CORRIGENDA

MACKEY FORMULA IN TYPE A

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The author recently noticed two errors in his paper [3] (from which we keep all the notation). They concern Theorem 4.1.1 and Formulas (5.1.7) and (5.1.8); however, they do not affect the validity of all other results in [3] as is explained in this note.

1. About formulas (5.1.7) and (5.1.8) in [3]

The sign ‘+’ in these formulas must be changed to ‘−’. This has no consequence concerning the results of [3] since both formulas are used for the induction argument: in each case where they are used, all the terms involved are equal to 0. Because of these errors, we provide here a complete proof for both formulas.

**Proposition 1.** Let \( P, P', Q \) and \( Q' \) be four parabolic subgroups of \( G \) and let \( L, L', M \) and \( M' \) be F-stable Levi subgroups of \( P, P', Q \) and \( Q' \) respectively. We assume that \( P \subseteq P' \), \( L \subseteq L' \), \( Q \subseteq Q' \) and \( M \subseteq M' \). Then

\[
\Delta^G_{L \subseteq P, M \subseteq Q} = \Delta^G_{L \subseteq P, M' \subseteq Q'} \circ R^M_{M \subseteq Q \cap M'} + \sum_{x \in L' \setminus \delta_G(L, M')} \Delta^L_{L' \cap \delta_G(L, M') \subseteq L' \cap M'} \circ (\text{ad} x)_{M'}.
\]

(b) \[
\Delta^G_{L \subseteq P, M \subseteq Q} = \Delta^G_{L' \subseteq P \cap L', M \subseteq Q} \circ R^M_{L' \subseteq P \cap L', L' \subseteq L' \cap M} + \sum_{x \in L' \setminus \delta_G(L, M')} \Delta^L_{L' \cap \delta_G(L, M') \subseteq L' \cap M} \circ (\text{ad} x)_{M'}.
\]

(c) \[
\Delta^G_{L \subseteq P, M \subseteq Q} = \Delta^G_{L' \subseteq P \cap L', M' \subseteq Q'} \circ R^M_{M \subseteq Q \cap M'} + \sum_{x \in L' \setminus \delta_G(L, M')} \Delta^L_{L' \cap \delta_G(L, M') \subseteq L' \cap M} \circ (\text{ad} x)_{M'}.
\]

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Proof. Note that (b) follows from (a) by adjunction and that (c) follows by applying (a) and (b) successively. Now, let us prove (a). Let $\Delta_0$ denote the right-hand side of the equality (a). By definition of the $\Delta$-maps, we easily get

$$
\Delta_0 = \gamma^G_{L_{\mathbb{C}}P} \circ \eta^G_{M_{\mathbb{C}}Q} \\
\quad + \sum_{g \in L^F \setminus \mathcal{G}(L, M')^F/M^F} R^L_{L \cap M' \subset L \cap M' \cap Q'} \circ \gamma^M_{L \cap M' \subset P \cap M'} \circ (\text{ad } g)_{M'} \\
\quad \cdot \eta^M_{M_{\mathbb{C}}Q \cap M'} + R^L_{L \cap M' \subset L \cap M' \cap Q'} \circ \gamma^M_{L \cap M' \subset P \cap M'} \circ (\text{ad } g)_{M'} \circ \eta^M_{M_{\mathbb{C}}Q \cap M'} \\
\quad - \sum_{g \in L^F \setminus \mathcal{G}(L, M')^F/M^F \setminus \mathcal{G}(L \cap M')^F/M^F} R^L_{L \cap M_{\mathbb{C}}L \cap M' \cap Q} \circ \eta^M_{M_{\mathbb{C}}P \cap M'.}
$$

