Quantum Dynamics and Non-Inertial Frames of Reference. III

Charged Particle in Time-Dependent Uniform Electromagnetic Field

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By successive applications of time-dependent rotation, translation and dilatation, it is shown that a charged time-dependent harmonic oscillator in a uniform time-dependent electromagnetic field is equivalent to a free particle. This equivalence is employed to present a simple derivation of the Feynman kernel as well as to obtain a concrete picture of the time evolution of a generic wave packet. A compact formula for an evolving anisotropic Gaussian packet is also derived. A special case of this formula gives a squeezed coherent state for a harmonic oscillator whose frequency is an arbitrary function of time. Finally we discuss an Aharonov-Bohm-Berry-like phase associated with a wave packet whose center traces a closed circuit.

§ 1. Introduction

In the first paper [Ref. 1], first paper, hereafter referred to as I] of this series general features of three non-inertial frames of reference as applied to non-relativistic classical and quantum dynamics have been discussed from a unified point of view; the three frames are the extended Galilean frame (EGF), the rotating frame (RF) and the comoving frame (CF, or the dilatating frame). Specific applications of the last frame to harmonic oscillators have been described in detail in the second paper [Ref. 1], second paper, hereafter referred to as II]. The present paper is devoted to the study of the system of a charged particle in a time-dependent uniform electromagnetic field such that the direction of the magnetic field is constant. The system includes a charged particle in a constant magnetic field and a time-dependent harmonic oscillator as special cases. We have already mentioned elsewhere that the system is equivalent to a free particle under the combined transformations to the three non-inertial frames. Here we shall give a precise statement of the equivalence theorem. We exploit this equivalence to provide a simple derivation of Green's function for Newton's equation in the classical case (§ 2) and of Feynman's kernel for Schrödinger's equation in the quantum case (§ 4). In the quantum case the equivalence gives a particularly straightforward method to find the time evolution of a wave packet. We shall illustrate this by deriving a compact expression for an anisotropic Gaussian packet. Expression (5·10) is perhaps a new result. Otherwise we do not necessarily claim originality of various formulae themselves given in §§ 2, 4 and 5, which may be derived by various (often laborious) ways, but hope to show that they emerge very simply in an elementary manner. The simplicity of the present method has an advantage of making the origin of the time-dependence quite transparent. This allows us to analyze with ease not only the amplitude but also the phase of the wave function. Specifically we show in § 5 for a generic wave packet that its center follows the classical trajectory, which
should be the case by the Ehrenfest theorem, and that its shape undergoes a *squeezing oscillation* (to be explained in the text). In addition we show in § 6 that a wave packet acquires a phase analogous to the Aharonov-Bohm (AB) phase when the packet is transported around a closed circuit by a linear external force; if the magnetic field is time-independent the magnitude of the phase is numerically identical to the AB phase associated with the closed circuit followed by the center of the packet, but the sign of the phase is opposite. We compare our consideration with Berry's gedanken experiment\(^3\) (to be explained in § 6) on AB effect. The comparison motivates us to seek a physical realization of Berry's gedanken experiment. This is the topic of § 7. The final section contains a few comments on nonlinear Schrödinger equations as well as related problems and works.

The system to be considered is a charged harmonic oscillator described by the following Schrödinger equation

\[
i \frac{\partial}{\partial t} \Psi(r, t) = H \Psi(r, t),
\]

\[
H = -\frac{1}{2} \left[ \nabla^2 - iA(r, t) \right]^2 - E(t) \cdot r + \frac{1}{2} \nu^2(t) r^2,
\]

\[
A(r, t) = \frac{1}{2} B(t) \times r,
\]

where \(\nabla\) denotes the gradient with respect to \(r\), \(\nu\) is an arbitrary function of time, and \(E\) and \(B\) represent a time-dependent electromagnetic field. In order to restore the ordinary (Gaussian) units, replace \(t, r, E, B\) and \(\nu\) by \(mt/\hbar, mr/\hbar, qE/m^2, qB/m^2 c\) and \(h\nu/m\) respectively, where \(m\) and \(q\) are particle's mass and charge. Let \((e_\parallel, e_\perp, n)\) be the orthonormal right-handed triad defining the inertial frame. We assume that

\[
E(t) = E_j(t) e_j = E_{i\parallel}(t) e_i + E_{i\perp}(t) e_\perp,
\]

