Zweibein Operator Formalism of Two-Dimensional Quantum Gravity*)

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The unitary, manifestly covariant operator formalism of two-dimensional quantum gravity, presented previously, is extended to the zweibein formalism. All the two-dimensional (anti)commutation relations between primary fields are obtained in closed form. The four degrees of freedom of the zweibein are shown to be realized as q-number transformation functions of the general coordinate transformation, the local Lorentz transformation and the Weyl transformation. As the result, the explicit expression for the gravitational extension of the Pauli-Jordan D-function is found in terms of the zweibein. Bosonized operator solutions known in solvable two-dimensional models are extended to the quantum-gravity case through the above q-number transformations.

§ 1. Introduction

In our previous paper,1) we have satisfactorily formulated the unitary theory of two-dimensional quantum gravity. The crucial ingredient of this formalism is the Weyl BRS transformation newly defined by making the gravitational FP ghosts play also the role of the Weyl FP ghosts so as to resolve ghost counting trouble peculiar to two dimensions. In this theory, we have shown that all the full (i.e., two-dimensional) (anti)commutation relations between primary fields are obtained in terms of the gravitational extension of the Pauli-Jordan D-function. This is due to the fact that the gravitational field $g_{\mu\nu}(x)$ is "almost c-number" in the sense that $g_{\mu\nu}(x)$ commutes two-dimensionally with all the primary fields except the gravitational B-field and the Weyl one.

Classically, the two-dimensional gravity is trivial in the sense that $g_{\mu\nu}(x)$ can be transformed into $\eta_{\mu\nu}$ by means of the general coordinate transformation and the Weyl transformation. Although we are considering quantum gravity, similar trivialization is expected to be possible, because $g_{\mu\nu}(x)$ is almost c-number as stated above, as far as the quantities which do not involve B-fields are concerned. But in contrast to the classical gravity, the transformation functions themselves must be expressible in terms of field operators involved in the theory. Furthermore, since we should maintain manifest $GL(2)$ covariance, the use of a non-tensor $\eta_{\mu\nu}$ is not adequate; it is better to replace it by the tangent-space metric $\eta_{ab}$. This idea suggests that the transformation functions are essentially provided by the zweibein $h^{a}(x)$.

The purpose of the present paper is to extend our previous theory of two-dimensional quantum gravity to the zweibein formalism. As before, we can obtain all the two-dimensional (anti)commutation relations between primary fields in the coupled system of the zweibein field and the massless Dirac field, where the masslessness is required by the Weyl invariance. We show that the spacetime parameter $x^\mu$...
in quantum gravity is related to the "flat" spacetime operator \( \hat{x}^a \) through the \( q \)-number general coordinate transformation, the \( q \)-number local Lorentz transformation and the \( q \)-number Weyl transformation whose transformation functions are explicitly expressible in terms of the zweibein. Since all components of the zweibein commute two-dimensionally with themselves, \( \hat{x}^a \) also commutes with \( \hat{x}^b \), though, of course, the \( \hat{x}^a \)'s are not \( c \)-number because of the noncommutativity between the zweibein and the B-fields. With the above transformation functions, it is possible to extend the known results in the usual Minkowskian two-dimensional models to those in quantum-gravitational ones. In particular, we find the explicit expression for the gravitational extension of the Pauli-Jordan \( D \)-function and that for the bosonization formula of the generally-covariant massless Dirac field.

The present paper is organized as follows. In § 2, we present the zweibein formalism of quantum gravity together with its canonical quantization. In § 3, we consider the symmetry generators of our theory. In § 4, we calculate the two-dimensional (anti)commutation relations. In § 5, the relation between \( x^a \) and \( \hat{x}^a \) is established. In § 6, we find the explicit expressions for the gravitational extensions of the Pauli-Jordan \( D \)-function and \( D^\pm \)-functions. In § 7, we consider the bosonization formula of the massless Dirac field and extend the results to the Thirring model. The final section is devoted to discussion. In Appendix A, we summarize the zweibein formalism and its peculiar features. In Appendix B, the proof of the unitarity is made in the zweibein formalism by extending the previous result.

§ 2. Zweibein formalism and canonical quantization

We consider the covariant zweibein formalism of two-dimensional quantum gravity based on the following Lagrangian density [see Appendix A for explanation]:

\[
\mathcal{L} = \mathcal{L}_c + \mathcal{L}_l + \mathcal{L}_d ,
\]
\[
\mathcal{L}_c = \partial_{\mu} \bar{\gamma}^{\mu} \cdot b_{\nu} - i \bar{\gamma}^{\mu} \bar{\partial}_{\mu} c_{\nu} \cdot \partial_{\nu} c^\sigma - 2 \bar{\gamma}^{\mu} \bar{\partial}_{\mu} \bar{\gamma}_{\nu} b_{\nu} ,
\]
\[
\mathcal{L}_l = 2 \bar{\gamma}^{\mu} \omega_{\mu} \partial_{\nu} s + 2i \bar{\gamma}^{\mu} \partial_{\mu} \bar{I} \cdot \partial_{\nu} t .
\]

In (2·2), \( b_{\nu} \) is the gravitational B-field, \( c^\sigma \) and \( \bar{c}_{\tau} \) are the gravitational FP ghost and antighost, respectively, and \( b \) is the Weyl B-field. The conjugate spin connection \( \bar{\omega}_{\mu} \) defined by (A·17) simplifies the terms involving \( b \) or \( \partial_{\nu} b \) in (4·2) of Ref. 1) into the last term in (2·2). In (2·3), \( s \) is the local Lorentz B-field, \( t \) and \( \bar{t} \) being the local Lorentz FP ghost and antighost, respectively. We define the local Lorentz Lagrangian density (2·3) by extrapolating that of the vierbein formalism of quantum Einstein gravity.\(^{2,3} \) The matter Lagrangian density must be Weyl invariant; for definiteness we take a massless Dirac field one \( \mathcal{L}_d \).

The Lagrangian density (2·1) is invariant under the following BRS transformations up to total divergence:

\[ \delta_b \eta_{\mu}^a = - \partial_{\mu} c^\sigma \cdot \eta_{\sigma}^a - c^\sigma \partial_{\sigma} \eta_{\mu}^a , \]
\[ \delta_\ast \bar{c} = ib_t - c^\mu \partial_\mu \bar{c}, \quad (2.5) \]
\[ \delta_\ast \Phi = -c^\mu \partial_\mu \Phi \quad \text{for other primary fields } \Phi; \quad (2.6) \]

\[ \delta^i h^a_\mu = \frac{1}{2} c^1 h^a_\mu, \quad \delta^i \bar{c} = \frac{1}{2} \partial_\mu c^i, \quad (2.7) \]
\[ \delta^i \bar{c} = i \delta^i b, \quad (2.8) \]
\[ \delta^i \phi = -\frac{1}{4} c^i \phi, \quad (2.9) \]
\[ \delta^i (\text{others}) = 0; \quad (2.10) \]

[3] Local Lorentz BRS transformation
\[ \delta_L h^a_\mu = -\epsilon^{ab} h^b_\mu, \quad \delta_L \omega_\mu = \partial_\mu t, \quad (2.11) \]
\[ \delta_L \bar{t} = is, \quad (2.12) \]
\[ \delta_L \phi = \frac{1}{2} t \gamma^5 \phi, \quad (2.13) \]
\[ \delta_L (\text{others}) = 0. \quad (2.14) \]

As emphasized in Ref. 1), \( \bar{L}_c \) is not the coboundary of \( \delta_\ast \) and \( \delta^i \), while \( \bar{L}_L \) is rewritten into \( \bar{L}_L = -2i \delta_L (\bar{g}^\mu_\nu \omega_\mu \partial_\nu \bar{t}) \).

The field equations derived from (2.1) are the following (see Appendix A for unexplained notation):

\[ 2(\nabla_\mu \nabla_\nu - g_{\mu \nu} \nabla^a \nabla_a) b - E_{\mu \nu} + \frac{1}{2} g_{\mu \nu} E = -(T_{\mu \nu} + V_{\mu \nu}), \quad (2.15) \]
\[ E_{\mu \nu} = \partial_\mu b_\nu + i \partial_\mu \bar{c} \cdot \partial_\nu c^e + (\mu \leftrightarrow \nu), \quad (2.15a) \]
\[ E = E^a_\mu, \quad (2.15b) \]
\[ T^{\mu \nu} = -\frac{1}{\hbar} \hbar^{\nu a} \frac{\delta}{\delta h^a_\mu} \bar{L}_c \]
\[ = i \frac{1}{4} g^{\mu \nu} g^{\rho \sigma} \left[ \bar{\psi} \gamma_\rho \partial_\rho \phi - \partial_\rho \bar{\psi} \gamma_\rho \phi + (\lambda \leftrightarrow \rho) \right], \quad (2.15c) \]
\[ V^{\mu \nu} = -\frac{1}{\hbar} \hbar^{\nu a} \frac{\delta}{\delta h^a_\mu} \bar{L}_L \]
\[ = -2 \epsilon^{\mu \nu} \hbar^{a \mu} \partial_a (h^a_\sigma \partial_\sigma s) + 2 g^{\mu \nu} g^{\rho \sigma} \omega_\sigma \partial_\rho s - 2i (g^{\mu \nu} g^{\rho \sigma} - g^{\mu \rho} g^{\nu \sigma} - g^{\mu \sigma} g^{\nu \rho}) \partial_a \bar{t} \cdot \partial_t, \quad (2.15d) \]
\[ \partial_\mu \bar{g}^{\mu \nu} = 0, \quad (2.16) \]
\[ \partial_\mu (\bar{g}^{\mu \nu} \omega_\nu) = 0, \quad (2.17) \]
\[ \partial_\mu (\bar{g}^{\mu \nu} \omega_\nu) = 0, \quad (2.18) \]
\[
\partial_\mu (\bar{g}^{\mu\nu} \partial_\nu X) = 0 \quad \text{for} \quad X = b_\rho, b, c_\sigma, \bar{c}_\tau, s, t \text{ and } \bar{t},
\]

\[
\gamma^{\nu}(\partial_\mu + \omega_\mu)\psi = 0.
\]

Here the second equality of (2·15c) and (2·19) for \(X = b_\rho, b, s\) are indirect results: We use (2·20) to derive (2·15c). From the antisymmetric part of (2·15), we obtain (2·19) for \(X = s\). As for \(X = b_\rho \) and \( b \) in (2·19), see Ref. 1. Because of (A·19), (2·17) is equivalent to \( R = 0.4\).

