Supersymmetric Quantum Mechanics and Inverse Scattering

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The connection between supersymmetric quantum mechanics and the inverse scattering problem is utilized to construct symmetric reflectionless potentials with an arbitrary number of bound state levels. A relation between Bäcklund transformations and supersymmetry is explored. The transformations of interest relate a potential with a given spectrum to one with an additional bound state below all the rest. This potential may be regarded as an instantaneous solution of the Korteweg-de Vries (KdV) equation. The Bäcklund transformation is just that which adds a soliton to a multi-soliton KdV solution.

§ 1. Introduction

Many physical systems exhibit behavior typical of supersymmetry, in which bosonic and fermionic excitations are related.\(^1\)\(^2\) The physicist being honored here has studied many of these systems, including one of the first.\(^3\) As a contribution inspired by his interest, we would like to explore still another manifestation of supersymmetry—the inverse scattering problem.\(^4\)\(^5\) The results are a simple application of some ideas expressed in terms of supersymmetric quantum mechanics relatively recently,\(^6\)\(^7\)\(^8\)\(^9\) but which go back to Schrödinger.\(^10\)

A problem which recurs in many areas of physics is to reconstruct a potential \(V(x)\) from information on its bound state energy levels. Usually such information is not enough, and one needs scattering data as well.\(^11\) However, this is not always available. For indefinitely rising potentials, such as those which confine quarks, only bound state information exists.

Symmetric, reflectionless potentials are unique in that they are specified entirely by their bound states.\(^12\) They have been found to provide good approximations to confining potentials, in the range of energies actually probed by the levels.\(^13\)\(^14\)\(^15\)\(^16\) Thus, a harmonic oscillator \(V = x^2/2\) with energies \(E_n = \hbar \omega (n+1/2)\) will be approximated well up to \(V \sim E_N\) by a symmetric, reflectionless potential with the same lowest \(N\) levels. We have found that supersymmetry simplifies the construction of such potentials to a great extent. While the supersymmetric aspects of the discussion are very rudimentary, they illustrate once more how widespread supersymmetry is in nature once we look.

We shall make use in § 2 of a factorization\(^10\) \(H_+=A^+A\) of the Schrödinger Hamiltonian \(H_+\) into the product of two first-order differential operators. The product \(AA^+=H_-\) will turn out to have the same spectrum as \(H_+\), except that the ground state of \(H_+\) is absent in \(H_-\). The connection of the Schrödinger factorization with supersym-
metric quantum mechanics is noted in § 3, and applied in § 4 to the construction of a symmetric, reflectionless potential with a single level. The result is then generalized to \( N \) levels (§ 5). Illustrations are given for known potentials (§ 4) and for hypothetical quarkonium levels (§ 7). The connection with Bäcklund transformations is described briefly in § 8. We conclude, with a list of future questions, in § 9.

\[ \text{§ 2. Schrödinger’s factorization} \]

Consider the 1-dimensional Schrödinger equation
\[
\left[ -\frac{d^2}{dx^2} + V(x) \right] \phi(x) = E \phi(x). \tag{2.1}\]

Let
\[
-\frac{d^2}{dx^2} + V_+(x) = A^+ A \equiv H_+, \tag{2.2}\]

where
\[
A = -\frac{d}{dx} + f(x). \tag{2.3}\]

We write \( V_+ \) and \( H_+ \) in Eq. (2.2) to distinguish them from a related potential and Hamiltonian to be defined below.

Since
\[
A^+ A = -\frac{d^2}{dx^2} + f^2 + f', \tag{2.4}\]

we must take
\[
V_+(x) = f^2 + f'. \tag{2.5}\]

Let us choose the zero of energy so that the ground state \( \phi_0^{(+)} \) in \( H_+ \) has zero energy:
\[
H_+ \phi_0^{(+)} = A^+ A \phi_0^{(+)} = 0, \tag{2.6}\]

implying
\[
A \phi_0^{(+)} = 0. \tag{2.7}\]

This is a first-order differential equation, constraining \( f \) to be the logarithmic derivative of the ground state wave function:
\[
f = \phi_0^{(+)} / \phi_0^{(+)} . \tag{2.8}\]

A “partner potential” \( V_- \) arises if we consider
\[
H_- \equiv AA^+ = -\frac{d^2}{dx^2} + V_-(x) . \tag{2.9}\]

From the form of \( A \),
\[
V_- = f^2 - f'. \tag{2.10}\]
Now suppose $\phi^{(+)}$ is any eigenfunction of $H_+$:

$$H_+ \phi^{(+)} = E_+ \phi^{(+)}.$$  \hspace{1cm} (2.11)