\[
B(t) = \omega_c(t) n = 2\omega_L(t) n,
\]

where the subscript \(j\) is supposed to run over 1 and 2 only, and \(E_j\) and \(\omega_c = 2\omega_L\) are arbitrary functions of time. (Such an electromagnetic field cannot exist in vacuo unless \(E\) and \(B\) are linear in \(t\), but may be approximately realized in the middle of a solenoid.) For brevity we shall call the above \(E\) and \(B\) the *uniform electromagnetic field*, although the actual electric field \((= E - \frac{1}{2} \dot{B} \times r)\) is not spatially uniform if time dependence is allowed. We shall ignore particle's motion in the \(n\)-direction; accordingly \(r\) is regarded as a two-dimensional vector such that \(r \cdot n = 0\).

In the text we refer to Eq. \((m \cdot n)\) of I as Eq. \((I \cdot m \cdot n)\), to § 3.1 of II as § II.3.1, and so on.

\section{2. Classical charged particle}

Let us begin with studying the classical equation of motion for the charged oscillator described by the classical version of Hamiltonian (1.1b) or the Lagrangian

\[
L(r, \dot{r}, t) = \frac{1}{2} \dot{r}^2 + A(r, t) \cdot \dot{r} - \frac{1}{2} \nu^2(t) r^2 + E(t) \cdot r,
\]
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that is

$$\dot{R} = E - \frac{1}{2} B \times R + \dot{R} \times B - \nu^2 R , \quad (2.2)$$

where we have denoted a classical trajectory by $R(= R(t))$. There are various techniques to solve this equation, for example, the use of the complex variable $(e_1 + i e_2) \cdot R$. Here we shall employ transformations to non-inertial frames. In any case the equation is so elementary that the main purpose of this section is to illustrate our method in the classical context and to establish the notation which will also be used in the quantum case; after all, the thesis of this series of papers is to seek an intuitive picture of a quantum dynamics by decomposing it into the corresponding classical dynamics and the quantum dynamics of a free particle, the decomposition being done with the aid of non-inertial frames.

First we introduce a special rotating frame, that is, the Larmor frame (LF) defined by the rotating triad $\{e_1(t), e_2(t), n\}$ which obeys

$$I B \dot{e}_j = -T X e_j, \quad e_j(0) = e_j. \quad (2.3)$$

In other words

$$e_1(t) + i e_2(t) = (e_1 + i e_2) \exp[it\theta(t)], \quad \theta(t) = \int_0^t dt' \omega_L(t'). \quad (2.4)$$

We perform the transformation from the inertial to the rotating frame by defining

$$X_j(t) = R(t) \cdot e_j(t). \quad (2.5)$$

Correspondingly we define

$$E_j(t) = E(t) \cdot e_j(t). \quad (2.6)$$

Let $X$ and $E$ stand for the doublet $(X_1, X_2)$ and $(E_1, E_2)$, respectively, then Eq. (2.2) is converted into

$$\dot{X} + \omega^2 X = E, \quad \omega^2 = \nu^2 + \omega_L^2. \quad (2.7)$$

In LF, terms linear in $B$ or $\dot{B}$ do not appear; they have been cancelled by the Coriolis force. The charged harmonic oscillator in the uniform electromagnetic field has thus been reduced to a forced harmonic oscillator. This is an instance of what may be called strong Larmor's theorem. (Larmor's theorem is usually stated in a much weaker form, for example, thus: "the sole effect of a weak (uniform) magnetic field is to cause a precession of the entire motion about $B."\) Note that the frequency appearing in Eq. (2.7) is not the cyclotron frequency $\omega_c$ but the Larmor frequency $\omega_L$.

Next one could try to eliminate the external force $E$ by transforming to an extended Galilean frame via $X' = X - \xi$ so that $\dot{X'} + \omega^2 X' = 0$; for this to be the case, however, the function $\xi$ must obey the same equation as $X$, and the procedure becomes a tautology. Instead we transform to a comoving frame (CF) by defining

$$\bar{X} = X(t)/\alpha(t), \quad (2.8)$$
where the scale factor $a$ is arbitrary at this stage, to find

$$a\ddot{X} + 2\dot{a}\dot{X} + (\ddot{a} + \omega^2 a)\dot{X} = E. \quad (2.9)$$

This may be simplified by choosing $a$ so that

$$\ddot{a} + \omega^2 a = 0, \quad (2.10)$$

and introducing the scaled time

$$\tau = \int_0^t dt' [a(t')]^{-2}, \quad (2.11)$$

thus

$$d^2\ddot{X}/dt^2 = E = a^2(t)E(t). \quad (2.12)$$

The harmonic inertial force associated with the CF has cancelled the original harmonic term, and our system has been reduced to a particle under linear external force.

