Action-Angle Representation of Complex Multisolitons

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Using the algebra of symmetries/mastersymmetries the action/angle scalar fields and related vector fields for complex multisolitons are constructed in explicit form in terms of the field variable.

Introduction

In the previous paper\(^1\) denoted hereafter as I we presented the systematic method of construction of the canonical action/angle variables, its gradients and the related vector fields on multisoliton submanifold, in terms of the original field variable. The complete method involves the following steps. First, reparametrize the multisoliton manifold in terms of scattering data of a recursion operator, i.e., its eigenvalues and the conjugate phases, in which the multisoliton flow is obviously linear. Second, transform such a linear system into another linear one represented by, more convenient to handle, asymptotic data variables. Third, examine the trivial structure of the linear flow on the reparametrized manifold and pullback the structure to the physical representation. In I we examined the examples of systems with real scattering data of recursion operator.\(^2\) For such systems the second step is unnecessary as asymptotic data coincide with scattering data. But this is not always the case for complex multisolitons. In this paper we transfer the method mentioned above to dynamical systems whose scattering data are complex variables. As in I the main result is the explicit construction of the canonical action/angle variables in terms of the physical field variable.

In § 1 we discuss the algebraic properties of complex systems on the whole manifold \(M\) and the projection of the structure onto the \(N\)-soliton submanifold. Then we shortly discuss the method of reparametrization of the multisoliton manifold in terms of the asymptotic data. In § 2 we examine the structure of a linear complex system and its transformation to the real one. We also show that under certain constraints our complex system reduces just to the real one considered in I. In § 3, taking advantage of the properties of a variable transformation, we pullback the algebraic objects we are interested in, to the physical representation. Finally, in § 4, our method is illustrated and applied to few exemplary systems.

\(^1\) The example of NLS in I fits to the scheme developed there only under some special reduction which is presented at the end of § 2.
§ 1. General aspects

1.1. Algebraic structure and N-soliton submanifold

On a suitable manifold $M$ we consider the evolution equation

$$u_t = K_1(u), \quad (1.1)$$

where $u(x, t) \in M$ denotes the complex field variable and $K_1(u)$ is a vector field on $M$. We are interested in such equations which admit two compatible Poisson operators $\theta_a$ and $\theta_{a+1}$ $(a \in \mathbb{Z})$ with

$$K_1(u) = \theta_a \partial H(u) = \theta_{a+1} \partial G(u), \quad (1.2)$$

where $H(u)$ and $G(u)$ are some scalar fields on $M$ and $\partial$ stands for the gradient operator. If $\theta_a$ is invertible then $\theta_a$ and $\theta_{a+1}$ give rise to a hereditary recursion operator $\phi = \theta_{a+1} \theta_a^{-1}$ which maps symmetries of Eq. (1.1) again onto the symmetries. As a consequence of the hereditariness of $\phi$ there exists a hierarchy of symmetries $K_n(u) = \phi^n(u) K_{-1}(u)$, where as the first symmetry $K_{-1}(u)$ we choose the generator of phase rotation $K_{-1}(u) = iu$. The role of $K_0(u)$ for complex multisolitons plays the generator of space translations $u_x$, just like for real multisolitons in I. The symmetries $K_n(u)$ are hamiltonian vector fields with respect to each Poisson operator $\theta_{m+a} = \phi^a \theta_a$, and they do commute in pairs $[K_m, K_n] = 0$, where $[\cdots, \cdots]$ is the usual commutator between vector fields. There is another recursive way of generating commuting symmetries via the so-called mastersymmetries. The very natural mastersymmetry is the scaling one $\tau_0$, for which we choose the following normalization:

$$L_{\tau_0} \phi = \phi, \quad L_{\tau_0} K_0 = a K_0, \quad a = \text{const.}, \quad (1.3)$$

where $L_v T$ denotes the Lie derivative of an arbitrary tensor $T$ into the direction of the vector field $v$. Defining $\tau_n = \phi^n \tau_0$ we obtain other mastersymmetries and using the product rule for Lie derivatives, we find

$$L_{K_m} K_n = [K_m, K_n] = 0, \quad L_{\tau_0} K_m = [\tau_0, K_m] = (m + a) K_{n+m},$$

$$L_{\tau_m} \tau_n = [\tau_n, \tau_m] = (m-n) \tau_{n+m}, \quad (1.4)$$

which is usually called a hereditary algebra. As for the most of the known complex systems (1.1) the scaling mastersymmetry $\tau_0$ commutes with the vector field $K_{-1}$, hence $a = 1$ and we confine in our paper to such a case.

Contrary to $K_n$ vector fields all but one $\tau_n$ vector fields are non-hamiltonian with respect to an arbitrary Poisson tensor $\theta_r$. Moreover, in addition to the conservation laws $H_n$ which are related with the vector fields $K_n$ in the following way,

$$K_n(u) = \theta_r \partial H_{n-r}, \quad (1.5a)$$

there exists one global scalar field $F$ related to the mastersymmetries $\tau_n$ through the relation
\[ \tau_n = \theta_n + b \varphi F, \]

where \( b \in \mathbb{N} \) is some fixed number. The so-called fundamental scalar field \( F \) played an important role in I as it completely determined the action/angle variables of real multisolitons. As we will find later, this is not always the case for complex multisolitons.

