Finite Probability of Inflation from Higher Dimensions

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(Received November 9, 1990)

We estimated the probability of sufficient inflation in a higher dimensional cosmological model in which there are two compact subspaces. Using a newly defined measure on the initial data surface, we found the probability to be finite and reasonably large for certain regions of the parameters of our model.

§ 1. Introduction

The inflationary universe scenario\(^1\),\(^2\) has been introduced to explain the particular initial conditions which are required for the standard big bang cosmology to be successful. It has been investigated extensively\(^9\) and particular solutions showing inflation have been found.\(^4\)\(^\text{–}^7\) One of the most important problems concerning the scenario is whether the inflation occurs naturally or not, i.e., whether the inflationary solutions, in turn, require particular initial conditions. This problem has been investigated by several authors.\(^8\)\(^\text{–}^\text{14}\)

In the investigation of this problem, a natural way is to introduce a positive measure for the set of solutions of the dynamical equations and to estimate the probability of inflation. Gibbons et al.\(^9\) have proposed conditions that such a measure should satisfy: (i) It should be a property of the set of the models, and independent of the dependent variables and parameters chosen. (ii) It should respect the symmetries of the space of solutions and should not make use of any extra ad hoc structures which do not arise from the equations themselves. They found that the so-called canonical measure satisfies these conditions. However, it leads to an infinite total measure. That is to say, the canonical measure, by itself, cannot give a finite probability other than zero. Therefore ad hoc limiting procedure has to be introduced for the ratio of the infinite measures. It is difficult to find a measure that satisfies the above conditions and leads to a finite total measure. No example of such a measure has been found (the existence of which has not been proven).

As an alternative approach, we relax partly the condition (ii) above and construct a measure which leads to a finite total measure. Then the probability is given definitely. We start from a set of dependent variables which is natural from a physical point of view. The resulting measure, however, is invariant under the transformations of the dynamical variables (the details are given in § 4).

In the study of inflation, scalar fields, the so-called inflaton fields, play a crucial
role. It is desirable that they are incorporated into the model naturally or from a unified theoretical point of view. One of the possibilities is to use the higher dimensional cosmological models. In such models, the scalar fields appear naturally and in addition their interactions are almost automatically determined.

Along these lines we investigated, in a previous paper,\(^{15}\) the probability of sufficient inflation using a simple higher dimensional cosmological model in which the higher dimensional spacetime is composed of a direct product of a lower dimensional spacetime and a compact subspace. The estimated measure of sufficient inflation was zero within the approximation used there. (In the estimation we used the canonical measure. But the result remains the same even if we used a measure which we propose in the present paper.) It was suggested there that the increase of the dynamical degrees of freedom of the system might give a larger probability of inflation.

Thus, in this paper, we estimate the probability of sufficient inflation by modifying the model used in Ref. 15) and by using a newly constructed measure on the initial data surface. In the modification, we increase the degrees of freedom of the system by adding one more compact subspace to the direct product composing the higher dimensional spacetime. This seems to be natural, since it is anticipated that the inflation is closely related to the properties of the vacuum. So we do not take into account excitations such as inhomogeneities or gauge fields. Furthermore we adhere to the theory of higher dimensional gravity which reduces to the Einstein gravity after compactification. We do not choose the theory of gravity with higher order terms in the curvatures.

The probability of sufficient inflation depends on the parameters of our model such as the higher-dimensional cosmological constant, curvatures of the compact subspaces and the dimensions of subspaces. For certain regions of their values, the probability of sufficient inflation is found to be about 10% which can be thought large enough for the inflation to be natural.

In § 2 we formulate the dynamics by making simplifying assumptions on the higher dimensional metric and derive field equations. We use non-dimensional dynamical variables which are convenient in discussing the new measure. In § 3 approximate solutions of the field equations are obtained. A new measure is derived in § 4. It is constructed geometrically as far as possible. In § 5 numerical results on the probability of inflation are presented and discussions are given on them and on some aspects of our model.

§ 2. Formulation

In this section, we derive field equations of our model. We start from the higher-dimensional Einstein-Hilbert action:

\[
S = \frac{1}{16\pi G} \int d^Dx \sqrt{-\det \bar{g}_{MN}} (\bar{R} - 2\bar{\Lambda}),
\]  

(2.1)

where \( D \) is the dimension of the higher-dimensional spacetime, \( \bar{G}, \bar{g}_{MN}, \bar{R} \) and \( \bar{\Lambda} \) are \( D \)-dimensional gravitational constant, metric, scalar curvature and cosmological
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constant, respectively. A hat denotes quantities in $D$ dimensions.

Now the following assumptions are made on the metric $\tilde{g}_{MN}$ which describes the ground state of the $D$-dimensional spacetime.

(i) $\tilde{g}_{MN}$ takes the form

$$
\tilde{g}_{MN} = \begin{pmatrix}
\tilde{g}_{\mu \nu} & 0 & 0 \\
0 & b_1^2 \tilde{g}_{m_1n_1} & 0 \\
0 & 0 & b_2^2 \tilde{g}_{m_2n_2}
\end{pmatrix},
$$

(2.2)

where $\mu, \nu$ run from 0 to $d$ and $m_i, n_i$ ($i=1,2$) run from 1 to $d_i$ ($i=1,2$), respectively, and $b_i$ ($i=1,2$) depend only on time.

(ii) Metrics $\tilde{g}_{m_{ni}}$ ($i=1,2$) describe $d_i$-dimensional subspaces of constant curvature, so that the Ricci tensors $\tilde{R}_{m_{ni}}$ are equal to $(\tilde{R}_i/d_i) \tilde{g}_{m_{ni}}$, where $\tilde{R}_i$ are constant scalar curvatures.

