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DISCUSSION

R. M. Olson[2]

The method of estimating hydrodynamic entrance lengths for laminar flow in ducts given in this paper was pointed out in a discussion of a paper presented at a meeting of the Fluids Engineering Division in Denver, Colo., in Apr. 1966.[3] The results in Table 3 for rectangular ducts are contained in a book published in Sept. 1966.[4] Entrance lengths for laminar flow in concentric annuli, elliptical, and triangular ducts are now available in the present paper.

Analyses generally overestimate the incremental pressure drop due to entrance effects.[3] Experimental values for K given in the paper by Sparrow, et al., are 0.89 and 0.99 compared with computed values of 1.00 and 1.38 for rectangular ducts of aspect ratio 5:1 and 2:1, respectively. Thus, if measured K-values are used, dimensionless entrance lengths X_e/D Re would be larger than those given in the paper—0.0317 versus 0.0255 for the 2:1 duct and 0.014 versus 0.012 for the 5:1 duct. Whether other entrance lengths given in the present paper are correspondingly low is not known. Experimental measurements would be needed to ascertain this.

The correctness of applying the Bernoulli equation along the duct axis in the developing region (assuming a flat velocity profile throughout the entire cross section at entrance, as well as requiring the velocity profile to have a finite flat portion with zero second derivatives at the duct axis throughout the entrance length) was questioned by the authors of footnote 3. Measured velocity profiles were actually flatter at the duct axis than calculated profiles in the downstream portion of the entrance region for rectangular ducts of aspect ratio 2:1 and 5:1. The method of estimating entrance lengths for laminar flow in ducts essentially involves the use of entrance incremental pressure drop over a length for which the second derivatives of the velocity profile in the core flow are zero or essentially zero. Results using measured K-values agree well with measurements of entrance lengths based on pressure gradient (7) and the paper by E. M. Sparrow, et al.,[3] and thus the method appears to be valid and to have useful applications.

E. M. Sparrow[5]

This paper is a very welcome addition to the literature on laminar duct flows in tha' it provides a rapid means for estimating hydrodynamic entrance lengths.

In appraising the results, it is of interest to inquire as to why the presently predicted entrance lengths are shorter than those provided by other analyses. The underlying cause for this finding may, at least in part, be explained by reconsidering the analytical model as expressed by equation (1). While the viscous effects (as they enter the momentum equation at the position of maximum velocity) may well be negligible in the immediate neighborhood of the duct inlet, there is ample evidence that they play an important role in the downstream portion of the hydrodynamic entrance region. It is fruitful to examine the sign of the viscous contribution. For this purpose, it is convenient (but by no means essential) to look at a simple flow, say, for instance, that in a parallel-plate channel. In this case, the equation of motion at the axis is

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

where $\mu \partial^2 u/\partial x^2$ has been omitted because it is negligible in the downstream region. It is readily verified that $\partial^2 u/\partial y^2$ is negative. Correspondingly, the viscous term, if included, would contribute a positive quantity to the right-hand side of equation (1) of the paper. If one were to retrace the steps of the analysis, it would be found that the viscous term would contribute a positive quantity to the numerator of equation (8). The absence of such a contribution might well account for the too-short entrance lengths presently predicted.

A rapid method for estimating hydrodynamic entrance lengths has recently been proposed by R. M. Olson, as set forth in a formal discussion of a recently published ASME paper.[6] It will be appreciated if the similarities and differences of the two computation methods were to be discussed. In addition, the author may wish to compare his predicted entrance lengths with the experimentally determined values presented in the previously mentioned paper.

[6] Sparrow, E. M., Hixon, C. W., and Shavit, G., "Experiments on Laminar Flow Development in Rectangular Ducts," Journal of Basic Engineering, Trans. ASME, Series D, Vol. 89, No. 1, Mar. 1967, pp. 116–124.

Author's Closure

I would like to thank both discussers for their interest and comments on this paper.

Professor Olson is quite correct in his statements related to the use of a more accurate value of K. The purpose of the analysis presented in this paper was to predict entrance lengths for ducts of arbitrary cross section for which no experimental data is available. For this reason, one must rely on the accuracy of K values as predicted by reference [1] or any subsequent predictions.

Professor Sparrow's appraisal of the effect of a viscous contribution on the entrance length is correct. However, the magnitude of this effect would seem to be quite small as indicated by experimental measurements [3]; the accuracy of K being more important.

The present predictions of entrance lengths for rectangular ducts of 5:1 and 2:1 aspect ratio are low but they are based on predicted K of reference [1]. As pointed out by Professor Olson, these lengths are increased to values which agree quite well with the experimental data [3] if the values of K from that paper are used. Also, for a circular tube, a value of 0.028 instead of the value 0.026 presented in this paper is obtained if the experimental K value of reference [7] is used instead of the predicted K value of reference [1]. This agrees well with Schiller's value of 0.0288 [2] and with the experimental observation of approximately 0.03–0.035 [7].

The proposed method of Olson in his discussion of reference [6] is found to be the same as in the present paper but he uses the experimental data of that reference and is limited to two rectangular duct channels. The present work expands the methods as earlier stated.

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[3] Sparrow, E. M., Hixon, C. W., and Shavit, G., "Experiments on Laminar Flow Development in Rectangular Ducts," Journal of Basic Engineering, Trans. ASME, Series D, Vol. 89, No. 1, Mar. 1967, p. 116.

[4] Olson, R. M., *Essentials of Engineering Fluid Mechanics*, 2nd ed., International Textbook Company, Scranton, Pa., 1966.

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