Therefore,

$$
\Delta_0 = \gamma^G_{L_{\mathbb{C}}P} \circ \eta^G_{M_{\mathbb{C}}Q} \\
\quad - \sum_{g \in L^F \setminus \mathcal{G}(L, M')^F/M^F \setminus \mathcal{G}(L \cap M')^F/M^F} R^L_{L \cap M' \subset L \cap M' \cap Q} \circ \eta^M_{M_{\mathbb{C}}L \cap M' \cap Q} \circ \eta^M_{M_{\mathbb{C}}P \cap M'.}
$$

The argument at the end of the proof of [1, Lemma 3.2.1] completes the proof of (a). \hfill \square

2. About Theorem 4.1.1 in [3]

The second error is much more serious: Theorem 4.1.1 is false! However, its corollary 4.1.2 is still correct; it follows from Theorem 3 below. Fortunately, we use only Corollary 4.1.2 in the rest of [3] (and not Theorem 4.1.1). This means that all the other results in [3] are valid.

Our mistake in the proof of [3, Theorem 4.1.1] is the following (here we keep the notation of this theorem): it may happen that $\omega$ stabilizes a cuspidal local system but that it acts on the characteristic function by multiplication by a scalar different from 1.

Let us fix some notation. If $\ell = (C, \mathcal{L}) \in \mathcal{U}(G)^F$, we fix once and for all an isomorphism $\varphi: F^+ \mathcal{L} \cong \mathcal{L}$ and we denote by $\mathfrak{U}_i$ (or $\mathfrak{U}_i^G$ if we need to make the ambient group precise) the characteristic function associated to this isomorphism. Let $\mathcal{U}_{\text{un}}(G^F)$ denote the $\mathfrak{U}_i$-vector subspace of $\text{Class}_{\text{un}}(G^F)$ generated by the functions $\mathfrak{U}_i$ (for $\ell \in \mathcal{U}(G)^F$). Let $\text{Aut}(G, F)$ denote the group of automorphisms of the algebraic group $G$ commuting with $F$. The group $\text{Inn}(G^F)$ of inner automorphisms of $G^F$ is a normal subgroup of $\text{Aut}(G, F)$. We set $\text{Out}(G, F) = \text{Aut}(G^F)/\text{Inn}(G^F)$. It is clear that $\text{Aut}(G, F)$ (or $\text{Out}(G, F)$) acts on the vector spaces $\text{Class}_{\text{un}}(G^F)$, $\text{Cus}_{\text{un}}(G^F)$ and $\mathcal{U}_{\text{un}}(G^F)$.

We fix an $F$-stable Borel subgroup $B$ of $G$ and an $F$-stable maximal torus $T$ of $B$. Let $W$ denote the Weyl group of $G$ relative to $T$ and let $S$ be the set of simple reflections in $W$ corresponding to the choice of $B$. If $I \subset S$, we denote by $W_I$ the subgroup of $W$ generated by $I$ and we set $P_I = BW_I B$. We denote by $L_I$ the Levi subgroup of $P_I$ containing $T$. 

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If $I$ is a subset of $S$, we denote by $A_I$ the stabilizer of $(B \cap L_I, T)$ in the group $\text{Aut}(L_I, F)$. We have $\text{Aut}(G, F) = \text{Inn}(G^F) \cdot A_S$. So studying the action of $\text{Aut}(G, F)$ on $\text{Class}_{\text{uni}}(G^F)$, $\text{Cus}_{\text{uni}}(G^F)$ or $\text{SF}_{\text{uni}}(G^F)$ is equivalent to studying the action of $A_S$.

2.A. Generalized Springer correspondence

We denote by $\mathcal{P}(S)$ the set of subsets of $S$ and by $\mathcal{P}(S)_{\text{cus}}$ the set of subsets $I$ of $S$ such that $\mathcal{U}(L_I)_{\text{cus}} \neq \emptyset$. Note that $F$ and $A_S$ act on $W$, $S$, $\mathcal{P}(S)$ and $\mathcal{P}(S)_{\text{cus}}$ and that these two actions commute.