Furthermore we assume that $(d+1)$-dimensional components $\tilde{g}_{\mu \nu}$ are related to the physical metric $g_{\mu \nu}$ of the $(d+1)$-dimensional spacetime which is isotropic and homogeneous as $^{13,16}$

$$
\tilde{g}_{\mu \nu} = w^2 g_{\mu \nu} \quad \text{with} \quad w = a^2 \prod_{i=1}^2 b_i^{-\delta d_i(d-1)},
$$

(2.3)

where $a$ is a constant.

On the assumptions made above, $(d+1)$-dimensional Lagrangian density $\mathcal{L}$ is obtained from the action (2.1):

$$
\mathcal{L} = \frac{1}{16 \pi G} \sqrt{-\det g_{\mu \nu}} \left[ R - \sum_{i=1}^2 d_i (\ln b_i)_d (\ln b_i)^d + \frac{1}{d-1} \sum_{i=1}^2 d_i d_i (\ln b_i)_d (\ln b_i)^d \right]
$$

$$
+ a^2 \prod_{i=1}^2 b_i^{-\delta d_i(d-1)} \sum_{j=1}^2 b_j^{-2} \tilde{R}_j - 2a^2 \prod_{i=1}^2 b_i^{-2 \delta d_i(d-1)},
$$

(2.4)

where $G = \tilde{G}/(d-1)^2 \prod_{i=1}^2 V_i$ and $V_i = \frac{1}{2} \left( \det \tilde{g}_{m_{ni}} \right)^{1/2} d^{d_i} y$. $R$ is the scalar curvature derived from $g_{\mu \nu}$. It is noted that generalization to arbitrary number of subspaces is obvious. Now we take a coordinate system with which $g_{\mu \nu}$ is written as

$$
g_{\mu \nu} = \begin{pmatrix}
-1 & 0 \\
0 & a^2(t) \tilde{g}_{mn}(x)
\end{pmatrix}.
$$

(2.5)

Then $R = 2d(\dot{a}/a) + d(d-1)(\dot{a}/a)^2 + a^{-2} \tilde{R}$ and $\sqrt{-\det g_{\mu \nu}} = a^d \sqrt{-\det \tilde{g}_{mn}}$, where $\tilde{R}$ is the scalar curvature derived from $\tilde{g}_{mn}$.

We further simplify the Lagrangian density $\mathcal{L}$ by diagonalizing its kinematical part. It is achieved by the following transformation of $\ln b_i$

$$
\begin{cases}
x_1 = \pm \sqrt{(D-2)/(d_1+d_2)}(d_1 \ln b_1 + d_2 \ln b_2), \\
x_2 = \mp \sqrt{d_1 d_2}/(d_1 + d_2) (\ln b_1 - \ln b_2).
\end{cases}
$$

(2.6)

Then, using a variable

$$
x = \sqrt{d(d-1)} \ln a
$$

(2.7)
the Lagrangian \( L = \int \mathcal{L} d^d x \) is expressed as follows:

\[
L = \bar{V} \kappa^2 \exp(\eta x)[ -\frac{1}{2} \dot{x}^2 + \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) - U(x, x_i)],
\]

where

\[
U(x, x_i) = U_0(x) + U_1(x_i).
\]

Here the upper signs in (2.6) are taken and

\[
\begin{align*}
U_0(x) &= (\frac{1}{2}) \tilde{R} \exp\{-2x/\sqrt{d(d-1)}\}, \\
U_1(x_i) &= a \tilde{R} \exp\{-2\sqrt{(d_1 + d_2)/(d-1)(D-2)} x_i\} \\
& - \frac{1}{2} \tilde{R}_1 \exp\{-2\eta \left( x_1 - \frac{X}{d_1} x_2 \right)\} + \tilde{R}_2 \exp\{-2\eta \left( x_1 + \frac{X}{d_1} x_2 \right)\}.
\end{align*}
\]

In the above equations \( \bar{V} \) is the volume of the \( d \)-dimensional subspace,

\[
\eta = \sqrt{d/(d-1)}, \quad \lambda = \sqrt{(D-2)/(d-1)(d_1 + d_2)}, \\
\chi = \sqrt{d_1 d_2 (d-1)/(D-2)} \quad \text{and} \quad \kappa = \sqrt{8\pi M_P^{-(d-1)/2}},
\]

where \( M_P = G^{-1/(d-1)} \) is the Planck mass.

Equations of motion derived from the Lagrangian (2.8) are written as

\[
\begin{align*}
\dot{x} + \eta \dot{x}^2 - 2\eta \left( U - \frac{1}{d} U_0 \right) &= 0, \\
\dot{x}_i + \eta \dot{x}_i + \partial U/\partial x_i &= 0. \quad (i = 1, 2)
\end{align*}
\]

The Hamiltonian constraint is expressed as

\[
\frac{1}{2} (-\dot{x}^2 + \dot{x}_1^2 + \dot{x}_2^2) + U = 0.
\]

Equations (2.12) together with Eq. (2.13) constitute the field equations of the system under consideration.