Comment: It is easy to incorporate a friction term $\eta(t)\dot{X}$ into the left-hand side of Eq. (2.7), where $\eta$ is an arbitrary function. Transformation (2.8) leads to Eq. (2.12) provided that $a$ is chosen to obey

$$\ddot{a} + \eta\dot{a} + \omega^2 a = 0 \quad (2.13)$$

and that the following definitions are adopted,

$$\tau = \int_0^t dt' [a(t')]^{-2}, \quad \ddot{E} = a^2(t)b^4(t)E(t), \quad (2.14)$$

where

$$b(t) = \exp\left\{\int_0^t dt'\eta(t')/2\right\}. \quad (2.15)$$

Transformations connecting Eqs. (2.7), (2.9) and (2.12) are of course well known in the theory of ordinary differential equations. One often starts from Eq. (2.9) and eliminates the second term either by ‘change of the dependent variable’ (2.8) or by ‘change of the independent variable’ (2.11). The use of $\tau$ is also closely related to ‘the method of variation of parameters’. We therefore see that the harmonic inertial force has been effectively known long since in the classical context. However notation and terminology of typical textbooks of differential equations are too prosaic as above to let us recognize the inertial force as such. By contrast the idea of the CF offers an intuitive picture of the transformations involved.

The initial condition

$$R(0) = r_0, \quad \dot{R}(0) = v_0, \quad (2.16)$$

in the inertial frame corresponds to the condition

$$\ddot{X}_j(0) = r_{0j}/a_0, \quad (d\ddot{X}_j/d\tau)(0) = a_0 p_{0j} - \dot{a}_0 r_{0j}, \quad (2.17)$$

where

$$r_{0j} = r_0 \cdot e_j, \quad p_{0j} = p_0 \cdot e_j, \quad a_0 = a(0), \quad \dot{a}_0 = \dot{a}(0) \quad (2.18)$$
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and

\[ \mathbf{p}_0 = \mathbf{v}_0 + A(r_0, 0). \]  

(2.19)

The corresponding solution of Eq. (2.12) may be translated via Eq. (2.8) into

\[ X_i(t) = p_0 \alpha(t) + r_0 \beta(t) + \xi_i(t), \]  

(2.20)

where

\[ \alpha(t) = a(t) \tau, \quad \beta(t) = (1 - a(t) \dot{a}(t)) a(t)/a_0 \]  

(2.21a)

are the fundamental solutions of force-free time-dependent harmonic oscillator (2.10) such that

\[ \alpha = \dot{\beta} = 0 \quad \text{and} \quad \dot{\alpha} = \beta = 1 \quad \text{at} \quad t = 0 \]  

(2.21b)

and

\[ \xi_i(t) = a(t) \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \bar{E}_i(\tau_2). \]  

(2.22)

Changing the order of integration we find

\[ \xi_i(t) = a(t) \int_0^t d\tau d\tau' E(t', t) s(t, t'), \]  

(2.23)

where

\[ s(t, t') = a(t) \{ \tau(t) - \tau(t') \} a(t') = a(t) \beta(t') - \beta(t) a(t'). \]  

(2.24)

Note that \( s(t, t') = \omega^{-1} \sin \omega(t - t') \) if \( \omega \) is constant.

In this way we have been automatically led to the explicit expression of the propagator \( s(t, t') \) for forced harmonic oscillator (2.7), this propagator being relevant in obtaining the particular solution \( \xi \) such that \( \xi = \dot{\xi} = 0 \) at \( t = 0 \). With this propagator in hand it is a trivial task to produce desired Green's function appropriate for a given boundary condition.