As was mentioned in the Introduction, we are interested in the dynamics (1.1) on a special submanifold of \( M \), i.e., on \( N \)-soliton submanifold \( M_N \). Two crucial questions appear when such a reduction is considered: How to define the \( N \)-soliton submanifold and whether the algebraic structure derived above survives the reduction? Both questions are nontrivial and below we only sketch the answer.

One of the possible ways of defining an \( N \)-soliton solution of (1.1) is to take a solution \( u \) of the ordinary differential equation

\[ (\phi - \lambda_i)(\phi - \lambda_i^*)\cdots(\phi - \lambda_N)(\phi - \lambda_N^*)K_0 + \sum_{n=0}^{2N} c_n K_n(u) = 0, \]

\[ (1.6) \]

where \( (\lambda_i, \lambda_i^*) \) are different complex conjugate constants and \( c_n(\lambda_i, \lambda_i^*, \ldots, \lambda_N, \lambda_N^*) \) are some real constants. Asymptotically, for \( t \to \infty \), an \( N \)-soliton solution of (1.1) divides into \( N \) noninteracting one-solitons of the form

\[ u_N \approx \sum_{k=1}^{N} u_k = \sum_{k=1}^{N} u(\text{im} \lambda_k x + \text{im} \lambda_k^2 t + \text{im} \gamma_k) \exp(\text{re} \lambda_k x + \text{re} \lambda_k^2 t + \text{re} \gamma_k), \]

\[ (1.7) \]

where the one soliton \( u_k \) of (1.1) is just a solution of \( (\phi - \lambda_k)(\phi - \lambda_k^*)i = 0 \).

The set of all \( u \) fulfilling equation (1.6) for different constants \( (\lambda_k, \lambda_k^*) \) we call the \( N \)-soliton manifold \( M_N \). The constants \( \text{re} \lambda, \text{im} \lambda \) and the phases \( \text{re} \gamma, \text{im} \gamma \) characterize the points on this manifold, hence in general we have to expect \( M_N \) considered as a manifold modelled over a real vector space.

Now, in analogy to the real \( N \)-soliton case, using the arguments presented in Ref. 5), we can prove that every vector field \( K_0(u_N) \) and \( \tau_0(u_N) \) is tangent to the \( N \)-soliton manifold \( M_N \). Of course only \( 2N \) of each kind of vectors are linearly independent. But it means that the recursion operator survives the reduction onto \( M_N \). Moreover one of Poisson structures \( \theta_r \) survives the reduction, hence, all the others do as well, since \( \phi^R(u_N) \theta_r \text{red} \) are Poisson structures on \( M_N \).

1.2. The method

For the submanifold \( M_N \) we are now going to give a new parametrization just as it was done in I. Let us define a map \( \Pi_1 \) which to each \( u_N \) assigns the set of complex scattering data of \( \phi \) operator \( (\gamma_1, \gamma_1^*, \ldots, \gamma_N, \gamma_N^*, \lambda_1, \lambda_1^*, \ldots, \lambda_N, \lambda_N^*) \). Next, we define a map \( \Pi_2 \) which transforms complex scattering data into real asymptotic data \( (\text{re} \gamma_1, \ldots, \text{re} \gamma_N, \text{im} \gamma_1, \ldots, \text{im} \gamma_N, \text{re} \lambda_1, \ldots, \text{re} \lambda_N, \text{im} \gamma_1, \ldots, \text{im} \gamma_N) \). Hence, the map \( \Pi = \Pi_2 \Pi_1 \) assigns to each \( u_N \) the set of asymptotic data. Of course we cannot write down this map explicitly, however this is not necessary for our further considerations. All algebraic properties of the objects we are interested in are invariant w. r. t. the choice of a special chart of the manifold \( M_N \), so what we only need are the appropriate transformation laws for these objects. The quantities \( \text{re} \gamma_i, \text{im} \gamma_i, \text{re} \lambda_i \) and \( \text{im} \lambda_i \) are scalar fields on the submanifold \( M_N \) and although we use the asymptotic form of the
N-solitons (1.7) this new parametrization is defined for any arbitrary time.

**Lemma 1**
For all \( i = 1, \ldots, N \) it holds

\[
\frac{\partial}{\partial t} \text{re} \gamma_i(t) = \text{re} \lambda_i^2, \quad \frac{\partial}{\partial t} \text{re} \lambda_i(t) = 0, \\
\frac{\partial}{\partial t} \text{im} \gamma_i(t) = \text{im} \lambda_i^2, \quad \frac{\partial}{\partial t} \text{im} \lambda_i(t) = 0.
\]  

(1.8)

Proof is the same as in I.

Lemma 1 shows that the flow (1.1) on the submanifold \( M_N \) is linearized in our new coordinates

\[
v_t = \begin{pmatrix}
\vdots \\
\text{re} \gamma_i \\
\vdots \\
\text{im} \gamma_i \\
\vdots \\
\text{re} \lambda_i \\
\vdots
\end{pmatrix},
= K_1.
\]  

(1.8a)

Because of this linear structure the dynamics given by (1.1) on \( M_N \) is trivial with respect to this parametrization. Hence, we have at our disposal an effective tool for the construction of the algebraic structure on \( M_N \) with respect to the \( u_N \) variable. The method is as follows.

Find the algebraic structure of the linear system (1.8). Relate the known algebraic objects in physical representation with the respective ones of the linear representation via the map \( \Pi \). Construct the unknown objects in the physical representation applying the transformation laws to the respective known objects in the linear representation.
In this paper we restrict our considerations to the construction of the canonical action/angle invariants in physical representation for complex multisolitons.