§ 3. Approximate solutions

In this section, we solve field equations (2.12) and (2.13) approximately up to the end of inflation following the procedure given in Ref. 15). We divide the time interval into two periods. The first is the initial period, the so-called stiff region, in which it is assumed that \( \dot{x} \) is large and \( (1/2) (\dot{x}_1^2 + \dot{x}_2^2) \gg |U(x, x_i)| \) and that the curvature term is neglected. It is shown in § 5 that, for sufficient inflation, the maximum of \( U(x, x_i) \) is much smaller than the initial value of \( (1/2) (\dot{x}_1^2 + \dot{x}_2^2) = 8\pi M_P^{-2} \) from (3.4) and (3.5). The second is the inflationary period where \( U(x, x_i) \gg (1/2)(\dot{x}_1^2 + \dot{x}_2^2) \). After solving the field equations in each period, we connect the two sets of solutions (see § 4).

(i) Initial period

In this period, it is assumed that the expansion rate \( \dot{a}/a \) is large and that \( (1/2)(\dot{x}_1^2 + \dot{x}_2^2) \gg U_i(x_i) \) (the stiff region). We also assume that the curvature term is negli-
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The equations of motion become

\[
\begin{align*}
\ddot{x} + \eta \dot{x}^2 &= 0, \\
\ddot{x}_i + \eta \dot{x}_i \dot{x}_i &= 0 \quad (i=1, 2)
\end{align*}
\]

and the Hamiltonian constraint becomes

\[
\frac{1}{2} (\ddot{x}^2 + \dot{x}_1^2 + \dot{x}_2^2) = 0.
\]

The approximate solutions thus obtained are as follows:

\[
\begin{align*}
X &= \ln[\eta \dot{x}(0) \frac{t+1}{\eta} + x_0], \\
x_i &= (\dot{x}_i(0)/\eta \dot{x}(0)) \ln[\eta \dot{x}(0) t + 1] + x_{i0}. \quad (i=1, 2)
\end{align*}
\]

Here \(x_0, x_{i0}, \dot{x}(0)\) and \(\dot{x}_i(0)\) are constants of integration. Among these constants, there exist two relations. One is, of course, the Hamiltonian constraint at \(t=0\) and is expressed as

\[
\dot{x}(0)^2 = \dot{x}_1(0)^2 + \dot{x}_2(0)^2.
\]

The other is the quantum boundary (QB) condition. The QB is defined to be a hypersurface in a phase space on which the energy density of the scalar fields \(\rho = (\dot{x}^2/2)(\ddot{x}_1^2 + \ddot{x}_2^2) + U_i(x_i)\) is equal to the Planck density. If we choose \(t=0\) on the QB, the condition is expressed, using (3·4) as

\[
\frac{1}{2} \dot{x}(0)^2 = 8\pi M_p^2.
\]

Since \(\rho\) is a monotonically decreasing function of the time, the QB is uniquely determined and has a definite physical meaning.

Now if we put the time \(t=t_i\) when \((1/2)(\ddot{x}_1^2 + \ddot{x}_2^2) = U_i(x_i)\), the approximate solutions (3·3) are valid only for \(t < t_i\). Here \(t_i\) is given by

\[
4\sqrt{\pi M_p t} = \eta^2 \sigma_0^{1/2(1+\gamma \cos \theta)} \exp \left[ \frac{\sqrt{(d_1 + d_2)/(d-1)(D-2)}}{1+\gamma \cos \theta} x_{i0} \right],
\]

where

\[
\sigma_0 = 16\pi M_p^2 / 2\bar{\Lambda} a, \quad \gamma = 1 - \sqrt{(d_1 + d_2)/d(D-2)} \quad \text{and} \quad \cos \theta = \dot{x}_1(0)/\dot{x}(0).
\]

(ii) Inflationary period

Inflation, i.e., nearly exponential expansion, is caused by the large values of the potential \(U_i(x_i)\) near its local maximum. It is easily shown that \(U_i\) has a local maximum only when \(\bar{\Lambda}\) as well as \(\bar{R}_i (i=1, 2)\) are positive, which we assume hereafter. Putting \(x_i = \bar{x}_i (i=1, 2)\) at the local maximum of \(U_i\), we approximate \(U_i\) as

\[
U_i(x_i) \approx U_i(\bar{x}_i) + \frac{1}{2} \sum_{i,j=1}^{2} \left. \frac{\partial^2 U_i}{\partial x_i \partial x_j} \right|_{x_i = \bar{x}_i} (x_i - \bar{x}_i)(x_j - \bar{x}_j).
\]

\(\bar{x}_i\) are given by
\[ \left\{ \begin{array}{l} \bar{x}_1 = \frac{(D-2)}{2\lambda(d-1)} \ln[(D-2)\frac{(\bar{R}_1/d_1)^{d_1/(d_1+d_2)}}{(\bar{R}_2/d_2)^{d_2/(d_1+d_2)}} / 2\bar{A}], \\ \bar{x}_2 = (\lambda^2/2) \ln[d_1\bar{R}_2/d_2\bar{R}_1]. \end{array} \] (3.9)

From (3.8), (3.9) and (2.10), we have
\[ U_i(x_i) \approx \frac{1}{2} H_0^2[1 - 2\sum_{i=1}^2 (x_i - \bar{x}_i)^2/(d-1)] , \] (3.10)

where
\[ H_0^2 = 2\bar{A}a \left( \frac{d-1}{D-2} \right) \left( \frac{d_1}{\bar{R}_1} \right)^{d_1/(d-1)} \left( \frac{d_2}{\bar{R}_2} \right)^{d_2/(d-1)}. \] (3.11)

We also assume that the curvature term is negligible as in the initial period.