The solution to original equation (2.2) is given by

\[ R(t) = X_i(t) e_i(t) = p_{0L}(t) \alpha(t) + r_{0L}(t) \beta(t) + \mathbf{p}(t), \]  

(2.25)

where

\[ \mathbf{p}(t) = \xi_i(t) e_i(t) = \int_0^t d\tau' e_i(t) s(t, t') e_i(t') \cdot E(t'), \]  

(2.26)

\[ r_{0L}(t) = r_0 e_i(t) = r_0 \cos \theta_L(t) - \mathbf{r}_0 \sin \theta_L(t), \]  

(2.27)

and we have defined \( \mathbf{r}_0 = \mathbf{n} \times \mathbf{r}_0 \) so that \( \{ \mathbf{r}_0, \mathbf{r}_0, \mathbf{n} \} \) forms a right-handed triad; \( r_{0L} \) is the Larmor-precessing counterpart of \( r_0 \), and \( p_{0L} \) and \( \mathbf{p}_0 \) are similarly defined.

When \( \omega_L \) and \( \nu \) are constant, we have

\[ \alpha = \omega^{-1} \sin \omega t, \quad \beta = \cos \omega t \]  

(2.28)

and
In the absence of the electric field \( E \), therefore, the particle executes the doubly periodic motion

\[
\mathbf{R}(t) = \mathbf{R}_+(t) + \mathbf{R}_-(t), \quad \mathbf{R}_\pm(t) = l_\pm \cos \omega_\pm t \mp \tilde{l}_\pm \sin \omega_\pm t,
\]

where

\[
l_\pm = \frac{1}{2} (\mathbf{r}_0 \pm \mathbf{p}_0 / \omega), \quad \tilde{l}_\pm = \mathbf{n} \times l_\pm = \frac{1}{2} (\mathbf{r}_0 \mp \mathbf{p}_0 / \omega), \quad \omega_\pm = \omega \pm \omega_L.
\]

The trajectory is an epitrochoid. For example, if \( 0 < |\nu| < \omega_L^2 \) and \( |\nu| < \omega_c |\nu_0| \), then it represents a cyclotron motion of radius \( |L| \approx |\nu_0| / \omega_c \) whose center drifts along a larger circle of radius \( |L| \approx |\nu_0| \). The sense of rotation of the former is opposite or the same to that of the latter depending on \( \nu > 0 \) or \( \nu < 0 \). In the case of a free particle (\( \nu = 0 \)) in a constant magnetic field, \( \mathbf{R}_- \) vanishes and the pure cyclotron motion is recovered of course.

### § 3. Quantum mechanical equivalence theorem

Let us now go back to quantum mechanics and solve Eq. (1·1) with an arbitrary initial condition \( \psi(r, 0) \). First we follow the procedure of § 1.2.3 to define the LF (i.e., Larmor-frame) wave function \( \Phi \) as

\[
\Phi(x, t) = \psi(x, \mathbf{e}_j(t), t),
\]

where \( x \) stands for the doublet \((x_1, x_2)\) and \( \mathbf{e}_j(t) \) are the rotating axes introduced at Eq. (2·3). The Schrödinger equation in the LF takes the form

\[
i \frac{\partial}{\partial t} \Phi(x, t) = \left[ -\frac{1}{2} \mathcal{P}^2 + \frac{1}{2} \omega^2 x^2 - \mathbf{E} \cdot \mathbf{x} \right] \Phi(x, t),
\]

where

\[
\mathcal{P} = (\partial / \partial x_i) (\partial / \partial x_i), \quad x^2 = x_i x_i, \quad \mathbf{E} \cdot \mathbf{x} = E_i x_i
\]

with \( E_i \) defined by Eq. (2·6). The LF wave function represents a forced harmonic oscillator; this result is a quantum-mechanical manifestation of strong Larmor's theorem. In the absence of \( E \), an equation analogous to (3·2) can be obtained by restricting \( \psi \) to an eigenstate of the \( n \)-component of angular momentum. But our equation holds without such restrictions.

Next we follow the procedure of § 1.2.1 to define the EGF (i.e., extended-Galilean-frame) wave function \( \phi \) as

\[
\phi(x, t) = \Phi(x + X(t), t) \exp \left[ -i \dot{X}(t) \cdot x - i S(t) \right],
\]

where \( X \) stands for the doublet \((X_1, X_2)\), which is a general solution for the classical forced harmonic oscillator (2·7), and \( S \) is to be determined below. We have generalized the time-dependent phase in Eq. (1·2·7) so as to take account of time-dependent \( c \)-number terms which arise when Eq. (1·2·8) is applied to the present model.
Substitution of Eq. (3.4) into (3.2) shows that it is convenient to choose $S$ to be the action associated with $X$, namely

$$S(t) = \int_0^t dt' \left\{ \frac{1}{2} \dot{X}^2(t') - \frac{1}{2} \omega^2(t') X^2(t') + E(t') \cdot X(t') \right\}$$