Consider $AH_+ \phi_+ = E_+(A \phi_+) = AA^+ A \phi_+$. Either $A \phi_+ = 0$ (so that $E_+ = 0$, and $\phi_+$ is the ground state), or $A \phi_+$ is an eigenfunction of $AA^+$ with the same eigenvalue $E_+$:

$$H_-(A \phi_+) = E_+(A \phi_+).$$  \hspace{1cm} (2.12)

Thus, the two spectra are related, as illustrated in Fig. 1. Every eigenstate of $H_+$ except the ground state gives rise (via $A$) to an eigenstate of $H_-$ with the same eigenvalue. The ground state in $H_+$, with zero energy, does not correspond to any eigenstate of $H_-$. Suppose we were to invent a solution $w$ of the zero-energy Schrödinger equation with $H_-:

$$\left[ -\frac{d^2}{dx^2} + V_- \right] w = -w'' + (f^2 - f') w = 0.$$ \hspace{1cm} (2.13)

Since zero energy is not an eigenvalue of $H_-$, $w$ will not obey appropriate bound-state conditions at $\infty$, but will have an exponentially growing term. What is the relation implied by Eq. (2.13) between $f$ and $w$? We can view (2.13) as a Riccati equation for $f$, which can be linearized by the substitution\(^\text{17}\)

$$f = -w'/w$$ \hspace{1cm} (2.14)

to yield

$$\frac{-w''}{w} + \frac{v''}{v} = 0.$$ \hspace{1cm} (2.15)

Clearly one solution of (2.15) is $v = w$, or

$$f = -w'/w.$$ \hspace{1cm} (2.16)

There are two independent solutions $w$, and hence two independent functions $f$. We may choose to specify a solution $w$ with definite parity if $V_-$ is symmetric, as will frequently be the case. Only the even-parity $w$ will give a function $f$ regular at the origin; $f$ will then be odd, and $V_+ = f^2 + f'$ will then be symmetric. Using $f$ to construct $V_+$ in this way, we obtain $H_+$.

The ground state $\phi^{(+)}$ of $H_+$ satisfies Eq. (2.8). By comparing with (2.16), we have

$$\phi^{(+)} \sim \frac{1}{w},$$ \hspace{1cm} (2.17)
so the “new” eigenfunction of $H_+$ is already known when we have solved $H_-$ for the “wrong” eigenvalue (zero)!
§ 3. Supersymmetry

Schrödinger’s factorization can be written in a form suggestive of supersymmetry. (See, e.g., Refs. 6~9)). Define

\[
Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}; \quad Q^+ = \begin{bmatrix} 0 & A^* \\ 0 & 0 \end{bmatrix}.
\]  

(3·1)

Then

\[
\{Q, Q^+\} = \begin{bmatrix} A^* A & 0 \\ 0 & A A^* \end{bmatrix} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} \equiv H_{ss}.
\]  

(3·2)

This relation is part of a supersymmetry algebra obeyed by the charges \(Q\) and \(Q^+\); other relations are

\[
Q^2 = Q^{+2} = 0,
\]  

(3·3)

\[
[H_{ss}, Q] = [H_{ss}, Q^+] = 0.
\]  

(3·4)

The eigenfunctions of the supersymmetric Hamiltonian \(H_{ss}\) are

\[
\psi_{ss} = \begin{bmatrix} \phi^{(+)} \\ \phi^{(-)} \end{bmatrix}.
\]  

(3·5)

They have the properties that

\[
Q\psi_{ss} = \begin{bmatrix} 0 \\ \phi^{(-)} \end{bmatrix} \quad \text{unless} \quad A\phi^{(+)} = 0;
\]  

(3·6)

\[
Q^+\psi_{ss} = \begin{bmatrix} \phi^{(+)} \\ 0 \end{bmatrix}.
\]  

(3·7)

We can call the levels \(\begin{bmatrix} \phi^{(+)} \\ 0 \end{bmatrix}\) “bosonic” and the levels \(\begin{bmatrix} 0 \\ \phi^{(-)} \end{bmatrix}\) “fermionic” in view of the fermionic nature of the algebra (3·2) and (3·3).\(^*)\) The operators \(Q\) and \(Q^+\) then act like fermionic charges. They connect Schrödinger solutions in one potential \(V_+\) with solutions in another potential \(V_-\). The function \(f\) (see § 2) plays the role of the superpotential, since \(V\) is obtained from a quadratic form in it.