§ 2. Linear representation

In this section we examine algebraic structure of the linear system (1.8). We give the bi-hamiltonian formulation, construct the hereditary recursion operator, examine its spectral properties and finally construct the canonical action/angle representation of (1.8). Let us begin with a complex linear system consisting of $4N$ scattering data $(\gamma_1, \cdots, \gamma_N, \gamma_1^*, \cdots, \gamma_N^*, \lambda_1, \cdots, \lambda_N, \lambda_1^*, \cdots, \lambda_N^*)$. We can construct the hierarchy of commuting hamiltonian flows of the form

$$\ddot{u} = \dot{K}_m = \overline{\theta}_0 \nabla \dot{H}_m,$$  \hspace{1cm} (2.1)

where the field variable $\ddot{u}$, an implectic (Poisson) operator $\overline{\theta}_0$, scalar fields $\dot{H}_m$ and vector fields $\dot{K}_m$ are as follows

$$\ddot{u} = (\gamma_1, \cdots, \gamma_N, \lambda_1, \cdots, \lambda_N, \gamma_1^*, \cdots, \gamma_N^*, \lambda_1^*, \cdots, \lambda_N^*)^T,$$

$$\overline{\theta}_0 = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ -I & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \hspace{1cm} I = N \times N \text{ unit matrix},$$

$$\dot{H}_{m-1} = \frac{1}{i(m+1)} \sum \lambda_k^{m+1} - \lambda_k^{*m+1},$$

$$\dot{K}_m = (\lambda_1^{m+1}, \cdots, \lambda_N^{m+1}, 0, \cdots, 0, \lambda_1^{*m+1}, \cdots, \lambda_N^{*m+1}, 0, \cdots, 0)^T \hspace{1cm} (2.1a)$$

and $\nabla = (\partial/\partial \gamma_1, \cdots, \partial/\partial \gamma_N)^T$ is the gradient operator. The Poisson operator $\overline{\theta}_0$ is canonical for our variables in the sense that the only nonvanishing Poisson brackets are the following:

$$\{\lambda_k, \gamma_l\}_{\overline{\theta}_0} = i\delta_{kl}, \hspace{0.5cm} \{\lambda_k^*, \gamma_l^*\}_{\overline{\theta}_0} = -i\delta_{kl}, \hspace{1cm} (2.2)$$

hence, the coordinates $(\gamma_k, \lambda_k)$ and $(\gamma_k^*, \lambda_k^*)$ are called the canonical action/angle variables. Provided that the flow $\ddot{u} = \dot{K}_1$ is our linear system, the remaining vector fields $\dot{K}_m$ are its symmetries and do commute in pairs: $[\dot{K}_m, \dot{K}_n] = 0$.

The hierarchy (2.1) admits the bi-hamiltonian formulation

$$\dot{K}_m = \overline{\theta}_0 \nabla \dot{H}_m = \overline{\theta}_1 \nabla \dot{H}_{m-1}, \hspace{1cm} (2.3)$$
where

\[
\begin{pmatrix}
0 & \lambda_k & 0 \\
-\lambda_k & 0 & 0 \\
0 & 0 & 0 \\
\lambda_k^* & 0 & 0 \\
\end{pmatrix}
\]

As the linear combination of $\bar{\theta}_0$ and $\bar{\theta}_1$ is again an implictic operator, these operators give rise to the hereditary recursion operator

\[
\phi = \bar{\theta}_1 \bar{\theta}_0^{-1} = \begin{pmatrix}
\lambda_1 & \cdots & 0 & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_N \\
\end{pmatrix}
\]

Since $\phi$ is in diagonal form the eigenvalues of $\phi$ are $\lambda_1, \cdots, \lambda_N, \lambda_1^*, \cdots, \lambda_N^*$, each of them occurs two times. For every $k=1, \cdots, N$ the eigenvectors $A_k(A_k^*)$ and $B_k(B_k^*)$ of $\phi$ w. r. t. $\lambda_k(\lambda_k^*)$ are given by partial derivatives

\[
A_k = \frac{\partial \bar{u}}{\partial \gamma_k}, \quad B_k = \frac{\partial \bar{u}}{\partial \lambda_k}, \quad \phi A_k(B_k) = \lambda_k A_k(B_k),
\]

\[
A_k^+ = \frac{\partial \bar{u}}{\partial \gamma_k^*}, \quad B_k^+ = \frac{\partial \bar{u}}{\partial \lambda_k^*}, \quad \phi A_k^+(B_k^+) = \lambda_k^* A_k^+(B_k^+),
\]

and may represent the basis of the tangent space to the considered phase space. Moreover, they are hamiltonian action/angle vector fields w. r. t. $\bar{\theta}_0$ as

\[
A_k = \bar{\theta}_0 \mathcal{V}(-i\lambda_k), \quad B_k = \bar{\theta}_0 \mathcal{V}(i\gamma_k),
\]

\[
A_k^+ = \bar{\theta}_0 \mathcal{V}(i\lambda_k^*), \quad B_k^+ = \bar{\theta}_0 \mathcal{V}(-i\gamma^*).
\]

For the sake of convenience let us collect all the above results in the following.

**Lemma 2**

(i) W. r. t. the Poisson bracket given by $\bar{\theta}_0$ the coordinates $(\gamma_k, \lambda_k)$ and $(\gamma_k^*, \lambda_k^*)$ are canonical action/angle variables (2.2).