The canonical variables are \( h_{\mu}^a, b, c_\sigma, \bar{c}_\tau, s, t, \bar{t} \) and \( \phi \). The canonical conjugates are as follows (\( \Phi = \partial_0 \Phi \)):

\[
\pi_{h_{\mu}^a} = \frac{\partial}{\partial h_{\mu}^a} \mathcal{L} = (h_{\alpha}^a \bar{g}^{\alpha\beta} - h_{\beta}^a \bar{g}^{\beta\alpha} - h_{\alpha}^b \bar{g}^{\alpha\beta}) b_{\beta} - 2 \delta_{\mu}^a \left( e^{\nu}_{\alpha b} \partial_\nu b + h_{\alpha}^c \partial_\nu s + \frac{i}{4} \varepsilon_{ab} \bar{\psi} \gamma^b \psi \right),
\]

\[
\pi_b = \frac{\partial}{\partial b} \mathcal{L} = -2 \bar{g}^{0\nu} \partial_\nu \phi,
\]

\[
\pi_{c_\sigma} = \frac{\partial}{\partial c_\sigma} \mathcal{L} = i \bar{g}^{0\nu} \partial_\nu c_\sigma,
\]

\[
\pi_{\bar{c}_\tau} = \frac{\partial}{\partial \bar{c}_\tau} \mathcal{L} = -i \bar{g}^{0\nu} \partial_\nu \bar{c}_\tau,
\]

\[
\pi_s = \frac{\partial}{\partial s} \mathcal{L} = 2 \bar{g}^{0\nu} \omega_\nu,
\]

\[
\pi_t = \frac{\partial}{\partial t} \mathcal{L} = -2i \bar{g}^{0\nu} \partial_\nu \bar{t},
\]

\[
\pi_{\bar{t}} = \frac{\partial}{\partial \bar{t}} \mathcal{L} = 2i \bar{g}^{0\nu} \partial_\nu t,
\]

\[
\pi_\phi = \frac{\partial}{\partial \phi} \mathcal{L} = -i h \bar{\psi} \gamma^0.
\]

Setting up the canonical (anti)commutation relations, we can calculate all the equal-time (anti)commutation relations concerning the primary fields in closed form.
Employing the following abbreviated notation,

\[ [A, B'] = [A(x), B(y)]_0, \tag{2.29} \]

\[ \delta = \delta(x^1 - y^1) \tag{2.30} \]

with the subscript 0 denoting to set \( x^0 = y^0 \), we have

\[ [h_{\mu}^a, b'^0] = -i \frac{1}{g_{00}} \delta^0_{\mu} h_{\mu}^a \delta, \tag{2.31} \]

\[ [h_{\mu}^a, b'^0] = i \left( \frac{1}{g_{00}} \partial_\mu h_{\mu}^a \cdot \delta - \delta^a_{\mu} h_{\mu}^a \left( \partial_0 \frac{1}{g_{00}} \cdot \delta + 2 \frac{\delta_{01}}{g_{00}} \partial_1 \left( \frac{\delta}{g_{00}} \right) \right) + \delta^\mu_{\mu} h_{\mu}^a \delta \left( \frac{\delta}{g_{00}} \right) \right), \tag{2.32} \]

\[ [\hat{h}_{\mu}^a, b'^0] = \delta_0 [h_{\mu}^a, b'^0] - [h_{\mu}^a, b'^0], \tag{2.33} \]

\[ [b_\mu, b'^0] = i \frac{1}{g_{00}} \left( \partial_\mu b_\rho + \partial_\rho b_\mu \right) \cdot \delta, \tag{2.34} \]

\[ [\Phi, b'^0] = -[\Phi, b'^0] = i \frac{1}{g_{00}} \partial_\mu \Phi \cdot \delta \quad \text{for} \quad \Phi = \psi, b, c^\sigma, \bar{c}^\sigma, s, t \text{ and } \bar{t}, \tag{2.35} \]

\[ [h_{\mu}^a, b'] = -[\hat{h}_{\mu}^a, b'] = -\frac{i}{2} \frac{1}{g_{00}} h_{\mu}^a \delta, \tag{2.36} \]

\[ [\psi, b'] = -[\hat{\psi}, b'] = \frac{i}{4} \frac{1}{g_{00}} \psi \delta, \tag{2.37} \]

\[ [h_{\mu}^a, s'] = -[\hat{h}_{\mu}^a, s'] = -\frac{i}{2} \frac{1}{g_{00}} e^{ab} h_{\mu}^b \delta, \tag{2.38} \]

\[ [\psi, s'] = -[\hat{\psi}, s'] = -\frac{i}{4} \frac{1}{g_{00}} \gamma^{\sigma} \psi \delta, \tag{2.39} \]

\[ \{c^\sigma, \bar{c}'^\sigma\} = -\{\bar{c}'^\sigma, c^\sigma\} = -\delta^\sigma_\tau \frac{1}{g_{00}} \delta, \tag{2.40} \]

\[ \{t, \bar{t}'\} = -\{\bar{t}', t\} = \frac{1}{2} \frac{1}{g_{00}} \delta, \tag{2.41} \]

\[ \{\psi_0, \bar{\psi}'\} = (\gamma^0)_{jk} \frac{1}{g_{00}} \delta \tag{2.42} \]

and others vanish. Especially, it is very important to note that

\[ [h_{\mu}^a, \hat{h}_{\nu}^{b\sigma}] = 0 \tag{2.43} \]

and

\[ [\psi, \hat{h}_{\nu}^{b\sigma}] = 0 \tag{2.44} \]

are satisfied since \( \hat{h}_{\nu}^{b\sigma} \) is expressible in terms of \( h_{\nu}^{c\mu}, \partial_\nu h_{\nu}^{c\mu}, \pi_0 \) and \( \pi_\sigma \) as seen from (2.16), (2.22) and (2.25). All the above results, except for (2.43) and (2.44), involving neither \( b \) nor \( \bar{b} \) are nothing but the two-dimensional extrapolations of those in the vierbein formalism of Einstein gravity.2)
Our theory is, of course, unitary. The proof is trivial extension of the one given in Ref. 1), but for completeness, we present it in Appendix B.

§ 3. Symmetry generators

In this section, we consider the symmetry of our theory by constructing symmetry generators from the field equations.

The field equations (2·16) and (2·19) are summarized in terms of the d'Alembert equations

\[ \partial_\mu \tilde{g}^{\mu\nu} \partial_\nu X = 0 \quad \text{for} \quad X = x^i, b_\nu, b, c^\sigma, \bar{c}_\tau, s, t \text{ and } \bar{t}. \]

From (3·1), we obtain the following conserved hermitian charges:\n
\[ P(X) = \int_{-\infty}^{+\infty} dx^1 \tilde{g}^{0\nu} \partial_\nu X, \]

\[ M(X, Y) = \sqrt{\epsilon(X, Y)} \int_{-\infty}^{+\infty} dx^1 \tilde{g}^{0\nu}(X \partial_\nu Y - \partial_\nu X \cdot Y), \]

where we represent the twelve-dimensional supercoordinate \{x^i, b_\nu, b, c^\sigma, \bar{c}_\tau, s, t, \bar{t}\} by X, Y, etc.; \( \epsilon(X, Y) = -1 \) for both X and Y fermionic, \( \epsilon(X, Y) = 1 \) otherwise, and \( \sqrt{+1} = +1 \) and \( \sqrt{-1} = +i \). From (2·17) and (2·18), we obtain two more conserved charges:

\[ P(\bar{\omega}) = \int_{-\infty}^{+\infty} dx^1 \tilde{g}^{0\nu} \bar{\omega}_\nu, \]

\[ P(\omega) = \int_{-\infty}^{+\infty} dx^1 \tilde{g}^{0\nu} \omega_\nu. \]

By means of the equal-time (anti)commutation relations, the (anti)commutators between generators and primary fields can be calculated. Let \( \varphi \) be a primary field other than \( b_\nu, b, c^\sigma, \bar{c}_\tau, s, t \) and \( \bar{t} \). The generators (3·2), (3·4) and (3·5) are denoted in a unified form as \( P(X) \) where \( X \) stands for \( X, \bar{\omega} \) and \( \omega \). Then we obtain

\[ [iP(X), \varphi] = \eta(\bar{X}, x^\nu) \partial_\nu \varphi - \eta(\bar{X}, \bar{\omega}) d(\varphi) \varphi - \eta(\bar{X}, \omega)[\varphi], \]

\[ [iP(X), Z] = i\sqrt{-\epsilon(X, Z)} \eta(X, Z) + \eta(\bar{X}, x^\nu) \partial_\nu Z, \]

\[ [iM(X, Y), \varphi] = \eta(Y, x^\nu) (X \partial_\nu \varphi - \partial_\nu X \varphi) - \eta(Y, \bar{\omega}) d(\varphi) X \varphi \]

\[ - \eta(Y, \omega) X[\varphi] - \epsilon(X, Y)(X \leftrightarrow Y), \]

\[ [iM(X, Y), Z] = i\sqrt{-\epsilon(XY, Z)} \eta(Y, Z) X + \eta(X, x^\nu) X \partial_\nu Z \]

\[ - \epsilon(X, Y)(X \leftrightarrow Y), \]

where \( \epsilon(X) = \epsilon(X, X) \); \( \eta(X, Y) \) and \( \eta(\bar{X}, \bar{Y}) \) are the twelve-dimensional (degenerate) supermetric and its (nondegenerate) extension defined by

\[ \eta(x^\nu, b_\nu) = \eta(b_\nu, x^\nu) = \eta(c^\nu, \bar{c}_\nu) = - \eta(\bar{c}_\nu, c^\nu) = \delta^\nu_\nu, \]
\[ -\eta(t, \bar{t}) = \eta(\bar{t}, t) = \frac{1}{2}, \]
\[ \eta(\bar{a}, b) = \eta(b, \bar{a}) = -\eta(\omega, s) = -\eta(s, \omega) = \frac{1}{2}, \]
\[ \eta(\bar{X}, \bar{Y}) = 0 \quad \text{otherwise}; \quad (3.10) \]

The numerical factor \( d(\varphi) \) is the Weyl dimension of \( \varphi \), for example, \( d(h_\alpha^a) = -d(h^\alpha_a) = 1, \) \( d(\phi) = -1/2, \) etc.; the quantities \([\varphi]_L\) and \([\varphi]_A\) are defined by the infinitesimal internal Lorentz transform of \( \varphi \) and the general linear one, respectively:

\[ [h_\alpha^a]_L = -\epsilon^{ab} h_{\mu b}, \quad (3.11) \]
\[ [\varphi]_L = -\frac{1}{2} \phi \phi; \quad (3.12) \]
\[ [h_\alpha^a]_A = -\delta^a_{\mu} h_{\mu}, \quad (3.13) \]
\[ [\varphi]_A = 0. \quad (3.14) \]

The important symmetry generators are the following: affine generators \( P_\mu = P(b_\mu) \) and \( \tilde{M}^a_{\nu} = M(x^a, b_\nu) - M(c^a, c_\nu); \) global Weyl generator \( W = P(b); \) internal Lorentz generator \( M_L = 2P(s); \) the gravitational BRS generator \( Q_b = M(b_\mu, c_\nu); \) the vector-type Weyl BRS generator \( Q_\mu^a = M(b, c_\phi); \) the local-Lorentz BRS generator \( Q_s = -2M(s, t); \) etc.

The symmetry generators constitute a degenerate twelve-dimensional Poincaré-like superalgebra containing two extra generators:

\[ [P(X), P(Y)]_z = 0, \quad (3.15) \]
\[ [M(X, Y), P(Z)]_z = \sqrt{-\epsilon(\bar{X}Y, \bar{Z})} \{ \eta(Y, \bar{Z}) P(X) - \epsilon(X, Y)(X \leftrightarrow Y) \}, \quad (3.16) \]
\[ [M(X, Y), M(U, V)]_z = \sqrt{-\epsilon(\bar{X}Y, \bar{U}V)} \{ [\eta(Y, \bar{U}) M(X, V) - \epsilon(X, Y)(X \leftrightarrow Y] - \epsilon(\bar{U}V)(U \leftrightarrow V) \}. \quad (3.17) \]

Note that \( W = P(b) \) and \( M_L = 2P(s) \) constitute the center of this superalgebra.

\section*{§ 4. Two-dimensional (anti)commutation relations}

From the field equations and the equal-time (anti)commutation relations, we can calculate the two-dimensional (anti)commutation relations.\(^{15}\) We postulate the uniqueness of solution to Cauchy problem even if its coefficients and initial data are \( q \)-number.