In this period, \( U_i \approx U \) is large compared to \( \dot{x}_i^2 \) and is nearly constant (nearly exponential expansion) as is seen from (2.13). However, if we put \( \dot{x}_i^2 \) to be constant in the equation for \( x \) (2.12), and neglect the \( \dot{x}_i^2 \) terms in (2.13) completely, then we obtain an exactly exponential expansion and cannot obtain any information on the duration of inflation. So we retain \( \dot{x}_i^2 \) in (2.12) as a function of time. On the other hand, \( \dot{x} \) in the equations for \( x_i \) (2.12) is approximated to be a constant, which is determined to be \( H_0 \) from (2.13).

Thus the equations of motion (2.12) become
\[ \left\{ \begin{array}{l} \ddot{x} + \eta(\dot{x}_1^2 + \dot{x}_2^2) = 0 , \\ \ddot{x}_i + \eta H_0 \dot{x}_i - \frac{1}{2} H_0^2 \left[ 1 - 2\sum_{i=1}^2 (x_i - \bar{x}_i)^2/(d-1) \right] = 0 , \end{array} \right. \] (3.12)

and the Hamiltonian constraint (2.13) becomes
\[ \frac{1}{2} (-\dot{x}^2 + \dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} H_0^2 \left[ 1 - 2\sum_{i=1}^2 (x_i - \bar{x}_i)^2/(d-1) \right] = 0 . \] (3.13)

Equation (3.13) has been used to derive the first one of Eq. (3.12). First we solve for \( x_i \). Substituting the solutions into the equation for \( x \), we obtain the solution for \( x \).

The approximate solutions are expressed as follows:
\[ x = H_0 t + C - \frac{1}{2} \eta \exp(-\eta H_0 t) \left[ A^2 \exp(\eta A H_0 t) + B^2 \exp(-\eta A H_0 t) - \frac{16}{d} A \cdot B \right] , \] (3.14a)
\[ x_i = \exp(-\eta H_0 t/2) [A_i \exp(\eta A H_0 t/2) + B_i \exp(-\eta A H_0 t/2)] + \bar{x}_i . \] (3.14b)

Here \( A_i, B_i \) \((i=1, 2)\) and \( C \) are constants of integration and \( A = \sqrt{1+8/d} \). For \( A \) and \( B \), vector notation is used:
\[ A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} , \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} . \] (3.15)

In the solution for \( x \) (3.14a), the number of constants of integration has been reduced by using the Hamiltonian constraint.
The approximate equations of motion for $x_i$ (3·12) and Hamiltonian constraint (3·13) are valid only when the approximation for $U_i(x_i)$, (3·8), is valid, i.e., only when the higher order terms in the power expansion of $U_i(x_i)$ are negligible. This leads to a finite time interval $t_a < t < \bar{t}_a$ in which the solutions (3·14a, b) are valid. Here $t_a$ and $\bar{t}_a$ are estimated from the equation

$$\left| \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 U_i}{\partial x_i \partial x_j} (x_i - \bar{x}_i)(x_j - \bar{x}_j) \right| = \left| \frac{1}{3!} \sum_{i,j,k=1}^{3} \frac{\partial^3 U_i}{\partial x_i \partial x_j \partial x_k} (x_i - \bar{x}_i)(x_j - \bar{x}_j)(x_k - \bar{x}_k) \right|,$$

where the partial derivatives are evaluated at the local maximum of $U_i(x_i)$. Then we obtain

$$\begin{align*}
  t_a &= 2H_0^{-1} \left( \frac{1}{\eta \Delta} \ln \left[ 2F^{(3)}(B)/3B^2 \right] \right), \\
  \bar{t}_a &= 2H_0^{-1} \left( \frac{1}{\eta \Delta} \ln \left[ 3A^2/2F^{(3)}(A) \right] \right), \quad (3·16)
\end{align*}$$

where

$$F^{(3)}(A) = \left\{ \frac{1}{3} \right\} \left[ (d_1 + d_2)(D-2) + 2\sqrt{(d_1 + d_2)(d_1 + d_2)} \right] A_1^3 \left( (d_1 + d_2)^2 \right) A_2^2 + 3\sqrt{(D-2)/(d_1 + d_2)} A_1 A_2^2. \quad (3·17)$$

We now estimate the duration of inflation by approximating that the inflationary period is the one during which $U_i(x_i) \geq (1/2)(\dot{x}_1^2 + \dot{x}_2^2)$ holds. Denoting the time interval as $t_i \leq t \leq \bar{t}_i$, we have

$$\begin{align*}
  t_i &= H_0^{-1} \left[ 1/\eta (\Delta + 1) \right] \ln \left[ \eta^2 \Delta (\Delta + 1) B^2/2 \right], \\
  \bar{t}_i &= H_0^{-1} \left[ 1/\eta (\Delta - 1) \right] \ln \left[ 2/\eta^2 \Delta (\Delta - 1) A^2 \right]. \quad (3·18)
\end{align*}$$

The above approximation may be permitted due to the result $\dot{x}(t_i) \approx \dot{x}(\bar{t}_i) \approx H_0$, where $H_0$ is the value of $\dot{x}$ at local maximum of $U_i(x_i)$. However, it is necessary that $t_i > t_a$ and $\bar{t}_i < \bar{t}_a$. This is confirmed for $d=3$.

§ 4. Probability of inflation

In this section, we first construct the measure on the initial data surface. Then we apply it to our model and give an explicit expression for the probability of sufficient inflation in a form of an integral. The region of integration is also given explicitly.