(3.5a)

$$= \int_0^t dt' L(R(t'), \dot{R}(t'), t'),$$

(3.5b)

where $L$ is the Lagrangian (2.1) and $R$ is the solution of (2.2) corresponding to $X$. The last equality holds because the action is known to be invariant under the transformation to a rotating frame (see § I.4). The Schrödinger equation then takes the form

$$i \frac{\partial}{\partial t} \psi(x, t) = \left[ -\frac{1}{2} \nabla^2 + \frac{1}{2} \omega^2 x^2 \right] \psi(x, t).$$

(3.6)

A further transformation to a comoving frame with a scale factor $a$ obeying Eq. (2.10) reduces the above equation to the two-dimensional free-particle Schrödinger equation. Since this step has been extensively discussed in II, we shall merely quote the result.

$$\psi(x, t) = a^{-1} \psi(x/a, t) \exp\left(ia\dot{x}^2/2a\right),$$

(3.7)

where $a$ is the function of $t$ defined by Eq. (2.11), and $\psi$ is the free wave packet satisfying the initial condition which is determined by $\Psi(r, 0)$ through Eqs. (3.1), (3.4) and (3.7).

We could have proceeded differently by defining the CF wave function

$$\tilde{\psi}(x, \tau) = a\Phi(ax, t) \exp\left(-ia\dot{x}^2/2a\right)$$

(3.8)

at the stage of Eq. (3.2), so that

$$i \frac{\partial}{\partial \tau} \tilde{\psi}(x, \tau) = \left[ -\frac{1}{2} \nabla^2 - \bar{E} \cdot x \right] \tilde{\psi}(x, \tau),$$

(3.9)

where $\bar{E}$ is defined at Eq. (2.12). We would then go over to EGF via the transformation

$$\phi(x, \tau) = \tilde{\psi}(x + \bar{X}(\tau), \tau) \exp\left[-i \frac{d\bar{X}(\tau)}{d\tau} \cdot x - i \tilde{S}(\tau)\right],$$

(3.10)

where $\bar{X}$ is a solution of Eq. (2.12) and $\tilde{S}$ is the associated action

$$\tilde{S}(\tau) = \int_0^\tau d\tau' \left\{ \frac{1}{2} \left[ \frac{d\bar{X}(\tau')}{d\tau'} \right]^2 + \bar{E}(\tau') \cdot \bar{X}(\tau') \right\}.$$

(3.11)

This alternative procedure is perhaps logically simpler than the previous one, while the latter is more convenient if wave function $\psi$ for the force-free harmonic oscillator is known beforehand.

We have arrived at the following precise statement of the equivalence theorem; the quantum dynamics of the time-dependent charged harmonic oscillator in the uniform electromagnetic field is equivalent to that of a free particle (i.e., $\phi$) plus the classical Larmor precession (i.e., $e_i$) and the classical dynamics of a forced time-dependent harmonic oscillator (i.e., $X$); the latter can in turn be decomposed into the
classical dynamics of a force-free time-dependent harmonic oscillator and that of a free particle in an external force, as we have seen in § 2.

As is well known, equivalence of a charged free particle in a constant uniform magnetic field to a harmonic oscillator follows from the identity \((\mathbf{p} - A) \times (\mathbf{p} - A) = i \mathbf{B}\), where \(\mathbf{p} = -i \nabla\). This familiar equivalence is strong in the sense that the spectrum of the Hamiltonian is preserved. By contrast our equivalence does not preserve the spectrum, but it is more general. In the present approach Landau energy levels emerge as follows. Suppose that \(\mathbf{E} = 0\) and \(\omega_L\) and \(\nu\) are constant, then a stationary state of (1.1) is found from Eqs. (3.1) and (3.4) by putting \(X = 0\) and letting \(\phi\) be a stationary state of (3.6) of the form

\[
\phi(x, t) = F(|x|) \exp[i m \varphi - i (2n' + |m| + 1) \omega t],
\]

where \(\varphi\) is the azimuthal angle, and \(n' \geq 0\) and \(m\) are integers.