Since \( \hat{h}_\mu^a \) is expressible in terms of \( h^a_\nu \), \( \hat{h}_\nu^b \) and their spatial derivatives because of (2.16)~(2.18), (2.43) and (2.44) lead to

\[ [(\bar{\partial}_0)^m h_\mu^a, (\bar{\partial}_0)^n h_\nu^b] = 0, \quad (4.1) \]
\[ [(\bar{\partial}_0)^m \phi, (\bar{\partial}_0)^n h_\nu^b] = 0 \quad (4.2) \]
for any non-negative integers $m$ and $n$. Those results may be understood as the manifestation of the two-dimensional commutativity:

\[[h^a_\mu(x), h^b_\nu(y)] = 0, \quad (4.3)\]
\[[\phi(x), h^b_\nu(y)] = 0. \quad (4.4)\]

To obtain other two-dimensional (anti)commutators, we introduce the quantum gravity $D$-function and massless $S$-function by the following Cauchy problems:

\[
\begin{align*}
\partial_\mu \tilde{g}^{\mu\nu}(x) \partial_\nu \tilde{D}(x, y) &= 0, \\
\tilde{D}(x, y)|_0 &= 0, \\
\partial_\mu \tilde{D}(x, y)|_0 &= -\tilde{g}^{00} \delta; \\
\gamma^\mu (\partial_\mu + \omega_\mu) S(x, y) &= 0, \\
S(x, y)|_0 &= -iv_0(\tilde{g}^{00})^{-1} \delta. 
\end{align*}
\]

(4·3)

(4·4)

(4·5)

(4·6)

(4·7)

(4·8)

(4·9)

Then (4·3), (4·4) and the uniqueness of Cauchy problem yield the two-dimensional commutativity between any one of $\{h^a_\mu(x), \phi(x), \tilde{D}(x, y), S(x, y)\}$ and either of $\{\tilde{D}(z, w), S(z, w)\}$. We can also prove

\[
\begin{align*}
\tilde{D}(x, y)' &= -D(y, x) = \tilde{D}(x, y), \\
S(x, y)' &= -\tilde{g}^{00} S(y, x) \tilde{g}^{00} = -\tilde{g}^{00} S(x, y) \tilde{g}^{00}, \\
\{\phi(x), \bar{\phi}(y)\} &= i S(x, y). 
\end{align*}
\]

(4·10)

(4·11)

(4·12)

Using the integral representation of $X(y)$:

\[
X(y) = \int_0^{\infty} dz [\partial_\mu \tilde{D}(y, z) \cdot \tilde{g}^{\mu\nu}(z) X(z) - \tilde{D}(y, z) \tilde{g}^{\mu\nu}(z) \partial_\nu X(z)], 
\]

(4·13)

we can show

\[
\begin{align*}
[\phi(x), X(y)] &= -iv(X, x^\nu) \mathcal{L}_\mu(\varphi) \tilde{D}(x, y) \\
&\quad -iv(X, \omega) d(\varphi) \phi(x) \tilde{D}(x, y) \\
&\quad -iv(X, \omega) [\varphi(x)] \partial_\nu \tilde{D}(x, y),
\end{align*}
\]

(4·14)

where the differential operator $\mathcal{L}_\mu(\varphi)$ is defined by the Lie derivative of $\varphi$ in the general coordinate transformation:

\[
\partial_\mu \varphi(x) = \mathcal{L}_\mu(\varphi) \varepsilon^\mu(x) \quad (4·15)
\]

with $\varepsilon^\mu(x)$ being an infinitesimal transformation function; explicitly,

\[
\mathcal{L}_\mu(\varphi) \equiv [\varphi]^\nu_\mu \partial_\nu - (\partial_\nu \varphi) \quad (4·16)
\]

since $\varphi$ is a world tensor.

From the uniqueness of the Cauchy problem, we can further show that

\[
[\tilde{D}(x, y), X(z)] = iv(X, x^\nu) [\partial_\nu \tilde{D}(x, y) \cdot \tilde{D}(x, z) + \partial_\nu \tilde{D}(x, y) \cdot \tilde{D}(y, z)]. \quad (4·17)
\]
Therefore, the integral representation (4.13) with (4.17) yields the (anti)commutators between two twelve-dimensional supercoordinates:

\[ [X(x), Y(y)] = -\sqrt{-g(x)} \eta(X, Y) \mathcal{D}(x, y) \]

\[ + i[\eta(Y, x^{\mu}) \partial_{\mu} X(x) + \eta(X, x^{\mu}) \partial_{\mu} Y(y)] \cdot \mathcal{D}(x, y). \]  \hspace{1cm} (4.18)

\section*{§ 5. Zweibein field as $q$-number transformation function}

As stated in the Introduction, our primary motivation of introducing the zweibein is to transform $g_{\mu \nu}$ into $\eta_{ab}$. The tangent-space canonical coordinate $x^a$ and the world coordinate $x^{\mu}$ are related through the general coordinate transformation up to the local Lorentz transformation and the Weyl transformation in terms of the zweibein:

\[ \frac{\partial x^a}{\partial x^{\mu}} = e^{-\epsilon_w(x)} [h^{a}_{\mu}(x)] \epsilon_l(x), \]  \hspace{1cm} (5.1)

\[ [h^{a}_{\mu}(x)] \epsilon_l(x) = h^{a}_{\mu}(x) \cosh \epsilon_l(x) + e^{ab} h^{b}_{\rho}(x) \sinh \epsilon_l(x) \]  \hspace{1cm} (5.2)

with $\epsilon_w(x)$ and $\epsilon_l(x)$ being the respective transformation functions. Integrability condition of (5.1) \( (e^{\mu \nu} \partial^2 x^a / \partial x^\mu \partial x^\nu = 0) \) is written as

\[ e^{\mu \nu} (\eta_{ab} \cosh \epsilon_L + e_{ab} \sinh \epsilon_L) (\partial_{\mu} h_{\nu} - \partial_{\nu} \varepsilon_{\mu} \cdot h_{\rho} + \partial_{\rho} \varepsilon_{\mu} \cdot e^{bc} h_{bc}) = 0. \]  \hspace{1cm} (5.3)

On the other hand, from (A.5) and (A.17), we have

\[ e^{\mu \nu} (\partial_{\mu} h^a - \tilde{\omega}_{\mu} h^a) = 0, \]  \hspace{1cm} (5.4)

\[ e^{\mu \nu} (\partial_{\mu} h^a + \omega_{\mu} e^{ab} h_{ab}) = 0. \]  \hspace{1cm} (5.5)

Furthermore, from (2.18) (i.e., $e^{\mu \nu} \partial_{\mu} \tilde{\omega}_{\nu} = 0$) and (2.17) (i.e., $e^{\mu \nu} \partial_{\mu} \omega_{\nu} = 0$), it is possible to integrate $\tilde{\omega}_{\mu}$ and $\omega_{\mu}$ path independently:

\[ \tilde{\omega}(x) = \int x^a \tilde{\omega} \mu, \quad \tilde{\omega}_{\mu} = \partial_{\mu} \tilde{\omega}; \]  \hspace{1cm} (5.6)

\[ \omega(x) = \int x^a \omega_{\mu}, \quad \omega_{\mu} = \partial_{\mu} \omega, \]  \hspace{1cm} (5.7)

where the initial point of integration is fixed arbitrarily. Thus the solution to (5.3) is found to be

\[ \varepsilon_w(x) = \alpha \tilde{\omega}(x), \]  \hspace{1cm} (5.8)

\[ \varepsilon_l(x) = \beta \omega(x), \]  \hspace{1cm} (5.9)

where $\alpha$ and $\beta$ are $c$-number constants satisfying

\[ \alpha + \beta = 1. \]  \hspace{1cm} (5.10)

Integrating (5.1), we obtain
\[ \dot{x}^a(x) = \int_p^x dz^\mu e^{-a \tilde{\omega}(z)} \left[ h_\mu^a(z) \right]_{\gamma \theta(z)}, \]  
\text{(5·11)}

where \( p \) denotes the point of contact, or the origin of the canonical coordinate. From (5·4), (5·5) and (A·4), it is easy to show that \( \dot{x}^a(x) \) satisfies the d'Alembert equation,
\[ \partial_\mu \tilde{g}^{\mu \nu} \partial_\nu \dot{x}^a = 0, \]  
\text{(5·12)}
as it should be.

For later use, we introduce the difference of two points in the canonical coordinate system by
\[ f^\pm(x, y) \equiv \dot{x}^a(x) - \dot{x}^a(y) = \int_p^x dz^\mu e^{-a \tilde{\omega}(z)} \left[ h_\mu^a(z) \right]_{\gamma \theta(z)}, \]  
\text{(5·13)}

which is independent of the point of contact, \( p \). The corresponding “light-cone” coordinate is
\[ f^\pm(x, y) \equiv f^0(x, y) \pm f^a(x, y) \]
\[ = \int_p^x dz^\mu (h_\mu^0 \pm h_\mu^a) e^{-a \tilde{\omega} \pm \theta}, \]  
\text{(5·14)}

which satisfies the following equalities:
\[ \partial_\mu \tilde{g}^{\mu \nu} \partial_\nu f^\pm(x, y) = e^{-2a \tilde{\omega}(x)} [ g_{\mu \nu}(x) - e_{\mu \nu}(x) ], \]  
\text{(5·15)}
\[ g^{\mu \nu}(x) \partial_\mu f^\pm(x, y) \partial_\nu f^\pm(x, y) = 0, \]  
\text{(5·16)}
\[ \partial_\mu \tilde{g}^{\mu \nu}(x) \partial_\nu f^\pm(x, y) = 0, \]  
\text{(5·17)}
\[ e_{\mu \nu}(x) \partial_\mu f^\pm(x, y) = - \partial_\mu f^\pm(x, y), \]  
\text{(5·18)}
\[ \frac{1}{2} \left[ \frac{\partial_\mu f^\pm(x, y)}{\partial_\mu f^\pm(x, y)} \right] = - \frac{1}{g^{\theta \theta}(x)}, \]  
\text{(5·19)}

with \( e_{\mu \nu}(x) \) being the antisymmetric tensor defined in (A·2).

Let \( \Gamma(x, y) \) denote the square of the (rescaled) geodesic distance of two points \( x^a \) and \( y^a \) defined by
\[ e^{\lambda(x)} g^{\mu \nu}(x) \partial_\mu \Gamma(x, y) \partial_\nu \Gamma(x, y) = 4 \Gamma(x, y) \]  
\text{(5·20)}

with an appropriate scale factor function \( \lambda(x) \). Since it should be simply given by \( \eta_{ab} (\dot{x}^a(x) - \dot{x}^a(y))(\dot{x}^b(x) - \dot{x}^b(y)) \), we obtain the following expression:
\[ \Gamma(x, y) = \eta_{ab} f^a(x, y) f^b(x, y) \]
\[ = f^+(x, y) f^-(x, y) \]
\[ = \int_p^x dz^\mu (h_\mu^0 + h_\mu^1) e^{-a \tilde{\omega} + \theta_0 \gamma} \int_p^y dw^\nu (h_\nu^0 - h_\nu^1) e^{-a \tilde{\omega} - \theta_0 \gamma}. \]  
\text{(5·21)}

Actually, (5·21) satisfies (5·20) with
\[ \lambda(x) = 2a \tilde{\omega}(x) \]  
\text{(5·22)}
because of (5·15) and (5·16). We can evaluate the sign of $\Gamma(x, y)|_0$ in the following way. From (A·5) with $\nu=1$, we have

$$\partial_\mu(h_0^0 \pm h_1^1) = (\Gamma_{\mu 1}^{h_0^0} + \omega_\mu)(h_0^0 \pm h_1^1) + \Gamma_{\mu 1}^0(h_0^0 \pm h_0^1).$$  

(5·23)

Substituting the identity

$$h_0^0 \pm h_0^1 = \frac{g_{01} + h}{g_{11}} (h_0^0 \pm h_1^1)$$  

(5·24)

into (5·23), we obtain

$$\frac{\partial_\mu(h_0^0 \pm h_1^1)}{(h_0^0 \pm h_1^1)} = \frac{1}{2} \frac{\partial_\mu g_{11} + \omega_\mu}{g_{11}} \left( \omega_\mu - \frac{\Gamma_{\mu 1}^0}{g_{00}} \right);$$  

(5·25)

hence

$$h_0^0 \pm h_0^1 = \pm \sqrt{-g_{11}} \exp \left[ \pm \omega \pm \int^x dz^\nu \frac{\Gamma_{\nu 1}^0}{g_{00}} \right],$$  

(5·26)

where the line integral is path independent because of (A·10) with $R=0$. The overall sign factor of (5·26) is determined so as to be consistent with the Minkowskian limit $h_\mu^a \rightarrow \delta_\mu^a$. Then, the path independence of the integral in (5·21) yields

$$\Gamma(x, y)|_0 = \int_{x_1}^{x_2} dz^1(h_0^0 + h_1^1)e^{-\omega \pm \beta \omega} \cdot \int_{y_1}^{y_2} dw^1(h_0^0 - h_1^1)e^{-\omega \pm \beta \omega} \frac{\partial_\mu g_{11} + \omega_\mu}{g_{11}} \left( \omega_\mu - \frac{\Gamma_{\mu 1}^0}{g_{00}} \right);$$

(5·27)

where the line integral is path independent because of (A·10) with $R=0$. The overall sign factor of (5·26) is determined so as to be consistent with the Minkowskian limit $h_\mu^a \rightarrow \delta_\mu^a$. Then, the path independence of the integral in (5·21) yields

$$\Gamma(x, y)|_0 = \int_{x_1}^{x_2} dz^1(h_0^0 + h_1^1)e^{-\omega \pm \beta \omega} \cdot \int_{y_1}^{y_2} dw^1(h_0^0 - h_1^1)e^{-\omega \pm \beta \omega} \frac{\partial_\mu g_{11} + \omega_\mu}{g_{11}} \left( \omega_\mu - \frac{\Gamma_{\mu 1}^0}{g_{00}} \right);$$