(i) Construction of the measure

In order to obtain the probability of inflation, we need a measure for a set of the solutions of the field equations. A solution corresponds uniquely to a set of initial data which is represented by a point on the initial data surface. So if a measure on the initial data surface is given, we can define the probability of inflation.*)

As the initial data surface, we choose a hypersurface in the phase space of the

*) If we require the measure to be invariant along the Hamiltonian flow, we have to find a measure whose Lie derivative along the flow vanishes and whose initial value is given on the initial data surface.
system that satisfies both the Hamiltonian constraint and the quantum boundary (QB) condition. As explained in § 3, the QB has a definite physical meaning. In addition, for larger energy density than the Planck density, the classical description would not be valid. So it is natural to impose the QB condition for the initial data surface. Thus it is reasonable to assume that the probability of inflation is estimated by using the data on the QB.

The distribution of such data would, in principle, be calculable if the quantum theory valid inside the QB were available. So far such a quantum theory has not been established. At present, it is therefore desirable that the measure is defined independently of the dynamics in the classical regime or in a geometrical way as far as possible. In this sense, the so-called canonical measure is favourable. However, it leads to an infinite total measure. A probability is given by the ratio of the relevant measure to the total measure. So, a finite probability comes only from a ratio of infinite measures. Therefore, an ad hoc limiting procedure has to be defined to obtain a probability. The procedure has not been given in a model independent way. In other words, a measure leading to a finite total measure has not been found in a purely geometrical manner.

Therefore we relax partly the purely geometrical nature of the measure and introduce physical considerations to obtain a measure which gives a finite total measure. The only non-geometrical point is that we start from a set of dynamical variables which are natural from a physical point of view. The resulting measure is, however, invariant under the change of dynamical variables. The construction of the measure is so systematic and geometrical that treatment of the model dependence is also geometrical. We first define a measure in the phase space in a geometrical way. This is of course model independent except for the choice of the natural dynamical variables. The dependence on the model enters through the Hamiltonian constraint and the QB condition which define the initial data surface. The reduction of the above measure to the initial data surface is also carried out in a geometrical way. We present the construction of such a measure in the following.

To begin with, we assume that a set of physically natural dynamical variables are chosen. Then we compactify the phase space by adding a point \( \infty \) and maps the resulting space onto a sphere \( S^n \) (\( n \) is the dimension of the phase space): stereographic projection. In terms of the Cartesian coordinate of the Euclidean space \( E_{n+1} \) into which the sphere \( S^n \) is imbedded, the north pole \( N \) of \( S^n \) can be given as \((0,0,\ldots,a)\) where \( a \) is the radius of \( S^n \), and a point of the phase space can be expressed as \((X_1, X_2, \ldots, X_n, 0)\). Using the spherical coordinates \((r, \theta_1, \ldots, \theta_n)\) of \( E_{n+1} \), the stereographic projection can be given as

\[
\begin{align*}
\sin \theta_1 &= \frac{X_1}{R_2}, & \theta_n &= 2 \cot^{-1}(R_n/a) \quad \text{and} \\
\sin \theta_k &= \frac{R_k}{R_{k+1}} \quad \text{or} \quad \cos \theta_k = \frac{X_{k+1}}{R_{k+1}}, & (k=1, \ldots, n-1)
\end{align*}
\]

where

\[
R_k = \left[ \sum_{j=1}^{k} X_j^2 \right]^{1/2}.
\]
The inverse mapping is expressed as
\[
\begin{align*}
X_1 &= \sin \theta \sin \phi_1 \sin \phi_2/(1 - \cos \theta), \\
X_k &= \sin \theta \sin \phi_{k-1} \cos \phi_k/(1 - \cos \theta). \quad (k=2, \ldots, n)
\end{align*}
\] (4.3)

As is well known, the Riemannian volume element of a manifold is invariant under the local coordinate transformations and is given as \(|\det g_{\alpha\beta}|^{1/2} \Pi d\xi^\alpha\) in terms of the metric \(g_{\alpha\beta}\) and the local coordinate \(\xi^\alpha\). In the case of \(S^n\), such a measure (denoting it as \(d^nQ\)) is given, in terms of the spherical coordinates, by
\[
d^nQ = a^n \sin^{n-1} \theta \sin^{n-2} \theta_1 \cdots \sin \theta_{n-1} d\theta d\phi_1 d\phi_2 \cdots d\theta_n.
\] (4.4)

From (4.4) and the stereographic projection (4.1), a measure in the phase space (denoting it as \(d^nQ\)) is derived as
\[
d^nQ = [2a^2/(R^2 + a^2)]^n dX_1 \cdots dX_n.
\] (4.5)

Now we induce the measure on the initial data surface using (4.5). The initial data surface is characterized by the Hamiltonian constraint and the QB condition. If we put \(Y^i(X_i) = H\) and \(Y^2(X_i) = \rho - M_{\alpha}^2\), the metric \(g_{ij}\) of the initial data surface which is characterized by the constraints \(Y^i(X_i) = 0\) (\(\vec{a} = 1, 2\)), is given by
\[
\bar{g}_{ij} = \frac{4a^4}{(R^2 + a^2)^2} \left[ \delta_{ij} + \frac{2}{\delta_{\alpha \beta}} \left( \bar{A}^{-1} \right)^{\alpha \beta} \frac{\partial Y^\alpha}{\partial X_i} \frac{\partial Y^\beta}{\partial X_j} \right].
\] (4.6)

Here
\[
\bar{A}^{-1} = \sum_{i=1}^{2} \frac{\partial Y^i}{\partial X^i},
\] (4.7)

where the coordinates of the phase space are denoted as \((X_1, X_2, \ldots, X_{n-2}, X^1, X^2)\). Then the invariant measure on the initial data surface \(d^{n-2}\omega\) is given by
\[
d^{n-2}\omega = \sqrt{\det \bar{g}_{ij}} dX_1 \cdots dX_{n-2}.
\] (4.8)

