(-k) + r\bar{b}(k) + s\bar{b}^\dagger(-k), \]
(6·13e)
For the dispersion relation, we have two branches, both of which have an energy gap (value at $k=0$)

$$\Delta_s = 4\sqrt{K_2(K_2-K_1)}.$$  

(6.14)

According to Table II, this energy gap takes values of order 100 MeV. At low temperatures, therefore, spin-wave excitations will have less importance in determining the nature of ALS nuclear matter. This is a natural result since $H^x$ in (6.7) explicitly breaks rotational symmetry. However, we may have to take the zero-point energy $\epsilon_{0,ms}$ in (6.13a) into account as the most important contribution to the ground state energy. It is given in Table III for each density. At $\rho=0.266$ fm$^{-3}$, the energy eigenvalues develop imaginary parts in some region of wavevector. The solid of ALS phase will become unstable near this density. At higher densities investigated, the zero-point energy due to the spin fluctuations turns out to lower the energy of the system by the amount of $O(100$ MeV) per nucleon. This feature is peculiar to the three-dimensional crystalline structure which has an antiferromagnetic spin ordering.

In passing, we note that the expectation value of $\sigma_z$ in one sublattice changes from its saturated value. For the spin in the sublattice $A$, for instance, one has

$$\sigma_{j,z} = 1 - 2a_j \alpha_j$$

$$= 1 - \frac{4}{N} \sum_k \{(u(k)q(k) + v(k)p(k))^2 + (u(k)s(k) + v(k)r(k))^2\}$$

$$- \frac{4}{N} \sum_k (\vec{a}, \vec{a}^+, \vec{b}, \vec{b}^+); \quad (6.15)$$

The last term has a normal ordered bilinear form of the operators $\vec{a}(k), \vec{a}^+(k)$, etc. In deriving (6.15) we used (6.12) and (6.13). The results of numerical calculations are presented in Table III, which shows the deviations from the classical value ($=1$). Accordingly, the effective $\pi NN$ coupling constant will be reduced by certain fractions. In §7, this fact will be used to evaluate the effect of pion-condensation to the ground state energy.

<table>
<thead>
<tr>
<th>$\rho$ (fm$^{-3}$)</th>
<th>0.266</th>
<th>0.364</th>
<th>0.466</th>
<th>0.584</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_{0,ms}$</td>
<td>67.2</td>
<td>88.5</td>
<td>122</td>
<td></td>
</tr>
<tr>
<td>$\langle \sigma_{j,z} \rangle$</td>
<td>0.950</td>
<td>0.957</td>
<td>0.960</td>
<td></td>
</tr>
</tbody>
</table>
§ 7. Effect of charged particles

Several authors\(^{38}\) have noted that the attractive \( P \)-wave interaction in pion-nucleon system will trigger the condensation of charged pions with finite momenta in the liquid phase. In this section, we investigate the typical effects of isomagnons on pion-condensations in a solid phase following the conventional spin-wave theory. In § 6, we have seen that the spin-wave has a large energy gap. We take the quantum mechanical effects of spins into account by incorporating the zero-point energy and the zero-point fluctuation of \(<\sigma_{j,z}\rangle\) in our calculations given below. No electrons are assumed to be present. The matrix elements of the Hamiltonian (2·2) in terms of the nucleonic state (3·2) are expressed as

\[
H_{NN} = \int (\prod_j dr_j) \sum_j \left[ \frac{1}{2M} \phi^* \xi_j \phi + \frac{e}{\mu} \phi^* \phi \sigma_j \cdot \partial_j (\phi^* (r_j) \xi_j^- + \phi (r_j) \xi_j^+) + \phi_0 (r_j) \xi_j \right] \\
+ \int dr \left[ |\phi|^2 + |\partial \phi|^2 + \mu^2 |\phi|^2 + \frac{1}{2} (\phi_0^2 + (\partial \phi_0)^2 + \mu^2 \phi_0^2) \right].
\]  
(7·1)

The Gaussian form (5·5) is assumed for the wave function \( \Phi \). The exchange terms have suppression factors of the order of \( \exp(-v\alpha^2) \), which takes the values of order \( 10^{-3} \) and are neglected.