(5·27)

In quantum gravity, the $x^a$'s are no longer c-number coordinates although they commute with themselves because of (4·3). There, we treat two-point operator $f^a(x, y)$, rather than $x^a$ itself; this is because quantum theory should be irrelevant to such a classical concept as a fixed point of contact. Using (3·6) and (3·8) for $\varphi=h_\mu^a$ together with (A·6) and (A·17), we have

$$[iP(\vec{X}), \omega_v(z)] = \eta(\vec{X}, x^\nu)\partial_\mu \omega_v(z),$$

(5·28)

$$[iM(X, Y), \omega_v(z)] = \eta(Y, x^\nu)X(z)\partial_\mu \omega_v(z) - \eta(Y, \omega)\partial_\mu X(z)$$

$$+ \eta(Y, \omega)e_\nu(z)\partial_\mu X(z) - \epsilon(X, Y)(X \leftrightarrow Y);$$

(5·29)

$$[iP(\vec{X}), \omega_v(z)] = \eta(\vec{X}, x^\nu)\partial_\mu \omega_v(z),$$

(5·30)

$$[iM(X, Y), \omega_v(z)] = \eta(Y, x^\nu)X(z)\partial_\mu \omega_v(z) + \eta(Y, \omega)e_\nu(z)\partial_\mu X(z)$$

$$- \eta(Y, \omega)\partial_\mu X(z) - \epsilon(X, Y)(X \leftrightarrow Y).$$

(5·31)

Hence we obtain the following transformation properties of $f^a(x, y)$:

$$[iP(\vec{X}), f^a(x, y)] = \eta(\vec{X}, x^\nu)(\partial_\mu \omega + \partial_\nu \omega)f^a(x, y)$$

$$- \eta(\vec{X}, \omega)f^a(x, y) \pm \eta(\vec{X}, \omega)f^\pm(x, y),$$

(5·32)
\[ [iM(X, Y), f^\pm(x, y)] = \eta(Y, x^\mu)(X(x)\partial_\mu^x + X(y)\partial_\mu^y)f^\pm(x, y) \]
\[ - \eta(Y, \omega)[\alpha C(X)f^\pm(x, y) + \beta \mathcal{F}^\pm(X; x, y)] \]
\[ \pm \eta(Y, \omega)[\beta C(X)f^\pm(x, y) + \alpha \mathcal{F}^\pm(X; x, y)] \]
\[ - \epsilon(X, Y)(X \leftrightarrow Y), \tag{5.33} \]

where

\[ C(X) = X(x) - \int^x dz \partial_\mu X(z), \tag{5.34} \]
\[ \mathcal{F}^\pm(X; x, y) = \int_x^y dz \partial^\mu \left[ X(z) \mp \int_z^x dw \epsilon_{\mu}^w \partial_\nu X \right] \partial_\nu^x \pm(z), \tag{5.35} \]
\[ \partial_\mu^x \pm(z) = (h_\mu^0(z) \pm h_\nu^1(z))e^{-a\omega^\pm(z)} \tag{5.36} \]

Here, \( C(X) \) is a nonzero constant operator \( \partial_\mu C(X) = 0 \) corresponding to the zero mode of \( X \). Since \( C(1) = 1, \mathcal{F}^\pm(1; x, y) = f^\pm(x, y) \) and \( \alpha + \beta = 1, (5.32) \) for \( \mathcal{X} = X \) is reproduced from (5.33) by setting \( P(X) = M(1, X) = -M(X, 1) \) and \( \eta(1, \mathcal{Z}) = \eta(\mathcal{Z}, 1) = \eta(1, 1) = 0 \).

§ 6. Expressions for the gravitational \( D \)-function and \( D^{(\pm)} \)-functions

To begin with, we recapitulate the invariant singular functions in the two-dimensional Minkowskian spacetime. For later convenience, we represent the Minkowskian spacetime indices by lowercase Latin letters as in § 5.

There are two odd invariant solutions to the Minkowskian d'Alembert equation:

\[ \Box D(x) = \Box \tilde{D}(x) = 0, \tag{6.1} \]
\[ D(x) = -\frac{1}{2} \epsilon(x^0) \theta(x^3), \tag{6.2} \]
\[ \tilde{D}(x) = -\frac{1}{2} \epsilon(x^1) \theta(-x^3), \tag{6.3} \]

where \( \Box \equiv (\partial_0)^2 - (\partial_3)^2, \ x^2 = (x^0)^2 - (x^1)^2, \ \epsilon(\xi) = \text{sign of } \xi \) and \( \theta(\xi) = (1/2)(1 + \epsilon(\xi)) \). The Heaviside function \( \theta(\xi) \) can also be expressed as

\[ \theta(\xi) = -\frac{1}{2\pi i} \log \frac{-\xi - i0}{\xi + i0}. \tag{6.4} \]

Although the right-hand side of (6.4) could formally be rewritten as

\[ \frac{1}{2\pi i} \log \frac{-\xi - i0}{\xi + i0} \tag{6.5} \]

by using the properties of logarithm, (6.5) is actually equal to \( -\theta(-\xi) = \theta(\xi) - 1 \) but not to \( \theta(\xi) \). We must be very careful for taking the correct branch of logarithm.

It is possible to define \( D(x) \) and \( \tilde{D}(x) \) uniquely by the following Cauchy problems:

\[ \Box D(x) = 0, \tag{6.6} \]
where $\delta$ denotes to set $x_{\alpha}=0$. From (6·2) and (6·3), or (6·6)~(6·11), $D(x)$ and $\bar{D}(x)$ are mutually related through

$$ D(x)=-D(\bar{x}) ,$$

$$ \partial_{\alpha}D(x)=-\epsilon_{abc}\partial_{c}D(x) ,$$

where $\bar{x}^a\equiv-\epsilon^{abc}\eta_{bc}x^c$. Integrating (6·13), we obtain

$$ D(x)=-\int_{-\infty}^{\infty}dz^a\epsilon_{abc}\partial_{c}\bar{D}(z) ,$$

$$ \bar{D}(x)=-\int_{-\infty}^{\infty}dz^a\epsilon_{abc}\partial_{c}D(z)+\frac{1}{2} .$$

In defining the positive/negative frequency part of $D(x)$, we introduce a positive constant $\mu$ to avoid infrared divergence:

$$ D(\pm)(x)=\mp(4\pi)^{-1}[\log(-\mu^2x^2\pm i0)] ,$$

where $x^{\pm}=x^0\pm x^1$. It has the following properties:

$$ D(-)(x)=-[D(+)(-x)]^* ,$$

$$ iD(x)=D(+)D(-)(x) ,$$

Given two functions $f$ and $g$ satisfying $\square f=\square g=0$ and $f\partial_{\alpha}g=0$ and $\partial_{\alpha}f\cdot g=0$ as $x^{\alpha}\rightarrow\pm\infty$, we define a one-dimensional convolution of $f$ and $g$ by

$$ (f\ast_{1}g)(x)=\int_{-\infty}^{\infty}dz^1[\partial_{\alpha}f(x-\bar{z})\cdot g(\bar{z})-f(x-\bar{z})\partial_{\alpha}g(\bar{z})] .$$

Then $\bar{D}(\pm)(x)$ is defined by

$$ \bar{D}(\pm)(x)\equiv D^{(\pm)}\ast\bar{D}(x) ,$$

$$ =\pm\frac{1}{4\pi}\log\frac{x^-\mp i0}{x^+\mp i0} ,$$

which satisfies (6·17)~(6·19) with tildes for all the $D$'s. One should note that $\bar{D}(\pm)(x)$ is not Lorentz invariant:
(x_0 \partial_1 - x_1 \partial_0) \bar{D}^{( \pm )}(x) = i (2 \pi)^{-1} . \quad (6.22)

We have the following formulae:

\[ f * D = f \quad \text{for any \( f \)}, \quad (6.23) \]

\[ 1 * D^{(\pm)} = \frac{i}{2}, \quad (6.24) \]

\[ 1 * \bar{D}^{(\pm)} = 0, \quad (6.25) \]

\[ D^{(\pm)} * D^{(\pm)} = i D^{(\pm)} , \quad (6.26) \]

\[ \bar{D}^{(\pm)} * D^{(\pm)} = i \bar{D}^{(\pm)} , \quad (6.27) \]

\[ \bar{D}^{(\pm)} * \bar{D}^{(\pm)} = i D^{(\pm)} . \quad (6.28) \]

Similarly, the massless \( S \)-function in the two-dimensional Minkowskian spacetime is an even invariant function defined by the Cauchy problem

\[ \gamma^a \partial_a S(x) = 0 , \quad (6.29) \]

\[ S(x)|_0 = - i \gamma^a \delta(x^1) \quad (6.30) \]

with the solution

\[ S(x) = i \gamma^a \partial_a D(x) , \quad (6.31) \]

where the \( \gamma^a \)'s are the two-dimensional flat-space gamma matrices [see Appendix A]. The positive/negative frequency part of \( S(x) \) is given by

\[ S^{(\pm)}(x) = D^{(\pm)} * S(x) \]

\[ = i \gamma^a \partial_a D^{(\pm)}(x) \]

\[ = S^{(\mp)}(-x) \]

\[ = \gamma^0 [S^{(\mp)}(x)]^* \gamma^0 , \quad (6.32) \]

\[ i S(x) = S^{(+)}(x) + S^{(-)}(x) . \quad (6.33) \]

Now, we consider the gravitational extensions of the above expressions. From (4.17) and the two-dimensional commutativity between \( h_{\mu}^a(z) \) and \( \mathcal{D}(x, y) \), we have

\[ [i P(X), \mathcal{D}(x, y)] = \eta(X, x^a)(\partial_a x^1 + \partial_1 x^a) \mathcal{D}(x, y) , \quad (6.34) \]

\[ [i M(X, Y), \mathcal{D}(x, y)] = \eta(Y, x^a)(X(x) \partial_a x^1 + X(x) \partial_1 x^a) \mathcal{D}(x, y) - \epsilon(X, Y)(X \leftrightarrow Y) , \quad (6.35) \]

whence the gravitational \( D \)-function \( \mathcal{D}(x, y) \) is a Weyl-invariant affine scalar. This fact infers that \( \mathcal{D}(x, y) \) can be rewritten into \( D(\hat{x}(x) - \hat{x}(y)) \) through the transformation defined by (5.13):

\[ \mathcal{D}(x, y) = D(\hat{x}(x) - \hat{x}(y)) \]

\[ = - \frac{1}{2} \epsilon(f^a(x, y)) \theta(\Gamma(x, y)) \]
Indeed, it is straightforward to see that (6·36) satisfies (4·5)~(4·7) by making use of (5·16), (5·17), (5·27)* and (5·19). One should note that $\mathcal{D}(x, y)$ is independent of the parameter $\alpha$ although $f^\pm(x, y)$ defined by (5·14) is dependent. This is due to the positive definiteness of the Weyl transformation under which both the sign function $\varepsilon$ and the Heaviside function $\theta$ are invariant.