This gives

\[
\Psi(r, t) = F(|r|) \exp[i m \varphi - i (2n' + |m| + 1) \omega - m \omega_L t].
\]

When \(\nu = 0\), one obtains energy eigenvalues \((n + 1/2) \omega_c\) with \(n\) being non-negative integers.

§ 4. Feynman Kernel

As an application of the general scheme described in § 3, let us compute the Feynman kernel (FK) for Eq. (1.1), denoted by \(K(r, t; r_0, 0)\), which is the solution with the initial condition

\[
\Psi(r, 0) = \delta^2(r - r_0),
\]

where \(r_0\) is a constant. In this section it is convenient to employ the particular solution \(\xi\) (defined by Eq. (2.22)) for \(X\) in Eq. (3.4). Then condition (4.1) is translated into

\[
\phi(x, 0) = C \delta^2 \left( x - \frac{r_0}{a_0} \right), \quad C = a_0^{-1} \exp(-i a_0 r_0^2 / 2 a_0).
\]

Note that \(r_0 = (r_{01}, r_{02})\), as defined by Eq. (2.18), which denotes components of \(r_0\) with respect to the inertial frame. It follows that

\[
\phi(x, t) = CK_{FP} \left( x, \tau; \frac{r_0}{a_0}, 0 \right) = C (2 \pi i \tau)^{-1} \exp \left[ i \left( x - \frac{r_0}{a_0} \right)^2 / 2 \tau \right],
\]

where \(K_{FP}\) is the FK for a free particle in two dimensions. By use of Eq. (3.7) we find that

\[
\phi(x, t) = K_{HO}(x, t; r_0, 0) = \left[ 2 \pi i a(t) \right]^{-1} \exp[i S_{HO}(x, t; r_0, 0)],
\]

where

\[
S_{HO}(x, t; x_0, 0) = (\dot{\alpha}(t)x^2 + \beta(t)x_0^2 - 2x \cdot x_0) / 2 \alpha(t).
\]

Here \(K_{HO}\) is the FK for the two-dimensional time-dependent harmonic oscillator and
agrees with the known result.\(^5\) Next we use Eq. (3.4) to obtain \(\Phi\). Since this part of the calculation is well known,\(^5\,^6\) we only give the result.

\[
\Phi(x, t) = K_{\text{FH0}}(x, t; r_0, 0) = [2\pi i a(t)]^{-1} \exp[i S_{\text{FH0}}(x, t; r_0, 0)],
\]

where

\[
S_{\text{FH0}}(x, t; r_0, 0) = \frac{1}{2a(t)} \left[ \dot{a}(t)x^2 + \beta(t)x_0^2 - 2x \cdot x_0 \right.
+ 2 \int_0^t dt' (s(t', 0)x + s(t', t')x_0) \cdot E(t')
- 2 \int_0^t dt' \int_0^{t'} dt'' s(t', t')E(t') \cdot E(t'')s(t'', 0) \bigg].
\]

\(K_{\text{FH0}}\) is the FK for the two-dimensional forced time-dependent harmonic oscillator. Finally we use transformation (3.1) and definition (2.18) to obtain

\[
K(r, t; r_0, 0) = [2\pi i a(t)]^{-1} \exp[i S(r, t; r_0, 0)],
\]

where

\[
S(r, t; r_0, 0) = \frac{1}{2a(t)} \left[ \dot{a}(t)r^2 + \beta(t)r_0^2 - 2r \cdot r_0 \cos \theta_L(t) - 2(r \times r_0) \cdot n \sin \theta_L(t) \right.
+ 2 \int_0^t dt' (e_j(t) \cdot r_0s(t', 0) + s(t, t')r_0 \cdot e_j(0))e_j(t') \cdot E(t')
- 2 \int_0^t dt' \int_0^{t'} dt'' s(t, t')E(t') \cdot e_j(t')e_j(t'') \cdot E(t'')s(t'', 0) \bigg].
\]

We have thus arrived at the explicit expression for the FK as a functional of the time-dependent parameters \(\omega_L, \nu\) and \(E\); recall that the functional forms of \(a, \beta\) (hence \(s\)) and \(\theta_L\) (hence \(e_j\)) are explicitly known via Eqs. (2.21) and (2.4). We emphasize that the result has been obtained by a simple algebra with the knowledge of the free-particle kernel only. If \(\omega_L\) is constant and \(\nu = E = 0\), then \(a = \omega_L^{-1} \sin \theta_L, \beta = \cos \theta_L, \theta_L = \omega_L t, \) and the well-known result\(^7\) is recovered,

\[
K(r, t; r_0, 0) = \frac{\omega_L}{2\pi i \sin \omega_L t} \exp \left\{ i \omega_L \left[ \frac{1}{2} (r - r_0)^2 \cot \omega_L t - (r \times r_0) \cdot n \right] \right\}.
\]