\( \phi \) and \( \phi_0 \) are the charged and neutral pion field, respectively. They are quantized as

\[
\phi(r) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left[ a(k) e^{i k \cdot r} + b^\dagger(k) e^{-i k \cdot r} \right], \quad \omega_\pi = \omega_\pi (k) = \sqrt{k^2 + \mu^2}, \quad (7·2a)
\]

\[
\phi_0(r) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left[ c(k) e^{i k \cdot r} + c^\dagger(k) e^{-i k \cdot r} \right], \quad (7·2b)
\]

\[ [a(k), a^\dagger(k')] = [b(k), b^\dagger(k')] = \delta_{k,k'}, \quad [a(k), a(k')] = [b(k), b(k')] = 0, \text{ etc.} \quad (7·2c)\]

(In this section, the notations \( a(k), b(k) \) and \( c(k) \) are used for the annihilation operators of \( \pi^\pm \) and \( \pi^0 \), respectively.) For classical spins, we set

\[
\sigma_{j,x} = \sigma_{j,y} = 0, \quad \sigma_{j,z} = \pm \vec{\xi} = \xi \exp(i\Delta \cdot \vec{R}_j), \quad \Delta \equiv (0, 0, \Delta), \quad \Delta = \pi/\alpha, \quad (7·3)
\]

where \( \vec{\xi} = <\sigma_{j,z}> \) is the expectation value of \( \sigma_{j,z} \) in the sublattice \( A \) (see Table III). The position vector \( \vec{R}_j \) is given by (3·1).

Inserting these expressions together with (6·4) and (6·5a) for isospin operators into (7·1), one obtains up to third order of the operator \( d(k) \) and \( d^\dagger(k) \),

\[
H_{NN} = \frac{3\nu}{4M} N + \sum_k \omega_\pi (k) [a^\dagger(k) a(k) + b^\dagger(k) b(k) + c^\dagger(k) c(k)] \\
- 2i \frac{\xi f}{\mu} \sqrt{\frac{N}{V}} \sum_k \frac{k_x}{\sqrt{2\omega_\pi(k)}} \left[ (a^\dagger(k) + b(-k)) (d(k) + \Delta) \right.
\]
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\[-\frac{1}{Nk',k''} d^*(k') d(k'')d(k' - k'' + \Delta)\]

-(a(k) + b^†(-k))(d^†(k + \Delta)

\[-\frac{1}{Nk',k''} d^*(k' + \Delta) d(k'') d(k' + k'')\]

+ \frac{i \xi_f}{\mu} \frac{N}{\sqrt{V}} \frac{\Delta}{\sqrt{2 \omega_\pi(\Delta)}} [c(\Delta) - c^†(\Delta)]

- \frac{2i \xi_f}{\mu} \frac{1}{\sqrt{V}} \sum \frac{k_z}{k' k'' \sqrt{2 \omega_\pi(k)}} [c(k) d^†(k') d(k' - k + \Delta)

- c^†(k)d^†(k')d(k' + k - \Delta)] ,

\[\omega_\pi(k) = \omega_\pi(k) e^{kz/\alpha} .\] (7·4)

The electric charge operator is

\[Q = \sum_k [a^†(k)a(k) + d^†(k)d(k) - b^†(k)b(k)] .\] (7·5)

We choose as the trial state the simplest one:

\[|\psi\rangle \propto \exp[\sqrt{N}(iAA^†(\Delta - \Delta) + iBB^†(\Delta - \Delta) + iCC^†(\Delta + DD^†(\Delta)))] |0\rangle .\] (7·6)

A, B, C and D are assumed to be real numbers. \(K = (0, 0, K)\) and \(K\) is also subject to variations. (7·6) means that spatial dependences of classical pion fields are supposed to be exp\[-i(\Delta - K)z\], exp\[i(\Delta - K)z\] and sin\((\Delta z)\) for \(\pi^+, \pi^-\) and \(\pi^0\) respectively. We do not give the explicit form of time-dependences here. In order to see them, we have to evaluate the time-dependences of the expectation values of Heisenberg fields \(\exp[-i(H + \mu Q)t]/\phi(r) \exp[-i(H + \mu Q)t]\), etc., where \(\mu\) is the chemical potential of \(\pi^0\), which seems quite a complicated task. Following the spin-wave theory, isomagnons are treated as Bosons in (7·6), so that an infinite number of unphysical states are involved. (7·6) can be a better approximation for the ground state when \(D^2\), the number of isomagnons per nucleon, is sufficiently smaller than unity. For this trial state, the expectation value of the effective Hamiltonian may be expressed as

\[\langle H' \rangle /N = \frac{3\nu}{4M} - \epsilon_0 + V_{bc}(a)