The gravitational extension of $\tilde{D}$, denoted by $\tilde{\mathcal{D}}(x, y)$, is defined by the Cauchy problem

$$
\partial^\mu \tilde{g}^{\mu\nu}(x) \partial^\nu \tilde{\mathcal{D}}(x, y) = 0, \tag{6·37}
$$

$$
\mathcal{D}(x, y)|_{0} = -\frac{1}{2} \varepsilon(x^1 - y^1), \tag{6·38}
$$

$$
\partial x^\nu \tilde{\mathcal{D}}(x, y)|_{0} = 0. \tag{6·39}
$$

Its solution is given by

$$
\tilde{\mathcal{D}}(x, y) = \tilde{D}(\tilde{x}(x) - \tilde{x}(y))
$$

$$
= -\frac{1}{2} \varepsilon(f^1(x, y)) \theta(-\Gamma(x, y))
$$

$$
= \frac{1}{4\pi i} \log \frac{(f^- (x, y) - i0)(f^+ (x, y) + i0)}{(f^- (x, y) + i0)(f^+ (x, y) - i0)}
$$

$$
= -\tilde{\mathcal{D}}(y, x)
$$

$$
= [\tilde{\mathcal{D}}(x, y)]^*. \tag{6·40}
$$

The relations between $\mathcal{D}(x, y)$ and $\tilde{\mathcal{D}}(x, y)$ are obtained by generalizing the corresponding ones in the Minkowskian case:

$$
\partial^\mu \tilde{g}^{\mu\nu}(x) \partial^\nu \mathcal{D}(x, y) = 0, \tag{6·41}
$$

$$
\mathcal{D}(x, y) = -\int_{-\infty}^{x} dz e_r(x, z) \partial^r \tilde{\mathcal{D}}(z, y), \tag{6·42}
$$

$$
\tilde{\mathcal{D}}(x, y) = -\int_{-\infty}^{x} dz e_r(x, z) \partial^r \mathcal{D}(z, y) + \frac{1}{2}. \tag{6·43}
$$

Although there is no intrinsic meaning to the positive/negative frequency at the operator level in quantum gravity, we may define $\mathcal{D}^{(\pm)}(x, y)$ as the quantity satisfying the following relations:

$$
\partial^\mu \hat{g}^{\mu\nu}(x) \partial^\nu \mathcal{D}^{(\pm)}(x, y) = 0, \tag{6·44}
$$

$$
\mathcal{D}(x, y) = \mathcal{D}^{(+)\pm}(x, y) - [\mathcal{D}^{(+)\pm}(x, y)]^*, \tag{6·45}
$$

$$
\mathcal{D}^{(\pm)} \mathcal{D}^{(\pm)}(x, y) = i \mathcal{D}^{(\pm)}(x, y). \tag{6·46}
$$

We then set

*) We assume that if $\theta(\Gamma(x, y))$ vanishes classically, it also vanishes in quantum gravity.
In (6.46), we have used the formal gravitational extension of the one-dimensional convolution, which is defined by

\[(\mathcal{A} \ast \mathcal{B})(x, y) = \int_{-\infty}^{\infty} dx \tilde{g}^\nu(x) \partial_\nu \mathcal{A}(x, z) \cdot \mathcal{B}(z, y) - \mathcal{A}(x, z) \partial_\nu \mathcal{B}(z, y)\]  

(6.48)

for two bilocal operators \(\mathcal{A}(x, y)\) and \(\mathcal{B}(x, y)\) satisfying the d'Alembert equation in \(y\) and in \(x\), respectively. In the same way as above, we find the solution to (6.44) \(~(\mathcal{D}(\pm)(x, y)\):

\[\mathcal{D}^{(\pm)}(x, y) = D^{(\pm)}(\dot{x}(x) - \dot{x}(y))\]

\[= \pm \frac{1}{4\pi} [\log \mu(f^-(x, y) \mp i0) + \log \mu(f^+(x, y) \mp i0)] + i\pi \]

\[= - \mathcal{D}^{(\mp)}(y, x).\]  

(6.49)

The positive/negative frequency part of \(\mathcal{D}(x, y)\) is given by

\[\mathcal{D}^{(\pm)}(x, y) = D^{(\pm)}(\dot{x}(x) - \dot{x}(y))\]

\[= \pm \frac{1}{4\pi} \log f^-(x, y) \mp i0 \]

\[= - \mathcal{D}^{(\mp)}(y, x)\]

\[= - [\mathcal{D}^{(\pm)}(x, y)]^+.\]  

(6.50)

The relations corresponding to (6.27) and (6.28) are, of course, satisfied.

As for the gravitational extension of the massless \(S\)-function, defined by (4.8) and (4.9), we should take care of the effects of the local Lorentz transformation and the Weyl transformation. Then, we obtain

\[S(x, y) = e^{-(\alpha/2) \tilde{\omega}(x) - (\beta/2) \omega(x)} \tilde{\gamma}^a S(\dot{x}(x) - \dot{x}(y)) e^{-(\alpha/2) \tilde{\omega}(y) + (\beta/2) \omega(y)} \tilde{\gamma}^a\]

\[= i e^{(\alpha/2) \tilde{\omega}(x,y) + (\beta/2) \omega(x,y)} \tilde{\gamma}^a \gamma^\mu(x) \partial_\mu \mathcal{D}(x, y)\]

\[= S(y, x)\]

\[= - \tilde{\gamma}^a [S(y, x)]^* \tilde{\gamma}^0,\]  

(6.51)

\[S^{(\pm)}(x, y) = i e^{(\alpha/2) \tilde{\omega}(x,y) + (\beta/2) \omega(x,y)} \tilde{\gamma}^a \gamma^\mu(x) \partial_\mu \mathcal{D}^{(\pm)}(x, y),\]  

(6.52)

where \(\tilde{\omega}(x, y) = \tilde{\omega}(x) - \tilde{\omega}(y)\), \(\omega(x, y) = \omega(x) - \omega(y)\) and use has been made of the following equality

\[\tilde{\gamma}^a \partial_\mu \mathcal{D}^{(\pm)} = \tilde{\gamma}^a e^{\sigma \tilde{\omega}(x)} [\gamma^\mu(x)] \mathcal{D}^{(\pm)}(x)\]

\[= e^{\sigma \tilde{\omega}(x)} \mathcal{D}(x) \tilde{\gamma}^a \gamma^\mu(x).\]  

(6.53)

From (5.32) and (5.33), it is possible to obtain the transformation properties of \(\mathcal{D}^{(\pm)}\), \(\mathcal{D}^{(\pm)}\) and \(S^{(\pm)}\).
\[ [iP(\bar{X}), \mathcal{D}^{(\pm)}(x, y)] = \eta(\bar{X}, x^\mu)(\partial_\mu x^\nu + \partial_\nu x^\mu) \mathcal{D}^{(\pm)}(x, y) \pm \frac{1}{2\pi} \eta(\bar{X}, \bar{\omega}), \quad (6.54) \]

\[ [iM(X, Y), \mathcal{D}^{(\pm)}(x, y)] = \eta(Y, x^\mu)(X(x)\partial_\mu x^\nu + X(y)\partial_\nu x^\mu) \mathcal{D}^{(\pm)}(x, y) \pm \frac{1}{2\pi} \eta(Y, \bar{\omega}), \quad (6.56) \]

As seen from (6.54) and (6.56), neither the global Weyl invariance of \( \mathcal{D}^{(\pm)}(x, y) \) nor the internal Lorentz invariance of \( \mathcal{M}^{(\pm)}(x, y) \) is satisfied.

§ 7. Bosonization formula

Due to the results obtained in § 6, it is straightforward to extend the bosonization formula in the two-dimensional Minkowskian spacetime to the quantum-gravity case, provided that the model considered is Weyl invariant. For this purpose, we summarize the properties of the massless scalar field coupled with gravity, in the first place.

Replacing \( \mathcal{L}_0 \) in (2.1) by \( \mathcal{L}_s \) defined by
we obtain the following:
\[ \partial_\mu \bar{\phi}^{\mu} \partial_\nu \phi = 0, \quad (7.2) \]
\[ [\phi(x), \phi(y)] = i \mathcal{D}(x, y), \quad (7.3) \]
\[ [\phi(x), \lambda(x)] = 0, \quad \phi(x), \mathcal{D}(y, z) = 0, \quad (7.4) \]
\[ [\phi(x), X(y)] = i \gamma(X, x^\nu) \partial_\nu \phi(x) \cdot \mathcal{D}(x, y), \quad (7.5) \]
\[ [i P(\bar{X}), \phi] = \gamma(\bar{X}, x^\nu) \partial_\nu \phi, \quad (7.6) \]
\[ [i M(X, Y), \phi] = \gamma(Y, x^\nu) X \partial_\nu \phi - \epsilon(X, Y)(X \leftrightarrow Y). \quad (7.7) \]

We introduce the conjugate field \( \bar{\phi} \), which is defined by
\[ \partial_\mu \bar{\phi} = - e_\mu^\nu \partial_\nu \phi. \quad (7.8) \]

Note that (7·2) is nothing but the integrability condition of (7·8). Hence, assuming \( \lim_{x \to -\infty} \phi(x) = 0 \), we can integrate (7·8) as
\[ \bar{\phi}(x) = - \int_x^{-\infty} dz^\nu e_\mu^\nu(z) \partial_\nu \phi(z), \quad (7.9) \]
where the initial point of integration is set to \( (x^0, -\infty) \). From (7·8), \( \bar{\phi} \) satisfies
\[ \partial_\mu \bar{\phi}^{\mu} \partial_\nu \bar{\phi} = 0. \quad (7.10) \]

Furthermore, from (6·43) and (6·42), we obtain
\[ [\phi(x), \bar{\phi}(y)] = i \mathcal{D}(x, y) + \frac{i}{2}, \quad (7.11) \]
\[ [\bar{\phi}(x), \bar{\phi}(y)] = i \mathcal{D}(x, y). \quad (7.12) \]

By means of \( \mathcal{D}^{(\pm)}(x, y) \) defined by (6·49), we formally define the positive/negative frequency parts of \( \phi \) and \( \bar{\phi} \) (note, however, that we do not yet introduce the state vector space):
\[ \phi^{(\pm)}(x) = - i \mathcal{D}^{(\pm)} \ast \phi(x) \quad (7.13) \]
\[ \phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x); \quad (7.14) \]
\[ \bar{\phi}^{(\pm)}(x) = - i \mathcal{D}^{(\pm)} \ast \bar{\phi}(x), \quad (7.15) \]
\[ \bar{\phi}(x) = \bar{\phi}^{(+)}(x) + \bar{\phi}^{(-)}(x). \quad (7.16) \]

From the convolution properties of \( \mathcal{D}^{(\pm)} \) and \( \bar{\mathcal{D}}^{(\pm)} \), we obtain
\[ [\phi^{(\pm)}(x), \phi^{(\mp)}(y)] = \mathcal{D}^{(\pm)}(x, y), \quad (7.17) \]
\[ [\phi^{(\pm)}(x), \phi^{(\pm)}(y)] = 0, \quad (7.18) \]
\[ \left[ \phi^{(\pm)}(x), \tilde{\phi}^{(\mp)}(y) \right] = \mathcal{D}^{(\pm)}(x, y) + \frac{i}{8} \,, \quad (7.19) \]

\[ \left[ \phi^{(\pm)}(x), \tilde{\phi}^{(\pm)}(y) \right] = \frac{i}{8} \,, \quad (7.20) \]

\[ \left[ \tilde{\phi}^{(\pm)}(x), \tilde{\phi}^{(\mp)}(y) \right] = \mathcal{D}^{(\pm)}(x, y) \,, \quad (7.21) \]

\[ \left[ \tilde{\phi}^{(\pm)}(x), \tilde{\phi}^{(\pm)}(y) \right] = 0 \,. \quad (7.22) \]

Now, we consider the bosonization formula for the massless Dirac field. Since the bosonized Dirac field is not identical with the genuine one \( \phi \), the former is denoted by \( \tilde{\phi} = (\tilde{\phi}_j) \), which satisfies

\[ i\gamma^\mu(x)(\partial_\mu + \omega_\mu)^\tau \tilde{\phi}(x) = 0 \,, \quad (7.23) \]

\[ \{ \tilde{\phi}_j(x), \tilde{\phi}_k(y) \} = \{ \tilde{\phi}_j(x), \tilde{\phi}_j(y) \} = 0 \quad \text{for} \quad \theta(\Gamma(x, y)) = 0 \,, \quad (7.24) \]

\[ \{ \tilde{\phi}_j, \tilde{\phi}_k \} = 0 \,, \quad \{ \tilde{\phi}_j, \tilde{\phi}_j(\Lambda) \} = (\gamma^0 \gamma^0)_{jk} \frac{\delta}{\delta \phi^0} \,. \quad (7.25) \]

The solution, in terms of \( \phi, \tilde{\phi} \) and the zweibein, is given by

\[ \tilde{\phi} = e^{i(\sqrt{\pi} - \gamma^0 \phi) - (2\sqrt{\pi}) \gamma^0 \phi} u \,, \quad (7.26) \]

where colons indicate to arrange the product in the order \( \tilde{\phi}^{(-)}, \tilde{\phi}^{(-)}, \tilde{\phi}^{(+)} \), \( \tilde{\phi}^{(+)} \) and \( u = (u_j) \) with \( u_j \) being a complex number such that

\[ |u_j|^2 = \frac{\mu}{2\pi} \,. \quad (7.27) \]

We can find (7.26) by the same \( q \)-number transformation discussed in § 5 from the corresponding formula in the Minkowskian case. Since it is straightforward to check that (7.26) satisfies (7.23), we have only to consider (7.24) and (7.25).