(ii) The symmetries $\mathcal{K}_k$ are bi-hamiltonian vector fields (2.3) and commute in pairs.

(iii) At each point of the phase space the tangent space is a space of eigenvectors of
the recursion operator $\vec{\phi}$.

(iv) The eigenvectors of $\vec{\phi}$ are given by partial derivatives of the field variable $\vec{u}$ w. r. t. action/angle variables (2.5).

(v) The eigenvectors of $\vec{\phi}$ are hamiltonian action/angle vector fields w. r. t. $\vec{\theta}$ (2.6).

**Remark 1**

Of course one can easily construct more interesting objects for the linear system (2.1). However, it turns out that only the mentioned ones do have a physical representation.

To develop the algebraic structure of our basic linear system (1.8) let us perform the linear change of variables from the scattering data to the asymptotic data

$$F: \vec{u}=(\gamma_1, \cdots, \lambda_N^*)^T \to \vec{v}=(\varphi_1, \cdots, \varphi_N, q_1, \cdots, q_N, \delta_1, \cdots, \delta_N, c_1, \cdots, c_N)^T, \quad (2.7)$$

where

$$\varphi_k = \text{re}\gamma_k, \quad q_k = \text{im}\gamma_k, \quad \delta_k = 2i\text{m}\lambda, \quad c_k = 2\text{re}\lambda_k,$$

$$\gamma_k = \varphi_k + iq_k, \quad \lambda_k = \frac{1}{2}(c_k + i\delta_k). \quad (2.7a)$$

Hence, according to transformation rules we find

$$\vec{\theta} = \begin{pmatrix}
0 & I & 0 \\
-I & 0 & I \\
0 & -I & 0
\end{pmatrix},$$

$$\vec{\theta}_1 = \frac{1}{2} \begin{pmatrix}
0 & 0 & \cdots & 0 \\
-c_N & \delta_1 & \cdots & 0 \\
0 & -c_N & 0 & \delta_N \\
0 & 0 & 0 & 0
\end{pmatrix},$$

$$\vec{\phi} = \frac{1}{2} \begin{pmatrix}
c_1 & \cdots & 0 & -\delta_1 & 0 \\
0 & c_N & \cdots & 0 & -\delta_N \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}. \quad (2.8)$$
where the eigenstates of $\vec{\phi}$ are

$$\vec{A}_k = \frac{1}{2} \left( \frac{\partial V}{\partial \varphi_k} - i \frac{\partial V}{\partial q_k} \right),$$

$$\vec{A}_k^+ = \frac{1}{2} \left( \frac{\partial V}{\partial \varphi_k} + i \frac{\partial V}{\partial q_k} \right),$$

$$\vec{B}_k = \left( \frac{\partial V}{\partial c_k} - i \frac{\partial V}{\partial \delta_k} \right),$$

$$\vec{B}_k^+ = \left( \frac{\partial V}{\partial c_k} + i \frac{\partial V}{\partial \delta_k} \right).$$

(2·9)

Our new variables are canonical action/angle variables w. r. t. the new Poisson operator $\vec{\theta}_0$

$$\{\delta_k, \varphi_l\} = \delta_{kl}, \quad \{c_k, q_l\} = \delta_{kl},$$

(2·10)

and the respective canonical action/angle Hamiltonian vector fields are

$$\frac{\partial V}{\partial \varphi_k} = (\vec{A}_k + \vec{A}_k^+) = \vec{\theta}_0 \varphi \delta_k,$$

$$\frac{\partial V}{\partial q_k} = i(\vec{A}_k - \vec{A}_k^+) = \vec{\theta}_0 \varphi c_k,$$

$$\frac{\partial V}{\partial \delta_k} = \frac{1}{2} i(\vec{B}_k - \vec{B}_k^+) = \vec{\theta}_0 \varphi (-\varphi_k),$$

$$\frac{\partial V}{\partial c_k} = \frac{1}{2} (\vec{B}_k + \vec{B}_k^+) = \vec{\theta}_0 \varphi (-q_k).$$

(2·11)

The hierarchy of commuting flows takes the form

$$v_{i} = \vec{K}_m = \vec{\theta}_0 \varphi \vec{H}_m,$$

(2·12)

where

$$\vec{K}_m = (\text{re} \lambda_1^{m+1}, \ldots, \text{re} \lambda_N^{m+1}, \text{im} \lambda_1^{m+1}, \ldots, \text{im} \lambda_N^{m+1}, 0, \ldots, 0)^T,$$

$$\vec{H}_{m-1} = \frac{2}{m+1} \sum_{k=1}^{N} \text{im} \lambda_k^{m+1}$$

and we made use of the following relations

$$\frac{\partial \text{im} \lambda_k^{m+1}}{\partial \text{re} \lambda_k} = (m+1) \text{im} \lambda_k^m, \quad \frac{\partial \text{im} \lambda_k^{m+1}}{\partial \text{im} \lambda_k} = (m+1) \text{re} \lambda_k^m.$$

(2·13)