We calculate the determinant \(\det \bar{g}_{ij}\) by making use of the well-known formula which holds for an arbitrary number \(x\), \(s\)-dimensional unit matrix \(E\) and an arbitrary \(n \times n\) matrix \(A\). It is written as
\[
f_A(x) = \det(xE + A) = \sum_{k=0}^{n} c_k x^{n-k},
\] (4.9)

where
\[
c_0 = 1, \quad c_1 = \text{Tr} A \quad \text{and} \quad c_k = \sum_{i_1 < i_2 < \cdots < i_k} \left| \begin{array}{cccc} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ \vdots & & & \vdots \\ a_{i_k i_1} & \cdots & \cdots & a_{i_k i_k} \end{array} \right|,
\] (4.10)

where \(a_{ij}\) is the matrix element of \(A\). Then we obtain
\[
\det \bar{g}_{ij} = \left( \frac{2a^2}{R^2 + a^2} \right)^{(n-2)} \sum_{k=0}^{n-2} c_k.
\] (4.11)
In calculating $c_k$, the matrix element $a_{ij}$ is set equal to the second term in the bracket of (4.6). Thus we obtain an explicit expression for $d^{n-2} \omega$ given by (4.8) that we propose as the measure on the initial data surface with which the probability of inflation should be calculated.

In our model, we have six dynamical degrees of freedom, i.e., $n = 6$. For $(X_1, \ldots, X_6)$ we choose $x_i, x_i (i = 1, 2)$ and the momenta $p_i, p_i$ canonically conjugate to them, in terms of which the descriptions are very simple. The momenta $p$ and $p_i$ are derived from (2.8) as follows:

\[
\begin{aligned}
\dot{p} &= \partial L / \partial \dot{x} = - \tilde{V} \kappa^2 \exp(\eta x) \dot{x}, \\
\dot{p}_i &= \partial L / \partial \dot{x}_i = \tilde{V} \kappa^2 \exp(\eta x) \dot{x}_i. \quad (i = 1, 2)
\end{aligned}
\]  

(4.12)

In terms of these canonical variables, the Hamiltonian constraint and the QB condition are expressed as follows:

\[
Y^1 \equiv H = \tilde{V}^{-1} \kappa^2 \exp(-\eta x)(-\dot{p}^2 + \sum_{i=1}^{2} p_i^2)/2 \\
+ \tilde{V} \kappa^2 \exp(\eta x) U(x, x_i) = 0, \quad (4.13a)
\]

\[
Y^2 \equiv \rho - M \kappa^2 = \kappa^2 \left[ \frac{1}{2} \tilde{V}^{-2} \kappa^4 \exp(-2\eta x) \sum_{i=1}^{2} p_i^2 + U_i(x_i) \right] - M \kappa^2 = 0. \quad (4.13b)
\]

It is seen that both $p_1$ and $p_2$ cannot be eliminated simultaneously as they appear only in the combination $p_1^2 + p_2^2$. Therefore we choose $X_1 = x, X_2 = x_1, X_3 = x_2, X_4 = p_1, X_5 = p$ and $X_6 = p_2$. Then the matrix $\tilde{A}^{-1}$ in (4.6) is expressed as

\[
\tilde{A}^{-1} = \tilde{V}^{-4} \kappa^2 \exp(6\eta x) p^{-2} \begin{pmatrix} \tilde{V}^{-2} \exp(-2\eta x) & - \tilde{V}^{-1} \exp(-\eta x) \\ - \tilde{V}^{-1} \exp(-\eta x) & 1 + \rho^2 / p_2^2 \end{pmatrix}, \quad (4.14)
\]

where $p^2$ and $p_2^2$ are written as follows by using (4.13a, b) and by approximating $U_0 \approx 0$ and $U_i \ll (1/2)(\dot{x}_1^2 + \dot{x}_2^2)$:

\[
\begin{aligned}
\dot{p}^2 &= K \exp(2\eta x), \\
\dot{p}_2^2 &= - \dot{p}_1^2 + K \exp(2\eta x),
\end{aligned} \quad (4.15)
\]

where $K = 16 \pi \tilde{V} \kappa^4 M \kappa^2$. From (4.13) ~ (4.15) and (4.6), we obtain $a_{ij} (i, j = 1 \sim 4)$, then from (4.11) and (4.10), we can calculate det $\tilde{g}_{ij}$. Finally we obtain $d^{n-2} \omega$ of (4.8) for our model:

\[
d^{4} \omega = \left[ 2 \dot{\sigma}^2 / \left( \dot{x}_0^2 + \sum_{i=1}^{2} x_i^2 + 2 \dot{K} \exp(\eta x_0) + \dot{a}^2 \right) \right]^{4} \\
\times \left[ K \left( 2 \dot{\eta}^2 \dot{K} \exp(\eta x_0) + 1 \right) \exp(\eta x_0) \right]^{1/2} dx_0 dx_1 dx_2 d\theta.
\]  

(4.16)

The probability of sufficient inflation from our model is obtained by integrating (4.16) over a region which will be determined later.