From (7.26), we have

\[ \tilde{\phi}_j = u_j \cdot e^{i(\sqrt{\pi} - \gamma^0 \phi) - (2\sqrt{\pi}) \gamma^0 \phi} \,, \quad (7.28) \]

\[ \tilde{\phi}_k = u_k \cdot e^{-i(\sqrt{\pi} - \gamma^0 \phi) - (2\sqrt{\pi}) \gamma^0 \phi} \,. \quad (7.29) \]

Since \( e^A e^B = e^{[A, B]} e^A e^B \) if \([ [A, B], A] = [[A, B], B] = 0 \), we have

\[ \tilde{\phi}(x) \tilde{\phi}(y) = e^{-\mathcal{M}(x, y)} \tilde{\phi}(y) \tilde{\phi}(x) \,, \quad (7.30) \]

\[ \tilde{\phi}(x) \tilde{\phi}(y) = e^{\mathcal{M}(x, y)} \tilde{\phi}(y) \tilde{\phi}(x) \,, \quad (7.31) \]

where

\[ \mathcal{M}(x, y) = i\tau \left[ 1 + (-1)^{j+k} \right] \mathcal{D}(x, y) \]

\[ -(-1)^{j} \left[ \mathcal{D}(x, y) - \frac{1}{2} \right] - (-1)^{k} \left[ \mathcal{D}(x, y) + \frac{1}{2} \right] \,. \quad (7.32) \]

Furthermore, since we have

\( ^* \) Here and hereafter no summation convention applies to indices \( j, k \).
\[ \mathcal{M}_{jk}(x, y) = i\pi[(-1)^{j}\theta(f^{j}(x, y)) - (-1)^{k}\theta(-f^{k}(x, y))] \quad \text{for} \quad \theta(f(x, y)) = 0 \]

(7.33)

from (6.36) and (6.40), \( \tilde{\phi} \) obeys Fermi statistics (7.24).

If both \([A^{(\pm)}, B^{(\pm)}] \) and \([A^{(\pm)}, B^{(\pm)}] \) commute with both \( A^{(\pm)} \) and \( B^{(\pm)} \), the identity

\[ e^{A^{(\pm)}}, e^{B^{(\pm)}} = e^{\left[(A^{(\pm)}, B^{(\pm)}) + 1/2(A^{(\pm)}, B^{(\pm)}) + 1/2(A^{(\pm)}, B^{(\pm)}) \right] e^{A^{(\pm)}}, e^{B^{(\pm)}}} \]

holds. From (7.17) \(-\) (7.22), therefore, we have

\[ \tilde{\phi}_{j}(x) \tilde{\phi}_{k}(y) = u_{j} u_{k} e^{-3\pi^{*}(x, y)} \tilde{\Phi}_{j}^{(+)}(x, y), \]

(7.35)

\[ \tilde{\phi}_{k}(y) \tilde{\phi}_{j}(x) = u_{j} u_{k} e^{-3\pi^{*}(x, y)} \tilde{\Phi}_{j}^{(+)}(x, y), \]

(7.36)

\[ \tilde{\phi}_{j}(x) \tilde{\phi}_{k}^{*}(y) = u_{j} u_{k} e^{3\pi^{*}(x, y)} \tilde{\Phi}_{j}^{(-)}(x, y), \]

(7.37)

\[ \tilde{\phi}_{k}^{*}(y) \tilde{\phi}_{j}(x) = u_{j} u_{k} e^{3\pi^{*}(x, y)} \tilde{\Phi}_{j}^{(-)}(x, y), \]

(7.38)

where

\[ \mathcal{M}^{(\pm)}_{jk}(x, y) = \pi \left\{ 1 + (-1)^{j+k} \right\} \mathcal{D}^{(\pm)}(x, y) \]

\[ = (-1)^{j} \left\{ \mathcal{D}^{(\pm)}(x, y) - \frac{i}{4} \right\} - (-1)^{k} \left\{ \mathcal{D}^{(\pm)}(x, y) + \frac{i}{4} \right\}, \]

(7.39)

\[ \mathcal{D}^{(\pm)}_{jk}(x, y) = \exp \left[ i\sqrt{\pi} (\phi(x) - (-1)^{j} \phi(x) \pm \phi(y) - (-1)^{k} \phi(y)) \right] \]

\[ = \frac{\alpha}{2} (\dot{\phi}(x) + \dot{\phi}(y)) - \frac{\beta}{2} ((-1)^{j} \phi(x) + (-1)^{k} \phi(y)) \].

(7.40)

For \( j \neq k \), we have

\[ \mathcal{M}^{(\pm)}_{jk}(x, y) = \frac{i}{2} \pi (-1)^{j} \]

\[ = \frac{i}{2} \pi (-1)^{j+1} + i\pi (-1)^{j} \]

\[ = \mathcal{M}^{(\pm)}_{ij}(y, x) + i\pi (-1)^{j}, \]

(7.41)

whence

\[ \{ \tilde{\phi}_{j}(x), \tilde{\phi}_{k}(y) \} = \{ \tilde{\phi}_{j}(x), \tilde{\phi}_{k}^{*}(y) \} = 0 \quad \text{for} \quad j \neq k. \]

(7.42)

For \( j = k \), we have

\[ \mathcal{M}^{(\pm)}_{jj}(x, y) = 2\pi [\mathcal{D}^{(\pm)}(x, y) - (-1)^{j} \mathcal{D}^{(\pm)}(x, y)] \]

\[ = -\log \mu(f^{(\pm)}(x, y) - i0) - \frac{i}{2} \pi. \]

(7.43)

Since \( e^{-3\pi^{*}(x, y)} + e^{-3\pi^{*}(y, x)} = 0 \), because \( f^{(\pm)}(x, y) = -f^{(\pm)}(y, x) \), we obtain

\[ \{ \tilde{\phi}_{j}(x), \tilde{\phi}_{j}(y) \} = 0. \]

(7.44)

On the other hand, since
\[
(e^{\gamma^{\mu\nu}(x,y)} + e^{\gamma^{\nu\mu}(y,x)})_0 = \frac{1}{2\pi i}|u|^2 \left[ \left( \int_{y_1}^{x_1} dz^1(h_{10} - (1)^{1/2}h_{11})e^{-a\omega - (1)^{1/2}h_{11}} - i0 \right)^{-1} \right. \\
\left. - \left( \int_{y_1}^{x_1} dz^1(h_{10} - (1)^{1/2}h_{11})e^{-a\omega - (1)^{1/2}h_{11}} + i0 \right)^{-1} \right]
\]
\[
= \frac{1}{|u|^2} \frac{e^{a\omega + (1)^{1/2}h_{11}}}{h(h_{00} + (-1)^1h_{01})} \delta(x^1 - y^1)
\]
\[
(7.45)
\]
we obtain
\[
\{ \tilde{\phi}_j(x), \tilde{\phi}_j(y) \}_0 = \frac{\delta(x^1 - y^1)}{h(h_{00} + (-1)^1h_{01})}
\]
\[
\times \hat{\delta}(x^1 - y^1)
\]
\[
(7.46)
\]
Thus the canonical anticommutation relations (7.25) are satisfied.

Next, we consider the gravitational extension of the bosonized current. First, we define the hermitian point-split current by
\[
\tilde{j}^\mu(x; \varepsilon) = \frac{1}{4} h(x) \{ \bar{\phi}(x + \varepsilon) \gamma^\mu(x) \phi(x) - \bar{\phi}(x - \varepsilon) \gamma^\mu(x) \} + \bar{\phi}(x) \gamma^\mu(x) \phi(x + \varepsilon) - \bar{\phi}(x - \varepsilon) \gamma^\mu(x)
\]
\[
= \frac{1}{4} h(x) \sum_{j=1}^2 (h_{00}(x) + (-1)^1h_{01}(x))
\]
\[
\times \{ \tilde{\phi}_j(x + \varepsilon) \tilde{\phi}_j(x) - \phi_j(x) \phi_j(x - \varepsilon)
\]
\[
+ \phi_j(x) \phi_j(x + \varepsilon) - \phi_j(x - \varepsilon) \phi_j(x) \}
\]
\[
= \left[ \tilde{j}^\mu(x; \varepsilon) \right]^t
\]
\[
(7.47)
\]
where a superscript \( t \) stands for matrix transposition. From (7.37) and (7.38), we obtain
\[
\tilde{\phi}_j(x) \tilde{\phi}_j(x + \varepsilon)
\]
\[
= \frac{1}{2\pi i} \left[ 1 \pm i \sqrt{\pi} \varepsilon^\mu (\delta_\mu^\nu + (-1)^1\varepsilon_\mu^\nu) \partial_\nu \phi \pm \frac{1}{2} \varepsilon^\mu (\varepsilon \delta_\nu^\nu - (1)^1\beta \delta_\nu^\nu) \omega_\nu + O(\varepsilon^2) \right]
\]
\[
= \frac{1}{2\pi i} \left[ \varepsilon^\mu (h_{00}(x) + (1)^1h_{01}(x)) \pm i0 \right]
\]
\[
(7.48)
\]
Hence
\[
\tilde{j}^\mu(x; \varepsilon) = -\frac{\hbar}{2\sqrt{\pi}} \sum_{j=1}^2 \frac{h_{00}(x) + (1)^1h_{01}(x)}{\varepsilon^\mu (h_{00}(x) + (1)^1h_{01}(x))} \left[ \varepsilon^\nu (\delta_\nu^\nu + (-1)^1\varepsilon_\nu^\nu) \partial_\nu \phi + O(\varepsilon^2) \right]
\]
\[ = -\frac{1}{\sqrt{\pi}} \tilde{g}^{\mu\nu} \partial_\nu \phi + O(\varepsilon), \]  

(7·50)

where use has been made of the equality

\[ \delta^\mu_\nu + (-1)^j e^\mu_\nu = (h^\mu_0 - (-1)^j h^\nu_0)(h^\nu_0 + (-1)^j h^\mu_0). \]  

(7·51)

Then we find

\[ \tilde{J}^\mu(x) = \lim_{\varepsilon \to 0} \tilde{J}^\mu(x; \varepsilon) \]

\[ = -\frac{1}{\sqrt{\pi}} \tilde{g}^{\mu\nu} \partial_\nu \phi, \]  

(7·52)

which is a natural extension of that in the Minkowskian case.

The above formulae are easily extended to the Thirring model since it is Weyl invariant. We set

\[ \tilde{\phi}_{\text{Th}} = Z^{1/2} e^{i(\alpha \phi - \beta \tilde{\phi}) - (\pi/2) \phi - (\pi/2) \tilde{\phi} \omega} \]  

(7·53)

where \( Z \) is a renormalization constant, and parameters \( a \) and \( b \) satisfy

\[ ab = \pi, \quad -a + b = \frac{\lambda}{2\pi} (a + b) \]  

(7·54)

with \( \lambda \) being a coupling constant. The bosonized current is defined by

\[ \tilde{J}^\mu_{\text{Th}}(x) = \frac{1}{2} \lim_{\varepsilon \to 0} (\tilde{J}^\mu(x; \varepsilon) + \tilde{J}^\mu_{\text{Th}}(x; \varepsilon)) \]

\[ = -\frac{a + b}{2\pi} \tilde{g}^{\mu\nu} \partial_\nu \phi, \]  

(7·55)

where \( \varepsilon \equiv -\varepsilon^\mu e^\mu_\nu \) and \( \tilde{J}^\mu_{\text{Th}}(x; \varepsilon) \) is defined by replacing \( \tilde{\phi} \) in (7·47) by \( \tilde{\phi}_{\text{Th}} \). Then, \( \tilde{\phi}_{\text{Th}} \) satisfies the following field equation:

\[ i\hbar \gamma^\mu (\partial_\mu + \omega_\mu) \tilde{\phi}_{\text{Th}} = -\lambda \gamma_\mu \tilde{J}^\mu_{\text{Th}} \tilde{\phi}_{\text{Th}}, \]  

(7·56)

as well as Fermi statistics and the canonical anticommutation relations.