Let $\mathcal{U}'(G)$ denote the set of triples $(I, \iota, \rho)$ where $I \subseteq S$, $\iota \in \mathcal{U}(L_I)_{\text{cus}}$ and $\rho \in \text{Irr} W_G(L_I)$. The generalized Springer correspondence [6, Theorems 6.5 and 9.2] is a well-defined bijection $\psi : \mathcal{U}'(G) \rightarrow \mathcal{U}(G)$. This bijection commutes with the actions of $F$ and $A_S$.

2.B. Action of automorphisms on characteristic functions of local systems

The vector space $\text{Class}_{\text{uni}}(G^F)$ admits $(\mathcal{Y}_i)_{i \in \mathcal{U}(G^F)}$ as a basis. With respect to this basis, the action of an element of $\text{Aut}(G, F)$ is monomial. We are interested here in the way to determine the non-zero coefficients of this monomial matrix. Since the characteristic function $\mathcal{Y}_i$ (for $i = (C, L') \in \mathcal{U}(G^F)$) depends on the choice of the isomorphism $\varphi : F' \cdot L' \rightarrow L'$ that we have fixed once and for all, the interesting question is the following: if $\sigma \in \text{Aut}(G, F)$ and if $i \in \mathcal{U}(G^F)$ are such that $\sigma(i) = i$, then what is the root of unity $\xi_{i, \sigma}$ (or $\xi_{i, \sigma}^G$ if we want to emphasize the ambient group) such that $\sigma(\mathcal{Y}_i) = \xi_{i, \sigma}^G \mathcal{Y}_i$?

2.B.1. Permutation of unipotent classes in $G^F$. Let $\sigma \in \text{Aut}(G, F)$ and let $i = (C, L') \in \mathcal{U}(G^F)$ be such that $\sigma(i) = i$. We fix $u \in C^F$ such that $\mathcal{Y}_i(u) \neq 0$ and we denote by $\tilde{\xi}$ the irreducible character of $A_G(u)$ defined by $L'$. Let $\tilde{\xi}$ denote the extension of $\xi$ to the semi-direct product $A_G(u) \rtimes (F)$ (here, $\langle F \rangle$ is viewed as an infinite cyclic group) associated to the isomorphism $\varphi$.

If $a \in H^1(F, A_G(u))$, we denote by $g_a$ an element of $G$ such that $g_a^{-1} F(g_a) \in C_G(u)$ and such that the image $\tilde{\eta}$ of $g_a^{-1} F(g_a)$ in $A_G(u)$ belongs to the class $a$. We set $u_a = g_a u g_a^{-1} \in C^F$. Then $\{u_a | a \in H^1(F, A_G(u))\}$ is a set of representatives of $G^F$-conjugacy classes in $C^F$ and

$$\mathcal{Y}_i(u_a) = \tilde{\xi}(\tilde{\eta} F).$$

Therefore, if $a_{\sigma}$ denotes the unique element of $H^1(F, A_G(u))$ such that $\sigma^{-1}(u)$ is $G^F$-conjugate to $u_a$, we have

$$\xi_{i, \sigma} = \tilde{\xi}(\tilde{\eta}_{a_{\sigma}} F) / \tilde{\xi}(F).$$

2.B.2. Going down to cuspidal local systems. Let $i \in \mathcal{U}(G^F)$. We denote by $A_{S, i}$ the stabilizer of $i$ in $A_S$ and we set $\tilde{\xi}_i = \xi_i^G : A_{S, i} \rightarrow \mathcal{U}^F$, $\sigma \mapsto \xi_{i, \sigma}$. It is clear that $\xi_i$ is a linear character. Now, let $(I, i_0, \rho) = \psi^{-1}(i)$. Then $A_{S, i}$ stabilizes $L_I$, $B \cap L_I$, $T$, $i_0$ and $\rho$. Therefore, we get a morphism $A_{S, i} \rightarrow A_{I, i_0}$.