§ 4. One-level potential

Suppose the potential \(V_-\) is given by a constant:

\[
V_- = f^2 - f' = \chi^2.
\]  

(4·1)

\(^*)\) The levels of \(H^+\) include a ground state not degenerate with any levels of \(H^-\) (see Fig. 1). This ground state must correspond to zero bosons and zero fermions, and hence must be bosonic.
leading to
\[ f = -x \tanh x(x - x_0) \] (4.4)

and
\[ V_+ = f^2 + f' = x^2[1 - 2 \text{sech}^2 x(x - x_0)]. \] (4.5)

The two potentials are compared in Fig. 2. The potential \( V_+ \) has a zero-energy bound state, whose eigenfunction is
\[ \phi^{(+)} \sim \frac{1}{w} \sim \frac{1}{\cosh x(x - x_0)}. \] (4.6)

A simple argument shows that \( V_+ \) will be reflectionless. Consider a plane wave solution \( \phi^{(-)} = e^{iqx} \) to the Schrödinger equation with \( H_- \) (so \( q = \pm \sqrt{E - x^2} \)). Now define
\[ \phi^{(+)} = A^+ \phi^{(-)} = \left( \frac{d}{dx} + f \right) e^{iqx} = [iq + f] e^{iqx} \] (4.7)
or
\[ \phi^{(+)} = [iq - x \tanh x(x - x_0)] e^{iqx}. \] (4.8)

The asymptotic behavior of this function is
\[ \phi^{(+)} \rightarrow (iq + x) e^{iqx}, \quad (x \rightarrow -\infty) \] (4.9)
\[ \phi^{(+)} \rightarrow (iq - x) e^{iqx}, \quad (x \rightarrow +\infty) \] (4.10)

The S-matrix is
\[ S = \frac{iq - x}{iq + x}. \] (4.11)

No term of the form \( e^{-iqx} \) arises, so there is no reflection.

§ 5. \( N \) levels

We shall now show how to construct a symmetric reflectionless potential with \( N \) levels. The levels must be added in order of their binding, with the shallowest level added first. (See Fig. 3.)
The construction of the previous section may be reexpressed as follows. Suppose we want a potential $V_1$ with
\[ \left( -\frac{d^2}{dx^2} + V_1 \right)\psi_1 = -x_1^2 \psi_1 . \] (5·1)

Then $V_1 + x_1^2$ has a zero-energy bound state. We require a function $f_1$ to satisfy
\[ f_1^2 + f_1' = V_1 + x_1^2 \] (5·2)
and obtain $f_1$ from the partner potential $V_0=0$:
\[ f_1^2 - f_1' = V_0 + x_1^2 = x_1^2 . \] (5·3)

Now let us impose the requirement that $V_1$ be symmetric (even in $x$). We shall discuss this in greater detail in § 7, where odd-parity levels in a symmetric potential $V(x) = V(-x)$, $-\infty < x < \infty$, will be shown to correspond to $S$ waves in the corresponding three-dimensional potential $V(r)(r>0)$. Then $f_1$ must be odd in $x$. Comparing with (4·4), we see
\[ f_1 = -x_1 \tanh x_1 x \] (5·4)
and
\[ V_1 = -2x_1^2 \operatorname{sech}^2 x_1 x . \] (5·5)

This is just the potential of § 4, but with $V(\pm \infty) = 0$.

For arbitrary $n$ we may now assume $V_{n-1}$ is known, and define $f_n$ by
\[ f_n^2 - f_n' = V_{n-1} + x_n^2 . \] (5·6)

To obtain a symmetric $V_n$, we must take $f_n(0) = 0$. Then
\[ f_n^2 + f_n' = V_n + x_n^2 \] (5·7)
defines $V_n$.

Just as for a single level, Eq. (5·6) may be interpreted as a Schrödinger equation with a "fake" eigenvalue $E = -x_n^2$. Set $f_n = -w_n'/w_n$, so (5·6) becomes
\[ -w_n'' + V_{n-1}w_n = -x_n^2 w_n . \] (5·8)

We solve (5·8) with an even function $w_n(x) = w_n(-x)$; $f_n$ is then uniquely specified. As for the 1-level case, $u_n \equiv 1/w_n$ then solves
\[ -u_n'' + V_n u_n = -x_n^2 u_n \] (5·9)
and, because $w_n$ blows up as $x \to \pm \infty$, $u_n$ has the correct bound-state behavior.

An amusing relation stems from Eqs. (5·6) and (5·7). If we add these two equations, we find
\[ 2f_n^2 = V_{n-1} + V_n + 2x_n^2. \]  

(5.10)

At \( x = 0 \), where we have taken \( f_n(0) = 0 \), we find

\[ V_n(0) = -V_{n-1}(0) - 2x_n^2, \]  

or

\[ V_n(0) = -2 \sum_{i=1}^{n} (-1)^{n-i} x_i^2. \]  

(5.11)

This result was noted before,\(^{10}\) but not derived so simply.