But the flow (2·12) with $m=1$ is just our basic linear system (1·8) the structure of which we were looking for. A few first Hamiltonians and the corresponding vector fields in our new variables are as follows:
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\[ \bar{K}_0 = \begin{pmatrix} \frac{1}{2} C_k \\ \vdots \\ \frac{1}{2} C_k \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \bar{K}_1 = \begin{pmatrix} \frac{1}{4} (C_k^2 - \delta_k^2) \\ \vdots \\ \frac{1}{4} (C_k^2 - \delta_k^2) \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots \]

\[ \bar{H}_{-1} = \sum_k \delta_k, \quad \bar{H}_0 = \frac{1}{2} \sum_k C_k \delta_k, \quad \bar{H}_1 = \frac{1}{4} \sum_k \left( \delta_k C_k^2 - \frac{1}{3} \delta_k^2 \right), \quad \ldots. \] (2.14)

To recover the hereditary algebra (1.4) with \( a = 1 \), we have to introduce a suitable hierarchy of mastersymmetries \( \tilde{r}_n \). Of course we would like to generate them from one fundamental scalar field \( \bar{F} \) by applying to its gradient the hierarchy of Poisson operators \( \bar{\theta}_n \), as it was done in (1.5b). But here we come up with some difference when compared to the real case considered in I. In the case of complex systems, the analogue of a fundamental scalar field found in I is the following scalar field:

\[ \bar{T}_0 = \sum_k (\gamma_k \lambda_k - \gamma_k \lambda_k^* ) = - \sum_k (q_k C_k + \varphi_k \delta_k ) = - 2 \sum_k \left( q_k \frac{\partial \bar{H}_0}{\partial \delta_k} + \varphi_k \frac{\partial \bar{H}_0}{\partial C_k} \right) \] (2.15)

which is the generating quantity for action/angle variables

\[ \varphi_k = - \frac{\partial \bar{T}_0}{\partial \delta_k}, \quad \delta_k = - \frac{\partial \bar{T}_0}{\partial \varphi_k}, \quad q_k = - \frac{\partial \bar{T}_0}{\partial q_k}, \quad c_k = - \frac{\partial \bar{T}_0}{\partial q_k}. \] (2.16)

Unfortunately, for complex soliton systems considered in this paper, the fundamental scalar field \( F(u) \) on \( N \)-soliton submanifold has the following value:

\[ F(u_N) = \beta (- i \sum (\gamma_k - \gamma_k^*) ) = \beta (- \sum 2 q_k ) = \beta \bar{T}_{-1}, \quad \beta = \text{const.}, \] (2.17)

hence it differs from \( \bar{T}_0 \). The scalar fields \( \bar{T}_{-1} \) and \( \bar{T}_0 \) belong to the hierarchy \( \bar{T}_n \) of the so-called noncanonical angle variables whose theory is beyond the scope of this paper. Then the hierarchy of mastersymmetries in physical representation (1.5b) is the one of the forms

\[ \bar{r}_n = \bar{\phi}^{n+1}, \quad \bar{r}_{-1} = \bar{\theta}_{n+1} \bar{T}_{-1}, \] (2.18)

and the respective hereditary algebra reads

\[ [K_n, K_m] = 0, \quad [\bar{r}_n, K_m] = (m+1) K_{n+m}, \quad [\bar{r}_n, \bar{r}_m] = (m-n) \bar{r}_{n+m}. \] (2.19)

**Remark 2**

Of course the scalar field \( \bar{T}_0 \) gives rise to another hierarchy of mastersymmetries

\[ \bar{r}'_n = \bar{\theta}_n \bar{T}_0 \] (2.18')

with a different representation in physical coordinates than the well-known (1.5b) one.

In the end of this section we would like to examine one interesting reduction. Let
us introduce the following constraint for our variables: \( \delta_k = c_k \) and \( \varphi_k = 0 \), and then change the remaining coordinates into the new ones: \( q_k \rightarrow (1/2)c_kq_k \), \( c_k \rightarrow c_k \). Under such a reduction one finds

\[
\hat{K}_n(\delta_k, c_k) \rightarrow a_n(\cdots, c_k''', 0, \cdots, 0)^T = a_n\hat{K}_n(c_k), \quad a_n = 2^{-n}\text{im}(1+i)^{n+1},
\]

\[
\hat{H}_{n-1}(\delta_k, c_k) \rightarrow a_n\frac{1}{n+1} \sum_k c_k^{n+1} = a_n\hat{H}_n(c_k, a=1),
\]

\[
\bar{T}_i(q_k) \rightarrow -\sum_k c_k'q_k' = \bar{F}(c_k', q_k'),
\]

\[
\bar{\theta}_i(\delta_k, c_k) \rightarrow \bar{\theta}_i(c_k'; a=1), \tag{2.20}
\]

where the quantities on the r. h. s. of (2.20) are nothing but respective tensor invariants found in I for the 2N dimensional real linear system \( (q', c'; a=1) \). Hence, the hierarchy of commuting flows (2.1) in the second hamiltonian representation is projected onto the hierarchy (2.8) in I in the first hamiltonian representation, where the first hamiltonian representation is understood as the one connected with the canonical Poisson tensor.