**Lemma 2.** With the above notation, we have $\xi_i^G = \text{Res}_{A_{S, i}}^{A_{I, i_0}} \xi_{i_0}^L$.

**Proof.** Let $X_{i, i_0}^L$ denote the characteristic function of the restriction to the unipotent elements of the $F$-stable perverse sheaf defined by induction from the
datum \((I, t_0)\). Then \(\sigma \in A_{S, t}\) acts on \(X^G_{I, t_0}\) by multiplication \(\xi_{I_0}(\sigma)\). Moreover,

\[
X^G_{I, t_0} = \sum_{\rho \in (\text{irr } W_G(L))^F} n_{\rho} X^G_{I, t_0, \rho},
\]

where \(X^G_{I, t_0, \rho}\) is the characteristic function of the \(F\)-stable perverse sheaf associated to \((I, t_0, \rho)\) via the generalized Springer correspondence and \(n_{\rho} \in \mathbb{Q}_{\ell}^\times\).

Therefore, if \(\rho\) is \(\sigma\)-invariant, then \(\sigma\) acts on \(X^G_{I, t_0, \rho}\) by multiplication by \(\xi_{I_0}(\sigma)\) (indeed, the family \((X^G_{I, t_0, \rho})_{\rho \in (\text{irr } W_G(L))^F}\) is linearly independent).

But \(X^G_{I, t_0, \rho}\) and \(\lambda \varphi^G_{I, t_0, \rho}\) coincide on \(C^F\) where \((C, \mathcal{L}) = \psi(I, t_0, \rho)\) for some \(\lambda \in \mathbb{Q}_{\ell}^\times\). So \(\sigma\) acts on \(\varphi^G_{I, t_0, \rho}\) by multiplication by \(\xi_{I_0}(\sigma)\). 

\[\blacksquare\]

2.B.3. About cuspidal local systems. Lemma 2 shows that, in order to determine the linear characters \(\xi_t\), we can restrict our attention to the case ofcuspidal local systems. The first result in this direction is the following.

**Lemma 3.** If \(L\) is a rational Levi subgroup of a parabolic subgroup of \(G\), then \(N^G_{G, t}(L)\) acts trivially on \(\mathcal{H}_{\text{uni}}^G(L^F)\).

**Proof.** Let \(n \in N^G_{G, t}(L)\), let \(t = (C, \mathcal{L}) \in \mathcal{H}(L)_{\text{cus}}\) and let \(v \in C^F\). Then, by [4, Proposition I.8.3], \(n v n^{-1}\) and \(v\) are \(L^F\)-conjugate. This proves Lemma 3. 

\[\blacksquare\]

We close this section with a result concerning geometrically conjugate \(F\)-stable Levi subgroups. We need some further notation. Let \(\mathcal{A}\) denote a set of representatives of \(G^F\)-conjugacy classes of \(F\)-stable Levi subgroups \(L\) of proper parabolic subgroups of \(G\) such that \(\mathcal{H}(L)_{\text{cus}}^F \neq \emptyset\). By [6, Theorem 9.2], we have the following.

**Lemma 4.** (a) If \(I, J \in \mathcal{P}(S)_{\text{cus}}\) and if there exists \(w \in W\) such that \(w I = J\), then \(I = J\).

(b) Every \(L \in \mathcal{A}\) is geometrically conjugate to a unique \(L_I\) with \(I \in \mathcal{P}(S)_{\text{cus}}^F\).

If \(I \in \mathcal{P}(S)_{\text{cus}}^F\), then the set of \(G^F\)-conjugacy classes of \(F\)-stable Levi subgroups (of parabolic subgroups of \(G\)) geometrically conjugate to \(L_I\) are parametrized by \(H^1(F, W_G(L_I))\) where \(W_G(L_I) = N_G(L_I)/L_I\). Let \(\mathcal{G}\) be the set of pairs \((I, w)\) such that \(I \in \mathcal{P}(S)_{\text{cus}}^F\), \(I \neq S\) and \(w \in H^1(F, W_G(L_I))\). We then have a bijection \(\mathcal{G} \rightarrow \mathcal{A}\) denoted by \((I, w) \mapsto L_I, w\).