In concluding this section, we give a comment on \( \tilde{\phi} \). As already emphasized in the Minkowskian case,\(^7\) we cannot identify the bosonized Dirac field \( \tilde{\phi} \) with the genuine one \( \phi \) in \$2$, though both of them satisfy the same field equation and the same equal-time anticommutation relations.\(^8\) Indeed, for example, their transformation properties are different as shown in the following. From (3·6), (6·54), (7·6), (7·13) and (7·15), we have

\(^8\) Hence both \( \varepsilon^\mu \) and \( \bar{\varepsilon}^\mu \) cannot be c-number.

\(^9\) This implies that \( \{ \tilde{\phi}(x), \tilde{\phi}(y) \} \) is not the correct solution to the Cauchy problem posed by the Dirac equation and the equal-time anticommutators. This pathology is due to the singular nature of \( \tilde{\phi}(x) \) relevant already in the Minkowskian case.\(^7\) For example, \( \tilde{\phi}(x) \) has its inverse \( (\tilde{\phi}(x))^{-1} \) in spite of the fact that it satisfies the Pauli principle.
\[ [iP(\bar{X}), \phi^{(\pm)}] = \eta(\bar{X}, x^{\mu})\partial_{\mu}\phi^{(\pm)} \pm \frac{i}{2\pi} \eta(\bar{X}, \bar{\omega}) \Phi, \quad (7.57) \]
\[ [iP(\bar{X}), \bar{\phi}^{(\pm)}] = \eta(\bar{X}, x^{\mu})\partial_{\mu}\bar{\phi}^{(\pm)}, \quad (7.58) \]

where
\[ \Phi = \int_{-\infty}^{\infty} dz^1 \tilde{g}^{0\nu} \partial_{\nu} \phi, \quad (7.59) \]
\[ [\Phi, \phi^{(\pm)}] = -\frac{i}{2}. \quad (7.60) \]

Therefore, we obtain\(^*)

\[ [iP(\bar{X}), e^{i\frac{\phi}{\sqrt{\pi}}} \bar{\phi}^{(\pm)}] = \eta(\bar{X}, x^{\mu})\partial_{\mu}e^{i\frac{\phi}{\sqrt{\pi}}} \Phi^{(\pm)}(\Phi + \frac{\sqrt{\pi}}{4}) \]
\[ = \eta(\bar{X}, x^{\mu})\partial_{\mu}e^{i\frac{\phi}{\sqrt{\pi}}} \bar{\phi}^{(\pm)}(\Phi - \frac{\sqrt{\pi}}{4}) e^{i\frac{\phi^{(\pm)}}{\sqrt{\pi}}}, \quad (7.61) \]
\[ [iP(\bar{X}), e^{i\frac{\phi}{\sqrt{\pi}}} \phi^{(\pm)}] = \eta(\bar{X}, x^{\mu})\partial_{\mu}\Phi^{(\pm)} e^{i\frac{\phi}{\sqrt{\pi}}}, \quad (7.62) \]

whence
\[ [iP(\bar{X}), \bar{\phi}] = \eta(\bar{X}, x^{\mu})\partial_{\mu}\bar{\phi} + \frac{1}{4} \eta(\bar{X}, \bar{\omega}) \bar{\phi}. \quad (7.63) \]

On the other hand, from (3.6) for \( \varphi = \phi \), we have
\[ [iP(\bar{X}), \phi] = \eta(\bar{X}, x^{\mu})\partial_{\mu}\phi + \frac{1}{2} \eta(\bar{X}, \omega) \phi + \frac{1}{2} \eta(\bar{X}, \omega) \bar{\phi}. \quad (7.64) \]

Therefore, we conclude that \( \phi \) and \( \bar{\phi} \) cannot transform in the same way.

\[ \text{§ 8. Discussion} \]

In the present paper, we have constructed the zweibein formalism of two-dimensional quantum gravity and obtained all the full (anti)commutation relations. Our theory has quite a large symmetry superalgebra consisting of 86 generators which unify the spacetime symmetry and the internal Lorentz symmetry.

Our most important result is that the spacetime parameter \( x^{\mu} \) in two-dimensional quantum gravity is related to the flat spacetime operator \( \hat{x}^{a} \) through the \( q \)-number transformation whose transformation functions are explicitly expressible in terms of the zweibein. Since \( \hat{x}^{a} \) commutes with the matter fields as well as \( \hat{x}^{b} \), we can treat the matter fields in this \( q \)-number coordinate system just like those in the usual Minkowskian spacetime. Using this \( q \)-number transformation, we have obtained the

\[^*3\] [A, e^{a}] = e^{a} \sum_{n=1}^{n} \frac{1}{n}[\ldots[(A, B), B], \ldots, B].\]
gravitational extensions of $D$-function, $D^{(\pm)}$-functions, $\tilde D$-function, $\tilde D^{(\pm)}$-functions, massless $S$-function, $S^{(\pm)}$-functions and the bosonization formulæ of the massless Dirac field and the Thirring model.

To define $\hat x^a$, no Weyl transformation is necessary in the case $a=0$. This is possible because the two-dimensional flatness condition $R=0$, which is nothing but the integrability condition, (2·17), of $\omega_\mu$, is satisfied. In the $d(\geq 3)$-dimensional Einstein gravity, we have no Weyl symmetry, but only in three dimensions, the flatness condition is satisfied if the Ricci tensor vanishes. Actually, we encounter this situation in the zeroth order approximation with respect to the gravitational coupling constant in a recently proposed approximation method for the operator solution to the quantum Einstein gravity. Therefore, the above $q$-number transformation is likely to be generalized to the three-dimensional case in this approximation.

Finally, we make some comments on Polyakov's work concerning the two-dimensional quantum gravity. He employed a nonlocal Lagrangian density

$$\mathcal L_{\text{Polyakov}} = \text{const} h R \square^{-1} R$$

with $\square = h^{-1} \partial_\mu \tilde g^{\mu\nu} \partial^\nu$ and worked in the light-cone gauge. His theory is not only non-covariant but also non-unitary. In what follows, we show that our theory can be regarded as a covariantized and unitarized version of Polyakov's theory.

In our Lagrangian density (2·1), we perform the field redefinition ($q$-number Weyl transformation) given by

$$h_\mu^a \to e^{(\pm i a) b} h_\mu^a, \quad \phi \to e^{(-a b) \phi},$$

others being invariant, where $a (\neq 0)$ is a real constant. Then only the last term of $\mathcal L_0$ is transformed into

$$-2 \tilde g^{\mu\nu} \partial_\mu \partial_\nu b - \frac{a}{2} \tilde g^{\mu\nu} \partial_\mu b \cdot \partial_\nu b$$

(see (A·18)). As stated in § 2 (see (A·19)), the first term of (8·3) can be rewritten as $h R b$ by neglecting total divergence. Hence if we eliminate $b$ (i.e., if we carry out the integration over $b$ in the sense of path integral), we formally obtain (8·1).

In Polyakov's light-cone gauge, we have

$$g_{\mu\nu} = \begin{pmatrix} g_{01} + 1 & g_{01} \\ g_{01} & g_{01} - 1 \end{pmatrix},$$

where $g_{01}$ is nothing but Polyakov's $h_{++}$. On the other hand, from (5·18) (with (5·13)) we have

$$\partial_\mu \hat x^\pm = \mp e_\mu^a \partial_\nu \hat x^\pm, \quad (\hat x^\pm = \hat x^0 \pm \hat x^1)$$

where $e_\mu^a = e_\mu \tilde g^{\lambda\nu}$. Substituting (8·4) into (8·5), we find

$$\partial_+ \hat x^- = "h_{++}" \partial_- \hat x^-, \quad \partial_- \hat x^+ = 0.$$  

Thus Polyakov's "$f$" coincides with our $\hat x^-$. 

Polyakov investigated $n$-point functions of "$h_{++}$". In our theory, however, $n$-point functions of $g_{\mu\nu}$ are trivial. We should investigate $n$-point functions involving $b_\nu$. Anyway, our next task is how to construct the representation of the two-dimensional (anti)commutation relations obtained in §4 on a state-vector space.

**Appendix A**

--- Zweibein Formalism and Its Peculiar Features ---

The zweibein field $h^a_{\mu}(\mu=0,1; a=0,1)$ is related to $g_{\mu\nu}$ through

$$g_{\mu\nu} = \eta_{ab} h^a_{\mu} h^b_{\nu} \quad (A\cdot1)$$

with $\eta_{ab} = \text{diag}(+, -)$ being the Minkowski metric of the internal space (tangent space). If $h^a_{\mu}$ is regarded as a $2\times2$ matrix, $h^T a$ is its transposed matrix inverse. Then, with the help of the antisymmetric tensors

$$e^{\mu\nu} = \frac{1}{h} e^{\mu\nu}, \quad e_{\mu\nu} = h e_{\mu\nu}, \quad (A\cdot2)$$

$$h = \det h^a_{\mu}, \quad (A\cdot3)$$

and the antisymmetric internal Lorentz tensors $\epsilon^{ab}$ and $\epsilon_{ab}$ with $\epsilon^{01} = \epsilon_{01} = 1$, we can write

$$h^a_{\mu} = - e^{\mu\nu} \epsilon_{ab} h^b_{\nu}. \quad (A\cdot4)$$

The spin connection denoted by $\omega_{\mu a} = \epsilon_{ab} \omega_{\mu}$ [a two-dimensional antisymmetric tensor is equivalent to a (pseudo)scalar] is defined by

$$\partial_{\mu} h^a_{\nu} - \Gamma^a_{\mu\nu} h^a_{\nu} + \omega_{\mu}^{ab} h^b_{\nu} = 0. \quad (A\cdot5)$$

Multiplying (A·5) by $e^{\mu\nu}$ and using (A·4), we have

$$\omega_{\mu} = h^a_{\mu} e^{\mu\nu} \partial_{\nu} h^a_{\mu} \quad (A\cdot6)$$

As is well known, the two-dimensional Einstein action is trivial because $hR$ is a total divergence. This is shown explicitly in the following three ways:*

$$hR = 2\partial_{\nu} [e^{\mu\nu}(g^{11})^{-1} \Gamma^0_0]$$

$$= -2\partial_{\nu} [e^{\mu\nu}(g^{00})^{-1} \Gamma^0_0]$$

$$= -2\partial_{\nu} (e^{\mu\nu} \omega_{\nu}). \quad (A\cdot7)$$

The first and second equalities are based on the identity

$$R^4_{\mu\nu} = \frac{1}{2} (\delta^4_{\nu} g_{\mu\nu} - \delta^4_{\nu} g_{\mu\nu}) R, \quad (A\cdot8)$$

from which we obtain

$$hR = \frac{2}{g^{11}} R^1_{010}$$

---

* We adopted none of those simple expressions in Ref. 1), unfortunately.
\[
\begin{align*}
&= \frac{2}{\bar{g}^{11}} (\partial_1 \Gamma_{00}^1 - \partial_0 \Gamma_{10}^1 + \Gamma_{11}^1 \Gamma_{00}^1 - \Gamma_{01}^1 \Gamma_{10}^1) \\
&= \frac{2}{\bar{g}^{11}} \left( \partial_1 \Gamma_{00}^1 - \partial_0 \Gamma_{10}^1 - \frac{\partial_1 \bar{g}^{11}}{\bar{g}^{11}} \Gamma_{00}^1 + \frac{\partial_0 \bar{g}^{11}}{\bar{g}^{11}} \Gamma_{10}^1 \right) \\
&= 2 \partial_\mu [\epsilon^{\mu \nu} (\bar{g}^{11})^{-1} \Gamma_{\nu 0}^1] 
\end{align*}
\]

and

\[
\begin{align*}
\hbar R &= \frac{2}{\bar{g}^{00}} R^0_{001} \\
&= -2 \partial_\mu [\epsilon^{\mu \nu} (\bar{g}^{00})^{-1} \Gamma_{\nu 1}^0], 
\end{align*}
\]

where \( \bar{g}^{\mu \nu} \equiv \frac{1}{\bar{g}} g^{\mu \nu} \) and the metric condition \( \nabla \bar{g}^{11} = 0 \) is used. As for the third equality of (A·7), it is due to the abelian nature of the two-dimensional Lorentz symmetry:

\[
\begin{align*}
&= \partial_\mu \left[ \epsilon^{\mu \nu} (\bar{g}^{00})^{-1} \Gamma_{\nu 0}^1 \right], 
\end{align*}
\]

The generally covariant and local Lorentz invariant Lagrangian density of the two-dimensional massless Dirac field \( \psi = (\phi) \) is given by

\[
\mathcal{L}_D = i \hbar \bar{\psi} \gamma^\mu (\partial_\mu + \omega_\mu) \psi, 
\]

where \( \gamma^\mu \equiv \hbar^a \gamma^a \), \( \omega_\mu \equiv (1/2) \omega_\mu \gamma^5 \) with \( \gamma^5 \equiv \gamma^0 \gamma^1 \) and \( \bar{\gamma} \equiv \psi^1 \gamma^0 \). Here the \( \gamma^a \)'s are the two-dimensional flat-spacetime gamma matrices defined by \( \gamma^0 = \sigma_1 \) and \( \gamma^1 = i \sigma_2 \) whence \( \gamma^5 = -\sigma_0 \) with the \( \sigma_i \)'s being the Pauli matrices.