(ii) Condition for sufficient inflation

Now we examine the condition for sufficient inflation. Whether a solution is inflationary or not is determined by the values of the constants of integration for the
solution. We approximate the expansion factor $Z$ during the inflationary period by $a(T_f)/a(t_i)$. Then we have

$$Z = C(d)f(A, B),$$

(4·17)

where

$$C(d) = 2^{-(d+3)/8}\sqrt{d+8}((d-1)/\sqrt{d+8})^{d/4}\exp(-1/8d),$$

$$f(A, B) = (A^2)^{-1/d(d-1)}(B^2)^{-1/d(d+1)}.$$  

(4·18)

The constant $C(d)$ is of order 1 (e.g., $C(3)=0.91$). Equation (4·17) gives the upper limit for $Z$. The condition for sufficient inflation can be taken as $Z > 10^{30}$. Then from (4·17) and (4·18), we have

$$10^{-30}C(d) > (A^2)^{(d+1)/8}(B^2)^{(d-1)/8}.$$  

(4·19)

We must translate this condition into the one on the initial data surface. In order to carry out the translation, we connect the two sets of solutions (3·3) and (3·14). First, we require that $t_i$ in (3·6) is equal to $t_i$ in (3·18), since both of them are defined by the same condition, $(1/2)(\dot{x}_1^2 + \dot{x}_2^2) = U_1(x_i)$. This requirement corresponds to the synchronization of clocks used in the initial and the inflationary periods. Thus we obtain the relation

$$\ln[\eta^2\Delta(A+1)B^2]/2 = (H_0/4\sqrt{\pi}M_P)\sigma_0^{1/2(1+\gamma \cos \theta)}((A+1)/\eta)$$

$$\times \exp[(d_1 + d_2)/(d-1)(D-2)x_{10}/(1+\gamma \cos \theta)].$$  

(4·20)

Second, at $t = t_i = t_i$, we connect the two sets of solutions (3·3) and (3·14) by requiring that the values of $x$, $x_i$ ($i=1, 2$) and $\dot{x}$, given by the two solutions are the same. These conditions and the relation (4·17) constitute the necessary and sufficient conditions to relate constants of integration in the initial and the inflationary periods. The relations thus obtained are somewhat complicated. However, we obtain $A^2 < 1$ by using the inequalities (4·19) and $A^2 < B^2$ which is derived from $t_i < t_i$. Furthermore, at $t = t_i$, $\dot{x}$ is much smaller than $x(0)$, i.e., $\dot{x}(0)t_i \gg 1$. Then, approximate expressions for $A^2$ and $B^2$ are obtained as follows,

$$A^2 = \exp[-(\tilde{H}_0/\sqrt{\sigma_0})((A-1)/\eta)I(\theta, x_{10})]/\eta^4I^2(\theta, x_{10}),$$

$$B^2 = 2\exp[(\tilde{H}_0/\sqrt{\sigma_0})((A+1)/\eta)I(\theta, x_{10})]/\eta^2\Delta(A+1),$$

(4·21)

where $\tilde{H}_0 = H_0/\sqrt{2\Lambda a}$ and

$$I(\theta, x_{10}) = \sigma_0^{1/2(1+\gamma \cos \theta)}\exp[\sqrt{(d_1 + d_2)/(d-1)(D-2)}x_{10}/(1+\gamma \cos \theta)].$$  

(4·22)

From Eqs. (4·20), (4·17) and (4·18), we can express the expansion factor $Z$ in terms of the initial values:

$$Z = C d I(\theta, x_{10})^{(d+1)/4},$$

(4·23)

where

$$C d = \exp(-1/8d)[2\eta^2/\Delta(\Delta-1)]^{(d+1)/8}.$$  

(4·24)
Thus the condition for sufficient inflation (4.19) is expressed as

\[ \sqrt{(d_1 + d_2)/(d-1)(D-2)} x_{10} + \frac{1}{2} \ln \sigma_0 > \frac{120 \ln 10}{A+1} (1 + \gamma \cos \theta). \]  

(4.25)

Now it is easily seen that if the solutions are connected, \( \dot{x}_i \) \( (i=1, 2) \) are positive at the end of the inflationary period unless the value of \( \theta \) lies in a range

\[ \pi \leq \theta \leq \frac{3}{2} \pi. \]  

(4.26)

Therefore, if we do not assume any mechanisms that stop the expansion of the internal subspace after inflation, we should restrict the range of \( \theta \) to (4.26).

The probability of inflation is given by integrating the measure (4.16). The ranges of integration are restricted by the inequalities (4.25), (4.26) and

\[ -(d_2/2\lambda \chi)[2\lambda x_{10} + \ln \sigma_1] < x_{20} < (d_1/2\lambda \chi)[2\lambda x_{10} + \ln \sigma_1]. \]  

(4.27)

Here \( \sigma_i = 16\pi M_p^2/\tilde{R}_i a_i \) \( (i=1, 2) \). Furthermore

\[ x_{10} \geq -[d_1 \ln \sigma_1 + d_2 \ln \sigma_2]/2\lambda (d_1 + d_2) + \ln N, \]  

(4.28)

where \( N \) is a number of the order of 10 or larger. Inequalities (4.27) and (4.28) come from the condition that initially \( (1/2)(\dot{x}_1^2 + \dot{x}_2^2) \geq U_i(x_i) \).

§ 5. Results and discussion

Our model contains two kinds of parameters. One of them expresses the dimensionality of the subspaces in the direct product composing the higher-dimensional spacetime: \( d, d_1 \) and \( d_2 \). We take the external space to be three dimensional, i.e., \( d = 3 \). The dimensions of the internal subspaces \( d_1 \) and \( d_2 \) are not fixed uniquely. Another kind of parameters sets the scale of our model: \( \tilde{\Lambda}, \tilde{R}_i \) and \( \tilde{V} \). However, these parameters appear only in dimensionless combinations \( \sigma_0, \sigma_i \) \( (i=1, 2) \) and \( K \), in the measure (4.16), in the condition for sufficient inflation (4.25) and in the equations for the boundary of integration (4.27) and (4.28), as they should. These dimensionless combinations are assumed to take several sets of values in numerical calculations.