We now fix in this subsection, and only in this subsection, an element \(\sigma \in A\), a subset \(I \subseteq S\), and an element \(w \in H^1(F, W_G(L_I))\) such that \(\sigma(I, w) = (I, w)\). Let \(g \in G\) be such that \(L_I, w = g L_I, w\). We set \(\tilde{w} = g^{-1} F(g) \in N_G(L_I)\) (\(\tilde{w}\) is a representative in \(N_G(L_I)\) of \(w\)). Then, conjugacy by \(g\) induces a bijection \(\mathcal{H}(L_I)_{\text{cus}}^F \simeq \mathcal{H}(L_I, w)_{\text{cus}}^F\), \(I \mapsto I\).

Since \(N_G(L_I)\) acts trivially on \(\mathcal{H}(L_I)_{\text{cus}}^F\) by [6, Theorem 9.2], we get a bijection \(\mathcal{H}(L_I)_{\text{cus}}^F \simeq \mathcal{H}(L_I, w)_{\text{cus}}^F\), \(I \mapsto I\).

Since \(\sigma\) stabilizes \(w\), there exists \(x \in G^F\) such that \(x L_I, w = L_I, w\). We then set \(\sigma' = \text{Inn}(x^{-1}) \circ \sigma\) so that \(x L_I, w = L_I, w\).

**Lemma 5.** Let \(I \in \mathcal{P}(S)_{\text{cus}}^F\). Then:

(a) \(\sigma(I) = I\) if and only if \(\sigma'(x^{-1}) I = I\);

(b) If \(\sigma(I) = I\), then \(\xi^L_{I, \sigma} = \xi^L_{I, \sigma'}\).
**Proof.** Let $\tau = \text{Inn}(g^{-1}) \circ \sigma \circ \text{Inn}(g)$. Then $\tau \in \text{Aut}(L, \text{Inn}(\tilde{w}) \circ F)$. Moreover, $\sigma$ stabilizes $\ell$ if and only if $\tau$ stabilizes $\ell$. But $\tau = \text{Inn}(g^{-1}x^{-1}\sigma g) \circ \sigma$, so $g^{-1}x^{-1}\sigma g \in N_G(L)$: this proves that $g^{-1}x^{-1}\sigma g$ acts trivially on $\mathfrak{U}(L)_\text{cus}$ by [6, Theorem 9.2]. Therefore, $\sigma$ stabilizes $\ell$ if and only if $\sigma$ stabilizes $\ell$. This proves (a).

Let us now prove (b). Let $\iota = (C, \mathscr{L}) \in \mathfrak{U}(L)_\text{cus}$ be such that $\sigma(\iota) = \iota$. We fix an element $v \in C^F$ such that $\iota_1^{-1}(v) \neq 0$.

We write $n = g^{-1}x^{-1}\sigma g \in N_G(L)$. Then $\tau = \text{Inn}(n) \circ \sigma$ commutes with $\text{Inn}(\tilde{w}) \circ F$. Since $N_G(L)$ stabilizes $C$ and since $A_{L_1}(v) = A_G(v)$ (see [2, Corollary to Proposition 1.1]), we may (and we will) assume that $\tilde{w} \in N_G(L) \cap C_G^*(v)$. Now, $\sigma$ and $n$ stabilize $C$. So there exist $l$ and $m$ in $L$ such that $\sigma(v) = l\tilde{w}l^{-1}$ and $nvn^{-1} = mvm^{-1}$. So $m^{-1}n \in C_G(v)$. Since $A_{L_1}(v) = A_G(v)$, we may (and we will) choose $m$ in such a way that $m^{-1}n \in C_G^*(v)$.