### § 3. The physical representation

In the previous section we found some algebraic properties of the linear system (1.8) which has been a more or less trivial task. But of course we would like to carry over the results to the physical representation of N-soliton manifold and express all desired quantities like action/angle variables, their gradients and respective vector fields, in terms of the field variable \( u_\nu(x, t) \). Using the properties of the coordinate transformation II we can formulate the following theorem:

**Theorem I.**

On the N-soliton manifold the following statements are true:

(i) The vector fields \( \partial u_N/\partial \varphi_k, \partial u_N/\partial q_k, \partial u_N/\partial \delta_k \) and \( \partial u_N/\partial c_k \) are hamiltonian vector fields w. r. t. the symplectic operator \( \theta_{\text{red}} \) determined by

\[
\tau_{-1} = \theta(u)\bar{F}(u), \tag{3.1}
\]

where \( \theta_{\text{red}}(u) \) is the reduction of \( \theta(u) \) to the soliton submanifold

\[
\theta_{\text{red}}(u) = \theta_0 = (\Pi')^{-1} \bar{\theta}_0 (\Pi'^{-1}). \tag{3.2}
\]

(ii) The corresponding potentials \( \Delta_k, E_k, G_k \) and \( \Omega_k \) of these vector fields are given by the partial derivatives

\[
\Delta_k = -\frac{\partial T_0}{\partial \varphi_k}, \quad E_k = -\frac{\partial T_0}{\partial q_k}, \quad G_k = -\frac{\partial T_0}{\partial \delta_k}, \quad \Omega_k = -\frac{\partial T_0}{\partial c_k}, \tag{3.3}
\]

where

\[
T_0(u_N) = -2\sum_k \left( \varphi_k \frac{\partial H_0(u_N)}{\partial c_k} + q_k \frac{\partial H_0(u_N)}{\partial \delta_k} \right). \tag{3.3a}
\]
(iii) \((Q_k, E_k)\) and \((G_k, \Delta_k)\) are canonical coordinates w. r. t. \(\theta_0\), i.e., for all \(i, j = 1, \ldots, N\) the only Poisson brackets different from zero are the following:
\[
\{E_i, Q_j\}_{\theta_0} = \delta_{ij}, \quad \{A_i, G_j\}_{\theta_0} = \delta_{ij},
\]
where \(\delta_{ij}\) is interpreted as the delta function.

(iv) \(\phi = (\Pi')^{-1}\tilde{\phi}\Pi'\) is a hereditary recursion operator in the physical representation with the following eigenvectors,
\[
A_k = \frac{1}{2} \left( \frac{\partial u_N}{\partial \varphi_k} - i \frac{\partial u_N}{\partial q_k} \right), \quad A_k^+ = \frac{1}{2} \left( \frac{\partial u_N}{\partial \varphi_k} + i \frac{\partial u_N}{\partial q_k} \right),
\]
\[
B_k = \left( \frac{\partial u_N}{\partial c_k} - i \frac{\partial u_N}{\partial \delta_k} \right), \quad B_k^+ = \left( \frac{\partial u_N}{\partial c_k} + i \frac{\partial u_N}{\partial \delta_k} \right),
\]
where
\[
\phi A_k(B_k) = \lambda_k A_k(B_k), \quad \phi A_k^+(B_k^+) = \lambda_k^* A_k^+(B_k^+).
\]

Proof

(i) As the pullback conserves hamiltonian structures, the (i) is a consequence of the fact that in the linear representation the corresponding vector fields (2·11) and mastersymmetry \(\tau_1\) (2·18) are hamiltonian vector fields only w. r. t. the same implictic operator \(\Theta_0\).

(ii) From (i) we know that there are scalar fields \(\Delta_k, E_k, G_k\) and \(\Omega_k\) with
\[
\frac{\partial u_N}{\partial \varphi_k} = \theta_0 \text{ grad } \Delta_k, \quad \frac{\partial u_N}{\partial \delta_k} = \theta_0 \text{ grad } (-G_k),
\]
\[
\frac{\partial u_N}{\partial q_k} = \theta_0 \text{ grad } E_k, \quad \frac{\partial u_N}{\partial c_k} = \theta_0 \text{ grad } (-\Omega_k).
\]
Since \(E_k\) are the conservation laws and \(\xi_0 = \theta_0 \nabla T_0\) (compare with (2·18')) is an admissible scaling vector field, we find
\[
E_k = L_{\xi_0}E_k = \langle \nabla E_k, \xi_0 \rangle = \langle \nabla E_k, \theta_0 \nabla T_0 \rangle = -\langle \nabla T_0, \theta_0 \nabla E_k \rangle
\]
\[
= -\left( \nabla T_0, \frac{\partial u_N}{\partial q_k} \right) = -\frac{\partial T_0(u_N)}{\partial q_k}.
\]
The remaining equalities of (3·3) are proved in the same way.

(iii) is a direct consequence of (i), Eqs. (2·10) and (2·11).

(iv) is a simple consequence of the definition of \(\Pi'\) and the fact that \(\Pi'\) is a Lie algebra isomorphism.

§ 4. Examples

4.1. The nonlinear Schrödinger equation (NLS)

Let us consider a dynamical system in the form
\[ u_t = iu_{xx} + 2i|u|^2u^*, \]
\[ v_t = -iu_{xx} - 2i|u|^2u^*, \]  
\[ \text{(4.1)} \]

where \( u(x, t): \mathbb{R} \rightarrow \mathbb{C} \) is a complex valued function rapidly decreasing at infinites and \( u^* \) stands for the complex conjugate. The hierarchy of commuting symmetries

\[ K_n = \frac{1}{i}(K_n - K_n^*) = \phi^{n+1}(u, u^*)K_1(u, u^*) \]
\[ = \left( \begin{array}{cc} -iD - 2iuD^{-1}u^* & -2iuD^{-1}u \\ 2iu^*D^{-1}u^* & iD + 2iu^*D^{-1}u \end{array} \right)^{n+1} \left( \begin{array}{c} -iu \\ iu^* \end{array} \right), \]  
\[ \text{(4.2)} \]