+ \omega_\pi(\Delta)(g(K)(A^2 + B^2) + C^2 - 4h(K)(A - B)(D - D^2)

- 2h(0)C(1 - D^2) + \lambda(A^2 - B^2 + D^2))

\equiv E + \lambda \omega_\pi(\Delta)(A^2 - B^2 + D^2) ,\] (7·7a)

\[g(K) \equiv \frac{\omega_\pi(\Delta - K)}{\omega_\pi(\Delta)} , \quad h(K) \equiv \frac{\xi_f}{\mu \omega_\pi(\Delta)} \frac{\Delta - K}{\sqrt{2a^2 \omega_\pi(\Delta - K)} .\] (7·7b)
\( \varepsilon_{\text{zero-point}} \), which has been calculated in § 6, is the energy shift due to the zero-point fluctuations of spins. We have also added to \( H_{\text{NN}} \) the contribution of the repulsive core \( V_{\text{RC}}(\alpha) \). It is conventionally assumed to be given by (5·4) which the one pion exchange potential (2·3) is subtracted from.

The variational principle implies the equation of motion of the following form:

\[
\begin{align*}
(g(K)+\lambda)A-2h(K)(D-D^3) &= 0, \\
(g(K)-\lambda)B+2h(K)(D-D^3) &= 0, \\
C-h(0)(1-2D^2) &= 0, \\
(4h(0)C+\lambda)D-2h(K)(A-B)(1-3D^2) &= 0, \\
A^2-B^2+D^2 &= 0.
\end{align*}
\]

(7·8a) (7·8b) (7·8c) (7·8d) (7·8e)

One solution is

\[
A=B=D=0, \quad C=h(0).
\]

(7·9)

This solution corresponds to the pure \( \pi^0 \)-condensation discussed by PS. However, by studying the second derivatives in terms of \( A, B, C \) and \( D \), it is easily checked that (7·9) is not a local minimum but a saddle point of the energy \( E \) defined in (7·7a).

Another solution with non-vanishing \( A, B \) and \( D \) is found by solving (7·8) with the condition

\[
\begin{align*}
g(K)+\lambda &= 0, \\
g(K)-\lambda &= 2h(K)(1-3D^2), \\
-2h(K)(1-2D^2) &= 2h(K)(1-D^2), \\
4h(0)C+\lambda &= 0. (7·10)
\end{align*}
\]

Numerical values of roots and the corresponding energy \( E \) are obtained for a given value of \( K \). Then, varying \( K \), we seek minima of energy, which are given in Table IV for potential 3 (in contrast to the solution (7·9), these solutions are the local minima of \( E \)). However, the absolute values of \( E \) should not be taken too seriously because of our qualitative treatments of nuclear force in this section. The energy differences

\[
\Delta E = E(\pi^0) - E(\text{isomagnon-} \pi^+ \cdot \pi^0)
\]

(7·11)

between the solutions (7·9) and (7·10) are more meaningful and are also given there. Physical solutions with \( \pi^\pm \)-condensation were not found at \( \rho=0.266 \text{ fm}^{-3} \). Values of \( D^2 \) are less than 0.25 in the density region investigated. These small (but not sufficiently small) numbers may nearly justify the use of the trial state (7·6).
Our estimation shows that the onset of isomagnon-\( \pi^\pm \) condensation is expected somewhere between 0.266 fm\(^{-3} \) and 0.364 fm\(^{-3} \) (i.e., between 1.6 \( \rho_0 \) and 2.1 \( \rho_0 \), where \( \rho_0 \) is the density of normal nuclear matter). The energy under the presence of isomagnon-\( \pi^\pm, \pi^0 \) condensation is lower than the energy of the pure \( \pi^0 \)-condensation. In other words, the isomagnons with a finite wavevector cause the condensation of charged pions, resulting in a farther lowering of the system's energy by the amount of \( O(100 \text{ MeV/nucleon}) \).