We have

$$l^{-1}F(l) \in C^*_L(v), \quad \tau(v) = \text{Inn}(nln^{-1}m)(v)$$

and

$$(nln^{-1}m)^{-1}\tilde{w}F(nln^{-1}m)\tilde{w}^{-1} \in C^*_L(v).$$

According to formula (2), and since $\tilde{w}$ acts trivially on $A_{L_1}(v)$ (see [4, Lemma 1.3.12]), it is sufficient to prove that $l^{-1}F(l)$ and $(nln^{-1}m)^{-1}\tilde{w}F(nln^{-1}m)\tilde{w}^{-1}$ represent the same element of $A_{L_1}(v)$. Since $A_{L_1}(v) = A_G(v)$, we need to determine the class in $A_G(v)$ of $\mu = (nln^{-1}m)^{-1}\tilde{w}F(nln^{-1}m)\tilde{w}^{-1}$. But

$$\mu = (m^{-1}n)l^{-1}n^{-1}\tilde{w}F(nl)\tilde{w}^{-1}(m\tilde{w}F(nm)\tilde{w}^{-1}),$$

$m^{-1}n \in C^*_G(v)$ and $\tilde{w}F(nm)\tilde{w}^{-1} \in C^*_G(v)$ because $\text{Inn}(\tilde{w}) \circ F$ stabilizes $v$. Therefore, the class of $\mu$ in $A_G(v)$ is equal to the class of $\mu' = l^{-1}n^{-1}\tilde{w}F(nl)\tilde{w}^{-1}$. It is also easily checked that $\tilde{w}F(nm) = n\tilde{w}$. Therefore,

$$\mu' = l^{-1}n^{-1}\tilde{w}F(l)\tilde{w}^{-1} = l^{-1}\sigma \tilde{w}l^{-1}F(l)\tilde{w}^{-1}.$$

But, $l^{-1}\sigma \tilde{w}l \in C^*_G(v)$ because $l^{-1}\sigma(v)l = v$ and $\tilde{w} \in C^*_G(v)$. So the class of $\mu'$ in $A_G(v)$ is equal to the class of $l^{-1}F(l)$, which is the desired result. \hfill \Box

2.C. The main result

We recall (see, for example, [3, Conjecture C]) that it is conjectured that $\text{Cus}_\text{uni}(G^F) = \mathfrak{M}/\mathfrak{S}_\text{uni}(G^F)$ whenever $p$ is almost good for $G$. The next theorem goes in this direction.

**Theorem 6.** *If the Mackey formula holds in $G$ (in the sense of [3, Definition 1.4.2]), then $\text{Cus}_\text{uni}(G^F)$ and $\mathfrak{M}/\mathfrak{S}_\text{uni}(G^F)$ are isomorphic as $\mathbb{Q}_p$-$\text{Out}(G, F)$-modules.*

**Proof.** We proceed as for the proof of [3, Theorem 4.1.1]. But we avoid the mistake mentioned above! So we assume that the Mackey formula holds in $G$. Note that this implies that the Lusztig induction and restriction maps do not depend on the choice of the parabolic subgroup. Therefore, if $L$ is an $F$-stable Levi subgroup of a parabolic subgroup $P$ of $G$, we will denote by $R_L^G$ and $\rho R_L^G$ the maps $R_L^G$ and $\rho R_L^G$. We argue by induction on dim $G$. The result is obvious if $G$ is a torus. Therefore, we may assume that Theorem 6 holds for every $F$-stable Levi subgroup of a proper parabolic subgroup of $G$. Since $\text{Out}(G, F)$ acts on $\text{Cus}_\text{uni}(G^F)$ and
\( \mathcal{M}_{\text{uni}}(G^F) \) through a finite quotient (namely its image in \( \text{Out}(G^F) \)), it is sufficient to prove the following: if \( \sigma \in A_S \), then

\[
\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(G^F)) = \text{Tr}(\sigma, \mathcal{M}_{\text{uni}}(G^F)).
\]  