The potential \( V \) of Eq. (5.5), as mentioned, vanishes as \( x \to \pm \infty \). This will be true of all the potentials \( V_n \), as one can show inductively by a simple argument based on (5.8) and (5.9). It is then not hard to see that since \( w_n \) in (5.8) must be monotonically increasing in \( |x| \), \( f_n = -w_n'/w_n \) has the following properties:

\[ f_n \to \begin{cases} +x_n \\ -x_n \end{cases} \quad f_n' \to 0; \quad \text{as} \quad x \to \begin{cases} -\infty \\ +\infty \end{cases}. \]  

(5.12)

Moreover, \( f_n \) will be monotonically decreasing.

The sequence of Eqs. (5.6) and (5.7) for \( V_n \) has a solution in closed form, given in one rendition in Ref. 15). That solution cannot be evaluated in practice rapidly enough for large numbers of levels, as it requires partitioning the set \( \{x_i : i = 1, \cdots, N\} \) into all possible subsets. In contrast, numerical evaluation of (5.6) and (5.7) requires only a single “sweep” through the integration range. Information obtained about \( f_n \) at the \( k \)th integration step may be passed on to learn about \( f_{n+1} \) at the \( (k-1) \)th step. In the next two sections we illustrate some experience with numerical evaluations of \( V_n \).

§ 6. \( N \) levels in oscillator

Consider the potential \( V(x) = (1/4)x^2 + V(0) \) in Eq. (2.1) with energy levels \( E_m = m + 1/2 + V(0), \) \( m = 0, 1, 2, \cdots \). Suppose we are given a symmetric, reflectionless potential with the same first \( N \) levels. How closely does it approximate the oscillator?

The answer depends on the assumed behavior of the potential for distances beyond the classical turning point \( x_N \) of the highest \( (N\text{th}) \) level. In Ref. 15) we have shown that the best approximation to a potential based on a symmetric, reflectionless potential with the same \( N \) lowest levels must tend quite rapidly to a flat potential beyond \( x_N \).

To be more quantitative, suppose we wish to construct a symmetric, reflectionless potential \( \tilde{V}(x) \) behaving as \( \lim_{x \to \pm \infty} \tilde{V}(x) = 0 \), with the spectrum

\[ x_n^2 = \delta + \left( n - \frac{1}{2} \right), \quad n = 1, 2, \cdots, N. \]  

(6.1)

There is a choice of \( \delta \) so that \( \tilde{V}(0) \) agrees with the value expected for the oscillator with the same first \( N \) levels, at \( E = -\delta - (N-1/2), -\delta - (N-3/2), \cdots, -\delta - 1/2: \)

\[ \tilde{V}(0) = -\delta - N. \]  

(6.2)

By Eq. (5.11), we find this choice to be \( \delta = 0 \). Thus the potential \( \tilde{V}(x) \) providing the best approximation to \( V(0) \) rises to only half a level-spacing above the highest specified level, as \( x \to \pm \infty \).
In Ref. 15) we showed an example with $N=8$ of an approximation to the harmonic oscillator based on $\delta=0$. We can now exhibit results for many more levels. An illustration with $N=25$ is shown in Fig. 4. Oscillations of dimension characteristic of the level spacing are still present, but the average potential is quite well reproduced. It is clear that the present numerical method is suitable for approximating potentials with large numbers of levels.

§ 7. $N$ levels in quarkonium

The $S$-wave states in a three-dimensional central potential are described by a reduced radial wave function $u_n(r) = r R_n(r)$ satisfying $u_n(r) \sim r(r\rightarrow 0)$, and obeying the 1-dimensional Schrödinger equation

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) \right] u_n(r) = E_n u_n(r), \quad (7.1)$$

where $\mu$ is the reduced mass. The levels in (7.1) may be viewed as the odd-parity levels $u_n(r) = \psi_{2n}(r)$ in a symmetric potential $V(-r) = V(r)$, in view of the $r\rightarrow 0$ restriction on $u_n(r)$.