where

\[ \phi = \theta_{a+1}\theta_a^{-1} = \left( \begin{array}{cc} 2uD^{-1}u & -D - 2uD^{-1}u^* \\ -D - 2u^*D^{-1}u & 2u^*D^{-1}u^* \end{array} \right) \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \]  
\[ \text{(4.2a)} \]

and the hierarchy of mastersymmetries

\[ \tau_n = \left( \begin{array}{c} \tau_n \\ \tau_n^* \end{array} \right) = \phi^{n+1}(u, u^*)\tau_1(u, u^*) = \phi^{n+1}(u, u^*) \left( \begin{array}{c} iux \\ -ixu^* \end{array} \right), \]  
\[ \text{(4.3)} \]

fulfill the commutation relations

\[ [K_n, K_m] = 0, \quad [\tau_n, K_m] = (m+1)K_{n+m}, \quad [\tau_n, \tau_m] = (m-n)\tau_{n+m}. \]  
\[ \text{(4.4)} \]

Here \( D \) denotes the differential operator w. r. t. the variable \( x \) and \( D^{-1} \) its inverse. The commuting vector fields are multihamiltonian, i.e., \( K_n = \theta_r \partial H_{n-r} \), where \( \theta_r = \phi^r \theta_0 \). The few simplest scalar fields are as follows:

\[ N = \int_{-\infty}^{+\infty} |u|^2 dx, \quad P = \frac{1}{2i} \int_{-\infty}^{+\infty} (u_x u^* - u_x^* u) dx, \quad H = \int_{-\infty}^{+\infty} (|u_x|^2 - |u|^4) dx, \ldots. \]  
\[ \text{(4.5)} \]

Mastersymmetries are non-hamiltonian vector fields with respect to \( \theta_r \) except the one

\[ \tau_r = \theta_{r+1}\partial F(u, u^*), \]  
\[ \text{(4.6)} \]

where

\[ F(u, u^*) = -\int_{-\infty}^{+\infty} x|u|^2 dx \]  
\[ \text{(4.6a)} \]

is the fundamental scalar field.

The \( N \)-soliton solution decomposes asymptotically with \( t \to \infty \) into one-solitons as follows:

\[ u_N \equiv \sum_{k=1}^{N} \delta_k(x; q_k, c_k, \varphi_k, \delta_k) = \sum_{k=1}^{N} \frac{1}{2}\delta_k \text{sech} \left( \frac{1}{2} \delta_k x - q_k(t) \right) \exp \left( i \left( \frac{1}{2} c_k x - \varphi_k(t) \right) \right), \]  
\[ \text{(4.7)} \]
where

\[ q_k(t) = \frac{1}{2} \delta_k c_k t + q_{0k}, \quad \varphi_k(t) = \frac{1}{4} (c_k^2 - \delta_k^2) + \varphi_{0k}. \tag{4.7a} \]

Hence the solution (4.7) fits into our case (1.7) with respect to the coordinates (2.7). The flow (4.1) on \( M_N \) is linearized in our new coordinates and is equivalent to the flow \( K_1 (1.8 a), \ (2.14) \) from the hierarchy (2.12). Applying the results of the previous sections we find \( F(u_N) = T_1 \) and \( \tau_1 = \partial \varphi F \), so, \( \beta = 1, \ a = 0, \ N = H_{-1}, \ P = H_0 \) and \( H = H_1 \). Theorem 1 gives the action/angle variables \( \Delta_k, E_k, G_k \) and \( \Omega_k \) explicitly by (3.3) where

\[ T_\theta(u_N) = i \sum_{k=1}^N \int_{-\infty}^{\infty} [\varphi_k(u_x u^* - u_x^* u) c_k + q_k(u_x u^* - u_x^* u) s_k] dx, \tag{4.8} \]

and the suitable action/angle vector fields w. r. t. \( \theta_{0123} \) in the form (3.6) where the two-component field variable reads \( u_N = (u_N, u_N^*)^T \).

**Remark 3**

The example of NLS considered in I is valid only under the reduction considered in the end of § 2. It means that in § 4.5 of I one should put \( \delta_k = c_k \) and \( \varphi_k = 0 \) to get the correct result.

### 4.2. A Heisenberg ferromagnet

Here we consider a three component system with the constraint 

\[ S_i = -S \times S_{xx}, \quad S \cdot S = 1, \quad S = (S_1, S_2, S_3)^T. \tag{4.9} \]

We can put it into the bi-hamiltonian form in the following way

\[ S_i = \theta_a \varphi H = \theta_{a+1} \varphi P, \tag{4.10} \]

where

\[ \begin{bmatrix} 0 & -S_3 & S_2 \\ S_3 & 0 & -S_1 \\ -S_2 & S_1 & 0 \end{bmatrix}, \quad H = \frac{1}{2} \int_{-\infty}^{\infty} (S_{1x}^2 + S_{2x}^2 + S_{3x}^2) dx, \]

\[ \theta_a = \begin{bmatrix} D - D S_1 D^{-1} S_1 D & -D S_1 D^{-1} S_2 D & -D S_1 D^{-1} S_3 D \\ -D S_2 D^{-1} S_1 D & D - D S_2 D^{-1} S_2 D & -D S_2 D^{-1} S_3 D \\ -D S_3 D^{-1} S_1 D & -D S_3 D^{-1} S_2 D & D - D S_3 D^{-1} S_3 D \end{bmatrix}, \]

\[ \theta_{a+1} = \begin{bmatrix} 0 & -S_3 & S_2 \\ S_3 & 0 & -S_1 \\ -S_2 & S_1 & 0 \end{bmatrix}, \quad P = \int_{-\infty}^{\infty} \frac{S_1 S_{2x} - S_2 S_{1x}}{1 + S_3} dx. \tag{4.10a} \]