(3)

**First step.** Let us first evaluate the right-hand side of (3). Let \( \mathcal{U}(G)^{\circ} \) denote the set of \( (I, \iota, \rho) \in \mathcal{U}(G) \) such that \( I \neq S \). Then, since \( (\mathcal{M}_{\psi}(I, \iota, \rho))_{(I, \iota, \rho) \in \mathcal{U}(G)^{\circ}} \) is a basis of \( \text{Class}_{\text{uni}}(G^F) \), we have

\[
\text{Tr}(\sigma, \text{Class}_{\text{uni}}(G^F)) = \text{Tr}(\sigma, \mathcal{M}_{\text{uni}}(G^F)) + \sum_{(I, \iota, \rho) \in \mathcal{U}(G)^{\circ}} \xi^G_{\psi(I, \iota, \rho)}(\sigma).
\]

If we denote by \( \mathcal{E} \) the set of pairs \( (I, \iota) \) such that \( I \in \mathfrak{A}(S)_{\text{cus}}, I \neq S \) and \( \iota \in \mathcal{U}(L_I)^{\text{cus}} \), and if we use Lemma 2, we get

\[
\text{Tr}(\sigma, \mathcal{M}_{\text{uni}}(G^F)) = \text{Tr}(\sigma, \text{Class}_{\text{uni}}(G^F)) - \sum_{(I, \iota) \in \mathcal{E}, \sigma(I, \iota) = (I, \iota)} \xi^L_{I, \iota}(\sigma) \cdot |\{\rho \in (\text{Irr} W_G(L_I))^F | \sigma(\rho) = \rho\}|.
\]  

(4)

**Second step.** We now evaluate the left-hand side of (3). If \( L \in \mathcal{A} \), then \( N_{G^F}(L) \) acts trivially on \( \mathcal{M}_{\text{uni}}(L^F) \) by Lemma 3, so it acts trivially on \( \text{Cus}_{\text{uni}}(G^F) \) by the induction hypothesis. So, since the Mackey formula holds in \( G \), we have

\[
\text{Class}_{\text{uni}}(G^F) = \text{Cus}_{\text{uni}}(G^F) \oplus \bigoplus_{(I, w) \in \mathcal{G}} R^G_{L_{I, w}}(\text{Cus}_{\text{uni}}(L^F_{I, w})),
\]

and the map \( R^G_{L_{I, w}} : \text{Cus}_{\text{uni}}(L^F_{I, w}) \to R^G_{L_{I, w}}(\text{Cus}_{\text{uni}}(L^F_{I, w})) \) is an isomorphism. Note that this isomorphism commutes with every element of \( \text{Aut}(G, F) \) stabilizing \( L_{I, w} \). Therefore,

\[
\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(G^F)) = \text{Tr}(\sigma, \text{Class}_{\text{uni}}(G^F)) - \sum_{(I, w) \in \mathcal{G}, \sigma(I, w) = (I, w)} \text{Tr}(\sigma, R^G_{L_{I, w}}(\text{Cus}_{\text{uni}}(L^F_{I, w}))).
\]  

(5)

Let \( (I, w) \in \mathcal{G} \) be such that \( \sigma(I, w) = (I, w) \). Then there exists \( x \in G^F \) such that \( ^{\sigma}L_{I, w} = ^{\sigma}L_{I, w} \). We set \( \sigma' = \text{Inn}(x)^{-1} \circ \sigma \). Then \( \sigma' \) stabilizes \( L_{I, w} \) and

\[
\text{Tr}(\sigma, R^G_{L_{I, w}}(\text{Cus}_{\text{uni}}(L^F_{I, w}))) = \text{Tr}(\sigma', R^G_{L_{I, w}}(\text{Cus}_{\text{uni}}(L^F_{I, w}))),
\]