It should be noted that our model leads to a finite probability of sufficient inflation as suggested in Ref. 15), since the probability is given by integrating the measure (4.16) over the region restricted by (4.25)~(4.28). This is in contrast to the case, investigated in Ref. 15), in which there exists only one internal subspace. In that case the probability of sufficient inflation is zero in our approximation. Thus our results show that inflation occurs more naturally in the case of two internal subspaces than the case of only one. It remains to be clarified whether inflation occurs more naturally in a spacetime with more internal subspaces.

In numerical calculations, we first fix the values of \( d_1 \) and \( d_2 \). Two typical cases are taken:

(i) symmetric case in which \( d_1 = d_2 = 5 \)

(ii) unsymmetric case in which \( d_1 = 6 \) and \( d_2 = 16 \).
Table I. The probability (percentage) of sufficient inflation for various sets of values of the
parameters of our model. We took $\sigma_i=\sigma$ and denoted them as $\sigma_i$. In case (i) $d_1=d_2=5$, and
in case (ii) $d_1=6$ and $d_2=16$.

<table>
<thead>
<tr>
<th>$\sigma_0=10^{10}$</th>
<th>$\sigma_1=10^{10}$</th>
<th>$\sigma_0=10^{10}$</th>
<th>$\sigma_1=10^{10}$</th>
<th>$\sigma_0=10^{10}$</th>
<th>$\sigma_1=10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$1.2\times10^{-7}$</td>
<td>$1.4\times10^{-4}$</td>
<td>$5.8\times10^{-2}$</td>
<td>$6.5$</td>
<td>$7.2$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$1.1\times10^{-7}$</td>
<td>$5.9\times10^{-5}$</td>
<td>$4.5\times10^{-2}$</td>
<td>$2.7$</td>
<td>$2.7$</td>
</tr>
</tbody>
</table>

As to the parameters $\sigma_0$, $\sigma_1$ and $\sigma_2$, we have taken various sets of values. The results,
which exhibit the characteristic features of our model, are shown in Table I.

From Table I we see the following. For each case, we obtain a finite probability
of sufficient inflation. The probability increases as $\sigma_0$ and /or $\sigma_1$ increases. In order
for the sufficient inflation to be natural, the probability should be at least a few
percent. Such a probability requires the value of the parameter $\sigma_0$ to be as large as
$10^{46}\sim10^{50}$ or larger. The inflationary period lasts approximately $\sqrt{\sigma_0/t_p}$, where $t_p$ is
the Planck time. This is fairly a long time interval for the above values of $\sigma_0$. In
other words, there might have been a long time between the quantum cosmology
regime and the stage to which the standard cosmological model is applicable. For
other sets of values for the parameters $d_1$, $d_2$, we obtain essentially the same results
as are shown in Table I. Thus we can say that the sufficient inflation is not an
accidental event, but the one with a reasonable probability. We show typical trajec-
tories which exhibit sufficient inflation in Fig. 1. The parameters are taken to be $d_1$
$=d_2=3$ and $\sigma_0=\sigma_1=10^{50}$.

Finally, we remark on some aspects of our model. After inflation, both of the
internal subspaces continue to contract, so that their potential energies decrease
exponentially as is seen from (2.10). The universe can, therefore, be reheated by
conversion of the energy of the scalar field to that of the radiation, i.e., gauge fields.

Interactions between the scalar fields
and the gauge fields are determined
uniquely through the standard procedure
of the Kaluza-Klein theory. In the clas-
sical theory, the temperature would
increase exponentially and unboundedly.
Thus the estimation of the temperature
after inflation requires the quantum the-
ory. It is expected that the radiations
are emitted when the scalar fields fall to
energy eigenstates. This problem will
be investigated separately. We adopted
our model by taking into account the
unified theoretical point of view, as
noted in § 1. As a unified theory, our
model may be ambiguous due to many
parameters contained in it. We also
restricted the number of the internal

---

Fig. 1. Typical trajectories exhibiting sufficient inflation. The solid line represents $x$, dashed
line represents $x_1$ and dash-dotted line represents $x_2$. $t\equiv t-t_i$ is used instead of $t$ for $t
>t_i$, i.e., for the inflationary period.
subspaces to two only for simplicity. However, it might be possible to obtain some clues for determining the dimensions of the internal subspaces by quantum theoretical treatment as made in the string theory. On the other hand, if the dimensional parameters $\hat{R}_1$ and $\hat{R}_2$ (or $\sigma_1$ and $\sigma_2$) have some relevance to the low energy particle physics, one of the possibilities is that they correspond to the parameters characterizing the energy scales of the spontaneous breaking of the gauge symmetries associated with internal subspaces. Unfortunately, the probability of inflation is not so sensitive to these as $\hat{A}$ that their values are restricted meaningfully.

If there are two kinds of internal subspaces as our results require, it follows that there exist two kinds of gauge fields. Since their coupling constants are not required to be the same, there could be two kinds of matter whose gravitational couplings are different. The strange gauge field might be a candidate for the dark matter, which is in contrast with the proposition that the scalar fields, the inflaton fields, are the candidates for the dark matter.$^{16}$

Acknowledgements

The authors would like to thank Professor U. Furukane for encouragement. They would also like to thank Professor K. Miyatani for reading the manuscript.

References

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