Under the finite Weyl transformation defined by

\[
\begin{align*}
h_\mu^\alpha(x) &\rightarrow e^{i(x)} h_\mu^\alpha(x), 
\end{align*}
\]

the spin connection (A·6) transforms as

\[
\omega_\mu \rightarrow \omega_\mu - e^\alpha_\mu \partial_\mu \epsilon. 
\]

If we define the Weyl transform of \( \psi \) by

\[
\psi \rightarrow e^{-(1/2)\epsilon} \psi, 
\]

then \( \mathcal{L}_D \) is Weyl invariant as easily shown by substituting the identity

\[
\gamma^\mu \omega_\mu = \frac{1}{2} \gamma^\mu \tilde{\omega}_\mu 
\]

into (A·12), where

\[
\begin{align*}
\tilde{\omega}_\mu &\equiv -e^\nu_\mu \omega_\nu \\
&= \epsilon^{ab} h_{\mu b} e^{\nu a} \partial_\nu h_{\rho a}, 
\end{align*}
\]

and (A·16) is a consequence of \( \gamma^a \gamma^5 = \epsilon^{ab} \gamma_b \) and (A·4). Indeed, the conjugate spin
connection $\tilde{\omega}_\mu$ plays precisely the role of the connection for the Weyl invariance:

$$\tilde{\omega}_\mu \rightarrow \tilde{\omega}_\mu + \partial_\mu \varepsilon,$$

whence one may rather call $\tilde{\omega}_\mu$ the Weyl connection.

It is interesting to note that the Weyl gauge fixing $R=0$ adopted in Ref. 1 corresponds to nothing but the Landau gauge for $\tilde{\omega}_\mu$, since we have

$$\partial_\mu (\tilde{g}^{\mu\nu} \tilde{\omega}_\nu) = \frac{1}{2} \hbar R$$

from (A-7) and (A-17). Conversely, the curvature for the connection $\tilde{\omega}_\mu$, given by

$$\left[ \partial_\mu + \frac{1}{2} \tilde{\omega}_\mu, \partial_\nu + \frac{1}{2} \tilde{\omega}_\nu \right] = \frac{1}{2} \epsilon_{\mu\nu\alpha} \partial_\alpha (\tilde{g}^{\rho\sigma} \omega_\rho),$$

vanishes if we adopt the Landau gauge for $\omega_\mu$.

**Appendix B**

--- Asymptotic Fields and Unitarity ---

It is straightforward to confirm the unitarity of our theory by extending the previous result$^1$ to the zweibein formalism in a similar way done in the vierbein formalism of quantum Einstein gravity.$^3$ In this appendix, we omit matter fields for simplicity since they are irrelevant to the following discussion.

Under the postulate of asymptotic completeness, we introduce the following asymptotic fields:

$$(h_\mu^a - \delta_\mu^a)/2 \rightarrow \chi_\mu^a,$$

$$b_\mu \rightarrow \beta_\mu, \quad b \rightarrow \beta, \quad c^\sigma \rightarrow \gamma^\sigma, \quad \bar{c}_\mu \rightarrow \bar{\gamma}_\mu,$$

$$s \rightarrow \sigma, \quad t \rightarrow \tau, \quad \bar{t} \rightarrow \bar{\tau}.$$  

(B-1)

Raising and lowering of indices of the asymptotic fields are carried out by the Minkowski metric. It is convenient to set

$$\varphi_{\mu\nu} \equiv \chi_{\mu\nu} + \chi_{\nu\mu}, \quad \varphi = 2 \chi^\mu_\mu,$$

$$\phi_{\mu\nu} \equiv \varphi_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \varphi,$$

$$\bar{\chi} \equiv e^{\mu\nu} \chi_{\nu\mu},$$

(B-2)

(B-3)

(B-4)

where $\varphi_{\mu\nu}$ coincides with the asymptotic field of $(g_{\mu\nu} - \eta_{\mu\nu})/2$.

The asymptotic fields are governed by the following linearized Lagrangian density:

$$\mathcal{L}^{as} = -2 \partial_\mu \phi^{\mu\nu} \cdot \tilde{\beta}_\nu - i \partial^\alpha \bar{\gamma}_\rho \cdot \partial_\rho \gamma^\alpha - \partial^\alpha \phi \cdot \partial_\mu \beta + 2 \partial^\alpha \bar{\chi} \cdot \partial_\mu \sigma + i \partial^\mu \bar{\tau} \cdot \partial_{\mu \tau},$$

(B-5)

where we set $\tilde{\beta}_\mu \equiv \beta_\mu - \partial_\mu \beta + \epsilon_{\mu\nu} \partial^\nu \sigma$ for simplicity of description. Field equations derived from (B-5) are

$$\partial^\mu \phi_{\mu\nu} = 0,$$

(B-6)
\[\partial_{\mu} \tilde{\beta}_{\nu} + \partial_{\nu} \tilde{\beta}_{\mu} - \eta_{\mu\nu}\partial^4 \tilde{\beta}_{\lambda} = 0\]  \hspace{1cm} (B\cdot 7)

and the d'Alembert equations for \(\beta, \gamma^\sigma, \bar{\gamma}, \sigma, \tau, \bar{\tau}\). From (B\cdot 6) and (B\cdot 7), we obtain the d'Alembert equation also for \(\phi_{\mu\nu}\) and \(\tilde{\beta}_{\mu}\), respectively, since \(\phi_{\mu\nu}\) is symmetric and traceless.

From the canonical (anti)commutation relations, we have the two-dimensional (anti)commutation relations:

\[\begin{align*}
[\phi_{\mu\nu}(x), \tilde{\beta}_{\lambda}(y)] &= \frac{i}{2}(\eta_{\mu\nu}\partial_{\lambda}^2 + \eta_{\nu\lambda}\partial_{\mu}^2 - \eta_{\mu\lambda}\partial_{\nu}^2)D(x-y), \\
[\phi(x), \beta(y)] &= -iD(x-y), \\
[\bar{\phi}(x), \sigma(y)] &= \frac{i}{2}D(x-y), \\
{\{\gamma^\sigma(x), \bar{\gamma}(y)\}} &= -\delta^\sigma D(x-y), \\
{\{\tau(x), \bar{\tau}(y)\}} &= \frac{1}{2}D(x-y),
\end{align*}\]  \hspace{1cm} (B\cdot 8 - B\cdot 12)

and others vanish.

The BRS transformations of asymptotic fields are as follows:

\[\begin{align*}
\delta_* \phi_{\mu\nu} &= -\frac{1}{2}(\partial_{\mu} \gamma_{\nu} + \partial_{\nu} \gamma_{\mu} - \eta_{\mu\nu}\partial_{\lambda} \gamma^\lambda), \\
\delta_* \phi &= -\partial_{\mu} \gamma^\mu, \\
\delta_* \bar{\phi} &= -\frac{1}{2}\epsilon_{\mu\nu}\partial_{\nu} \gamma_{\mu}, \\
\delta_* \bar{\gamma} &= i\beta_t = i(\tilde{\beta} + \partial_t \beta - \epsilon_{\tau\rho}\partial^\rho \tau); \\
\delta^4 \phi &= \gamma^4, \\
\delta^4 \bar{\gamma} &= i\delta^4 \beta; \\
\delta_{\lambda} \bar{\phi} &= \tau, \\
\delta_{\lambda} \bar{\gamma} &= i\sigma,
\end{align*}\]  \hspace{1cm} (B\cdot 13 - B\cdot 20)

and others vanish. Since every asymptotic field, denoted generically by \(\Phi(x)\), satisfies the d'Alembert equation, we can decompose \(\Phi(x)\) into a sum of two independent modes each of which depends only on either \(x^0 + x^1\) or \(x^0 - x^1\). For the mode of all the asymptotic fields dependent on \(x^0 \pm x^1\), we find the following triples of \(\delta_*\), \(\delta^0\)- and \(\delta_{\lambda}\)-quartets and \(\delta_*\), \(\delta^1\)- and \(\delta_{\lambda}\)-quartets:

(i) \(\delta_*\), \(\delta^0\)- and \(\delta_{\lambda}\)-quartets

\[\begin{align*}
\delta_* \phi_{00} &= -\frac{1}{2}\partial_0 (\gamma^0 + \gamma^1), \\
\delta_* \bar{\gamma}_1 &= i(\tilde{\beta} + \partial_t \beta + \partial_0 \sigma);
\end{align*}\]  \hspace{1cm} (B\cdot 21)
\[
\begin{align*}
\begin{cases}
\delta^0 (2\phi_{00} + \varphi) = \gamma^0, \\
\delta^0 (\bar{\gamma}_0 \pm \bar{\gamma}_1) = i\beta,
\end{cases}
\end{align*}
\tag{B.22}
\]

\[
\begin{align*}
\begin{cases}
\delta_L (\phi_{00} \mp \bar{x}) = \pm \tau, \\
\delta_L \bar{\tau} = i\sigma,
\end{cases}
\end{align*}
\tag{B.23}
\]

(ii) $\delta_\star$, $\delta^1$- and $\delta_L$-quartets

\[
\begin{align*}
\begin{cases}
\delta_\star \phi_{00} = -\frac{1}{2} \partial_0 (\gamma^0 \mp \gamma^1), \\
\delta_\star \bar{\gamma}_0 = i(\bar{\beta}_0 + \delta_0 \beta + \partial_1 \sigma);
\end{cases}
\end{align*}
\tag{B.24}
\]

\[
\begin{align*}
\begin{cases}
\delta^1 (2\phi_{00} - \varphi) = -\gamma^1, \\
\delta^1 (\bar{\gamma}_0 \pm \bar{\gamma}_1) = \pm i\beta;
\end{cases}
\end{align*}
\tag{B.25}
\]

\[
\begin{align*}
\begin{cases}
\delta_L (\phi_{00} \mp \bar{x}) = \pm \tau, \\
\delta_L \bar{\tau} = i\sigma,
\end{cases}
\end{align*}
\tag{B.26}
\]

where (B.21)~(B.23) generate mutually orthogonal states as seen from their vanishing two-dimensional (anti)commutation relations; likewise for (B.24)~(B.26).

Independent degrees of freedom of the asymptotic fields are 1 for each of $\varphi$, $\phi_{\mu\nu}$, $\bar{x}$, $\bar{\beta}_1$, $\beta$, $\sigma$, $\tau$ and $\bar{\tau}$, and 2 for each of $\gamma^0$ and $\bar{\gamma}$, as seen from (B.6) and (B.7). These degrees of freedom constitute three BRS quartets which are orthogonal to one another as shown above. Thus the unitarity of the physical $S$-matrix is guaranteed by the Kugo-Ojima quartet mechanism, if we define the physical states by the following subsidiary conditions:

\[
Q_\lambda |\text{phys}\rangle = 0, \quad Q^\lambda |\text{phys}\rangle = 0, \quad Q_\lambda |\text{phys}\rangle = 0
\tag{B.27}
\]

for $\lambda = 0$ and/or 1.

References

4) K. Sato, Preprint EPHOU87, APR003 (Hokkaido).
7) For a review, see the book of Ref. 2), sec. 2.5.