so \( \text{Tr}(\sigma, R^G_{L_{I, w}}(\text{Cus}_{\text{uni}}(L^F_{I, w}))) = \text{Tr}(\sigma', \text{Cus}_{\text{uni}}(L^F_{I, w})). \) But, by the induction hypothesis, we get \( \text{Tr}(\sigma', \text{Cus}_{\text{uni}}(L^F_{I, w})) = \text{Tr}(\sigma', \mathcal{M}_{\text{uni}}(L^F_{I, w})). \) Moreover, by Lemma 5, we have \( \text{Tr}(\sigma', \mathcal{M}_{\text{uni}}(L^F_{I, w})) = \text{Tr}(\sigma, \mathcal{M}_{\text{uni}}(L^F_{I, w})). \) So we deduce from (5) that

\[
\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(G^F)) = \text{Tr}(\sigma, \text{Class}_{\text{uni}}(G^F)) - \sum_{(I, w) \in \mathcal{G}, \sigma(I, w) = (I, w)} \text{Tr}(\sigma, \mathcal{M}_{\text{uni}}(L^F_{I, w})).
\]
In other words,
\[
\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(G^F)) = \text{Tr}(\sigma, \text{Class}_{\text{uni}}(G^F)) - \sum_{l \in P(S^F_\text{uni}), l \neq S} \text{Tr}(\sigma, \mathcal{W}_l^\text{uni}(L^F_l)) \cdot |\{w \in H^1(F, W_G(L_l)) | \sigma(w) = w\}|.
\]

Finally, we get
\[
\text{Tr}(\sigma, \text{Cus}_{\text{uni}}(G^F)) = \text{Tr}(\sigma, \text{Class}_{\text{uni}}(G^F)) - \sum_{(l, i) \in E} \xi_{l, i} \cdot |\{w \in H^1(F, W_G(L_l)) | \sigma(w) = w\}|. \quad (6)
\]

Third step. Let \(l \in P(S)^F\) be such that \(\sigma(l) = l\). Then \(\sigma\) acts on \(W_G(L_l)\) and this action commutes with the action of \(F\). Therefore,
\[
|\{w \in H^1(F, W_G(L_l)) | \sigma(w) = w\}| = |\{\rho \in (\text{Irr}(W_G(L_l))^F | \sigma(\rho) = \rho\}|. \quad (7)
\]

The proof of (7) is similar to the proof of the well-known theorem of Brauer [5, Theorem 6.32]. By applying (4), (6) and (7), we get (3).

2.D. Some consequences of Theorem 6

In [3, § 1.8], we defined a morphism \(H^1(F, Z) \to \text{Out}(G, F)\). So Theorem 6 immediately implies the following result.

**Corollary 7.** Let \(\xi \in H^1(F, Z)^\wedge\). If the Mackey formula holds in \(G\), then
\[
\dim \text{Cus}_{\text{uni}}(G^F)_\xi = \dim \mathcal{W}_\xi^\text{uni}(G^F)_\xi.
\]

The last result says that [3, Corollary 4.1.2] is correct. It is just a straightforward consequence of Theorem 6 and Lemma 3. Note that in [3, Corollary 4.1.2(b)], the term ‘cuspidal function’ must be replaced by ‘absolutely cuspidal function’.

**Corollary 8.** If the Mackey formula holds in \(G\), then:
(a) \(\dim \text{Cus}_{\text{uni}}(G^F) = |\mathcal{W}(G)_{\text{cus}}^F|\);

(b) if \(G\) is a rational Levi subgroup of a parabolic subgroup of a connected reductive group \(H\) (endowed with a Frobenius endomorphism also denoted by \(F\)) then all absolutely cuspidal functions on \(G^F\) with unipotent support are invariant under the action of \(N^F_H(G)\).

**References**


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