\section*{§ 8. Concluding remarks}

The arguments in the text began on three assumptions: the PS potentials for nucleon interactions, the simple cubic lattice structure and ALS for the classical configuration of spins and isospins. Then, we have seen that the dense solid nuclear matter is likely in the condensed state of \( \pi^\pm, \pi^0 \) and isomagnons. The outline of the formation mechanism of these condensates is as follows: A small fraction of neutrons can be converted to protons as isospin-waves having long wavelengths without affecting very much the energy of solid which consists of nucleons only (remember that the flipping of \( T_3 \) from \(-1\) to \(+1\) at a certain fixed lattice site costs an energy of \( O(100 \text{ MeV}) \); but the excitation of one isospin-wave with infinite wavelength costs no energy). Then, isospin-waves act as powerful sources of charged pions, and then trigger their condensations. Important is that the isomagnon spectrum is gapless at the tree level.

At low temperatures, spin-wave excitations are expected unimportant because they have a large energy gap of \( O(100 \text{ MeV}) \), owing to the explicit breakdown of the \( O(3) \) symmetry of the effective Hamiltonian via the tensor interaction. However, as was shown in § 6, one of the most important contributions to the stabilization comes from the zero-point energy of antiferromagnetic spin system: The energy difference between phonon-\( \pi^0 \) system and phonon-magnon-\( \pi^0 \) system is \( O(100 \text{ MeV/nucleon}) \). The condensation of isomagnons and \( \pi^\pm \) seem to lower the energy further by the amount of \( O(100 \text{ MeV/nucleon}) \). The net energy gains are 140 MeV (at \( \rho=0.364 \text{ fm}^{-3} \)), 200 MeV (0.466 fm\(^{-3} \)) and 280 MeV (0.584 fm\(^{-3} \)) for potential 3. If these values are simply subtracted from the energies obtained by PS\(^4 \) or \( E \)'s obtained in § 7 in the text, then we arrive at a remarkable conclusion that the pressure gets negative or the system is unstable against contraction. The situation is essentially the same for potential 2. The nuclear matter may not solidify for PS's potential in the region \( 2\rho_0 < \rho < 4\rho_0 \).

One of the problems is that isomagnons are neither Bosons nor Fermions, while they were treated as Bosons in § 7. Although the conventional spin wave theory is known to (accidentally) yield fairly accurate values for the ground state energy, the treatment of isomagnons based on the correct statistics will be necessary for more quantitative discussion (and will be presented in a separate paper). Another one is that we may have to adopt a more realistic trial state of isomagnon-pion system which includes a wider range of wavevectors. In fact, several authors have pointed out a possibility of coexistence of neutral and charged pion condensates which minimize mutual interference by carrying wavevectors perpendicular to each other.\(^{39} \) It is
interesting to compare energies of systems with various wavevectors of pion condensates in the light of elementary excitations.

The method used in this article to study the stability of lattice can be applied to any other potentials and lattice structures of nucleonic matter. The results obtained in §§ 6 and 7 imply that, once a region of density exists in which the lattice is at least quasi-stable against distortions, then the energy of the system is possibly lowered further through isomagnon-pion interactions. Reinvestigations of models based on other sets of potentials and/or crystalline structures may be worthwhile.

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Appendix

—— Quantum Heisenberg Model and Holstein-Primakoff Transformation ——

The Hamiltonian of the Heisenberg model with the nearest-neighbor interactions is given by

\[ H = -\frac{1}{2} \sum_{\langle jk \rangle} J_{jk} S_j \cdot S_k, \quad [S_{j,i}, S_{k,j}] = \delta_{jk} \varepsilon \delta_{ijk} S_{j,k}, \quad S_j^2 = S(S+1), \tag{A\cdot1} \]

where \( S_j \) is the spin operator located at the \( j \)th lattice. The sum is taken over all pairs of \( z \) nearest-neighbors. The model is ferromagnetic (antiferromagnetic) when the nearest-neighbor couplings \( J_{jk} \) are positive (negative). We assume each lattice site is the center of symmetry.