The simplest mastersymmetries are

\[ \tau_0 = x S_x, \quad \tau_1 = x S \times S_{xx} + S \times S_x. \tag{4.11} \]

Now we pass to the equivalent two-component system without constraint

\[ \phi = i \psi [(1 - \psi \psi^*)^{1/2}]_{xx} + i \psi_{xx} (1 - \psi \psi^*)^{1/2}, \]
where
\[ \phi = S_1 + iS_2, \quad \phi^* = S_1 - iS_2, \quad S_3 = (1 - \phi\phi^*)^{1/2}. \]

With respect to this new representation the hierarchy of commuting symmetries reads
\[ K_n = \phi^{n+1}(\phi, \phi^*)K_{-1} = \theta_{a+1}\mathcal{P}H_{n-a} = \theta_{a+1}\mathcal{P}H_{n-a-1}, \]

where the recursion operator \( \phi \) is of the form
\[
\begin{pmatrix}
-D\phi D^{-1}\phi D & 2D - D\phi D^{-1}\phi^* D \\
2D - D\phi^* D^{-1}\phi D & -D\phi^* D^{-1}\phi^* D
\end{pmatrix}
\begin{pmatrix}
0 & 2i(1 - \phi\phi^*)^{1/2} \\
-2i(1 - \phi\phi^*)^{1/2} & 0
\end{pmatrix}
\]

The few first \( K_n \) symmetries are
\[
K_{-1} = \begin{pmatrix} i\phi \\ -i\phi^* \end{pmatrix}, \quad K_0 = \begin{pmatrix} \phi_x \\ \phi_x^* \end{pmatrix}, \quad K_1 = \begin{pmatrix} i\phi[(1 - \phi\phi^*)^{1/2}]_{xx} + i\phi_{xx}(1 - \phi\phi^*)^{1/2} \\ -i\phi^*[(1 - \phi\phi^*)^{1/2}]_{xx} - i\phi^*_{xx}(1 - \phi\phi^*)^{1/2} \end{pmatrix}, \ldots
\]

and the few simplest conservation laws are of the form
\[
H_{-1-a} = \int_{-\infty}^{\infty} [1 - (1 - \phi\phi^*)^{1/2}] dx = \int_{-\infty}^{\infty} (1 - S_3) dx = N,
\]
\[
H_{-a} = \frac{1}{2} i \int_{-\infty}^{\infty} (\phi^* \phi_x - \phi \phi_x^*)[1 + (1 - \phi\phi^*)^{1/2}] dx = P,
\]
\[
H_{1-a} = \frac{1}{2} \int_{-\infty}^{\infty} \left[ |\phi_x|^2 + \frac{1}{4} (1 - |\phi|^2)^{-1}(|\phi|_x)^2 \right] dx = H, \ldots
\]

Then, the hierarchy of mastersymmetries reads
\[ \tau_n = \phi^n\tau_0 = \theta_{a+n-1}\mathcal{P}F, \]

where
\[
\tau_0 = \begin{pmatrix} x\phi_h \\ x\phi_h^* \end{pmatrix}, \quad F = \int_{-\infty}^{\infty} xh(\phi, \phi^*) dx
\]

and \( h \) denotes the density of \( H_{1-a} \) functional. As \( L_{\phi}\phi = \phi \) and \( L_{\phi}K_{-1} = 0 \) we find the hereditary algebra in the form (4.4).

The \( N \)-soliton solution of (4.12) decomposes asymptotically for \( t \to \infty \) into one-solitons in the following way:
\[ \phi_N \approx \sum_{k=1}^{N} \frac{2c_k}{c_k^2 + \delta_k^2} \text{sech}^2 \left[ \frac{1}{2} c_k x - q_k(t) \right] \left[ -c_k \text{sh} \left[ \frac{1}{2} c_k x - q_k(t) \right] \right] + i\delta_k \text{ch} \left[ \frac{1}{2} c_k x - q_k(t) \right] \text{exp}(i\omega_k), \]
where
\[ \omega_n = \frac{1}{2} \delta_k x - \varphi_k(t), \quad \varphi_k(t) = \frac{1}{4} (c_k^2 - \delta_k^2) t + \varphi_0 k, \quad q_k(t) = \frac{1}{2} c_k \delta_k t + q_0 k. \quad (4.16a) \]

Again the quantities \((c_k, q_k)\) and \((\delta_k, \varphi_k)\) can be considered as new independent variables and are related to the field \(\psi\) through an appropriate map \(\Pi\). Comparing equalities \((2.18)\) and \((4.15)\), we find \(a = 2\) and from \(F(\psi_n) = 2T_1\) we get \(\beta = 2\). Now, when we have recognized \(\theta_0\) Poisson structure and the \(H_1\) conservation law, we can construct the canonical action/angle invariants according to Theorem 1.

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References

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