The corresponding even-parity levels may also be reconstructed for a symmetric, reflectionless potential $\tilde{V}(x)$ if one knows the values $u_n'(0)$. For $n^3S_1$ bound quarkonium states, this information is provided by leptonic widths:

$$\Gamma(nS\rightarrow e^+e^-) = \frac{4\alpha^2 e_e^2}{[M(nS)]^2} |u_n'(0)|^2. \quad (7.2)$$

The function

$$f(E) = 1 + \sum_{k=1}^{N} \frac{|u_k'(0)|^2}{\chi_k(E-E_k)} \quad (7.3)$$

with

$$\chi_k^2 = 2\mu [\tilde{V}(\infty) - E_k], \quad (7.4)$$

has zeroes at the positions of the desired even-parity levels.\textsuperscript{14,15)}

We have constructed an approximation to a quarkonium potential\textsuperscript{18)} on the basis of information about its $nS$ levels. For $m_t=40$ GeV, one expects at least 10 levels below flavor threshold. These levels and their expected leptonic widths are summarized in Table I. We now choose $\tilde{V}(\infty)$ in Eq. (7.4) to be just above the 10th level, corresponding
to the 20th level in a symmetric potential. This is done in such a way that the even-parity levels (zeroes of $(7\cdot3)$) and odd-parity levels (poles of $(7\cdot3)$) form a smooth progression for large $n$, as one would expect in the semiclassical limit. The resulting $\tilde{V}(\infty)=80.76$ GeV leads to the levels and potential shown in Fig. 5. The agreement with the original potential is excellent over the range of $r$ probed by the levels. From this exercise we learn the distance scale over which $t\bar{t}$ bound-state information is likely to shed light on the interquark force.

§ 8. Bäcklund transformation

A potential of the form

$$V(x) = -2x^2 \text{sech}^2 x(x-x_0)$$

(8.1)

can be regarded as an instantaneous soliton $V(x)=v(x,0)$ of the Korteweg-de Vries (KdV) equation:20–22

$$v_t - 6vv_x + v_{xxx} = 0.$$  

(8.2)

Here $v_t = \partial v/\partial t$, $v_x = \partial v/\partial x$, etc. The solution for all $t$ is

$$v(x,t) = -2x^2 \text{sech}^2 x(x-4x^2t-\delta);$$

(8.3)

the soliton travels with a velocity proportional to its amplitude.

The construction of reflectionless potentials outlined in §§ 2–5 has an analogue in the construction of multisoliton solutions to equations such as

(8.2). Transformations of solutions possessing $N$ solitons to those with $N+1$ solitons are known as Bäcklund transformations.23 We exhibit such a transformation for the KdV equation and show its close similarity to ones examined above.

Suppose that we define a function $f(x,t)$ satisfying a modified KdV equation21

$$h(x,t) \equiv f_t + 6(x^2-f^2)f_x + f_{xxx} = 0.$$  

(8.4)

Then

$$v_-(x,t) = f^2 - f_x - x^2$$

(8.5)
satisfies the KdV equation (8·2), which may be transcribed using (8·5) as
\[ 2f_t - h_x = 0. \]  
(8·6)

But the related function
\[ v_+(x, t) \equiv f^2 + f_x - x^2 \]  
(8·7)

then also satisfies the KdV equation, since
\[ 2f_t + h_x = 0. \]  
(8·8)

With suitable boundary conditions on \( f \), one can interpret \( v_+ \) as an \( N+1 \)-soliton solution if \( v_- \) is an \( N \)-soliton solution. One can eliminate \( f \) and write the relation between \( v_- \) and \( v_+ \) as
\[ \frac{\partial}{\partial x} \sqrt{\frac{v_- + v_+}{2} + x^2} = \frac{v_- - v_+}{2}. \]  
(8·9)

We note that the general \( N \)-soliton solution to the KdV equation has been constructed previously.\(^{24}\) Our purpose here has been merely to point out how supersymmetry can be of use in constructing this solution.

§ 9. Conclusions

Supersymmetric quantum mechanics has an intimate connection with the inverse scattering problem. This connection has been exploited in the construction of symmetric, reflectionless potentials with an arbitrary spectrum, of practical use in approximating potentials given only bound-state information. While our present applications have been to quarkonium, one can envision a broader set of possibilities. With the ability to handle large numbers of levels, we can even hope to describe periodic structures in terms of their band spectrum.

The connection between supersymmetric quantum mechanics and Bäcklund transformations also has further possibilities. The symmetries which lead to an infinite number of conservation laws\(^{29}\) in integrable systems such as the KdV equation have already been recognized as related to Kac-Moody algebras.\(^{28}\) We now add supersymmetry to the list of properties connected with such systems. The suggestion of Ref. 1) — that supersymmetry is widespread in nature — is borne out once more.

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Note added in proof: Since submitting this manuscript we have become aware of other interesting work on the present subject. The inverse scattering problem in supersymmetric quantum mechanics has been treated in several recent articles with careful attention to historical origins spanning more than a century. The Bäcklund transformation for the KdV equation was constructed in Ref. 28.)