Ferromagnetism

For the ferromagnetism, the Holstein-Primakoff (HP) transformation is defined by

\[
\begin{align*}
S_j^+ &= S_{jk} + i S_{jk} = (2S - a_j^\dagger a_j)^{1/2} a_j, \quad &\text{(A\cdot2a)} \\
S_j^- &= S_{jk} - i S_{jk} = a_j^\dagger (2S - a_j^\dagger a_j)^{1/2}, \quad &\text{(A\cdot2b)} \\
S_{jk} &= S - a_j^\dagger a_j, \quad &\text{(A\cdot2c)} \\
[a_j, a_k^\dagger] &= \delta_{jk}, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0. \quad &\text{(A\cdot2d)}
\end{align*}
\]

This representation corresponds to the case in which spins are classically aligned to the positive \( z \)-direction.

Antiferromagnetism

The whole lattice is divided into two sublattices \( A \) and \( B \). The classical spin directions in \( A \) and \( B \) opposite. The HP transformation for spins in the sublattice \( A \) is then defined as before, while, for spins in the sublattice \( B \), the HP transformation
is defined by

\begin{align}
S_{k\pm} &= b_k^\dagger (2S - b_k b_k)^{1/2} b_k , \\
S_{kz} &= -S + b_k b_k , \\
[b_{k}, b_{k'}^\dagger] &= \delta_{kk'}, \quad [b_{k}, b_{k'}] = [b_{k'}, b_{k}] = 0 .
\end{align}

This representation corresponds to the case in which spins are classically aligned to the negative z-direction.

**Constraint on physical states**

We consider the representation (A·2). The eigenvalues of $a_j a_j$ are 0 and 1, corresponding to the state of $S_z=1/2$ and $-1/2$, respectively. This means that, although operators $a_j$ and $a_j^\dagger$ obey Bose commutation relations, the physical space must be spanned by states which satisfy the condition

\begin{equation}
(a_j^\dagger)^2 |\text{phys}\rangle = 0 , \quad \langle \text{phys}|a_j^2|\text{phys}\rangle = 0 .
\end{equation}

We expand $a_j$ and $a_j^\dagger$ into Fourier series,

\begin{equation}
a_j = \frac{1}{\sqrt{N}} \sum_k a(k) e^{i k \cdot R_j} , \quad a_j^\dagger = \frac{1}{\sqrt{N}} \sum_k a^\dagger(k) e^{-i k \cdot R_j} .
\end{equation}

The Fourier coefficients satisfy the commutation relation

\begin{equation}
[a(k), a^\dagger(k')] = \delta_{kk'} , \quad [a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 .
\end{equation}

Then, the above constraint can be expressed as

\begin{equation}
Z_{a_j} |\text{phys}\rangle = 0 , \quad Z_a = \sum_k a(k) a(q - k)
\end{equation}

for all \( q \). Then, we see that the whole Hilbert space constructed by many-boson states like

\begin{equation}
\sum_k (1/\sqrt{N}) (a^\dagger(k))^q |0\rangle ,
\end{equation}

contains unphysical states. In this sense, the coherent state (7·6) in the text is also unphysical.

In the subspace constrained as (A·4), the operators $S_j^\pm$ are equivalent to\(^36,37\)

\begin{align}
S_j^+ &= (1 - a_j a_j^\dagger) a_j , \quad S_j^- = a_j^\dagger (1 - a_j^\dagger a_j) \\
or
S_j^+ &= a_j , \quad S_j^- = a_j^\dagger (1 - a_j^\dagger a_j) .
\end{align}

The Hamiltonian with these representations gives correct time developments of ket vectors. The representation

\begin{align}
S_j^+ &= a_j , \quad S_j^- = a_j^\dagger
\end{align}

is also available as long as the matrix elements of the Hamiltonian in terms of the
physical states are concerned. The representation \((A \cdot 9a)\) is used in the variational calculation in § 7 in order to keep the Hamiltonian hermite.

References

30) See, for example, C. Kittel, Quantum Theory of Solids (John Wiley & Sons, New York, 1963).
36) F. J. Dyson, Phys. Rev. 102 (1956), 1217, 1230.