§ 5. Motion of wave packets

With the knowledge of the Feynman kernel, one can in principle calculate \(\Psi(r, t)\) for any given \(\Psi(r, 0)\). The nature of \(\Psi(r, t)\), however, can be inferred more directly by tracing the transformations backward once the corresponding wave packet \(\phi\) for the free particle or \(\psi\) for the harmonic oscillator is known. It is most convenient to start from the formula

\[
\Psi(r, t) = \phi(x - X(t), t) \exp[i P(t) \cdot (r - R(t)) + i S(t)],
\]

which follows from Eqs. (3.1) and (3.4), where it is understood that \(x = x = (x_1 - X_1,\)

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$x_2 - X_2$ and $x_j - X_j(t) = (r - R(t)) \cdot e_j(t)$. We have introduced the canonical momentum

$$P(t) = \dot{R}(t) + A(R(t), t).$$

(5.2)

Suppose that the initial packet represented the particle at $r = r_0$ with velocity $v_0$, that is,

$$\Psi(r, 0) = f(r - r_0)\exp[ip_0 \cdot (r - r_0)],$$

(5.3)

where $f(r)$ is peaked and real around $r = 0$, and $p_0$ is the canonical momentum (2.19) corresponding to $v_0$. Let us take the $R$ given by Eq. (2.25) as the classical trajectory appearing in Eq. (5.1), and identify $(r_0, p_0)$ in Eq. (2.25) as those in Eq. (5.3). Then $\phi$ satisfies the initial condition

$$\phi(x, 0) = f(x) e_j.$$  

(5.4)

Equations (5.1) and (5.4) lead to the following result. Since the center of $\phi(x, t)$ will remain at $x \sim 0$, that of $\Psi(r, t)$ will follow the classical trajectory $R$, as it should in view of the Ehrenfest theorem. From the result of II we know that the width of $\phi(x, t)$ pulsates. This pulsation will be inherited to $\Psi(r, t)$. If the initial packet is anisotropic, then the anisotropy axis will rotate with (instantaneous) Larmor frequency $\omega_L$.

We may summarize the motion of a generic wave packet thus; a wave packet executes two kinds of internal motion, rotation of anisotropy axis and pulsation (or squeezing oscillation), while its center traces the classical trajectory.

As a specific example let us find the wave packet corresponding to the initial condition (5.3) with

$$f(r) = (\Delta_0 \Delta_0^*)^{-1/2} \exp\left[-\frac{1}{2} r \cdot \Lambda_0 \cdot r\right],$$

(5.5)

where $\Delta_0$ and $\Delta_0^*$ are positive parameters and $\Lambda_0$ is a diadic

$$\Lambda_0 = \sum_j \Delta_0^* e_j \otimes e_j.$$  

(5.6)

Accordingly

$$\phi(x, 0) = \prod_j (\Delta_0^* \exp(-x_j^2/2\Delta_0^*)).$$

(5.7)

It follows that $\phi(x, t)$ is just the product of one-dimensional squeezed states (II.3.23), now generalized to the case of time-dependent frequency,

$$\phi(x, t) = \prod_j \phi_j(x_j, t; \Delta_0),$$

(5.8)

where

$$\phi_j(r, t; \Delta_0) = \Delta^{-1/2} \exp(i\Delta r^2/2\Delta), \quad \Delta = \Delta_0 \beta + i\Delta_0^{-1} \alpha.$$  

(5.9a)

The appearance of $\beta$ is not inconsistent with expression (4.5) for $K_{ho}$ which does not contain it, because there exists Wronskian relation $\dot{\alpha} \beta - \dot{\beta} \alpha = 1$. It is useful to note
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Transforming back via Eq. (5.1) we obtain

\[ \Psi(r, t) = (\Delta_0 \Delta_b)^{-1/2} \exp \left[ -\frac{1}{2} (r - R) \cdot \Lambda \cdot (r - R) + iP \cdot (r - R) + iS(t) \right], \]

where

\[ \Lambda = -i \sum_j \frac{\Delta_j}{\Lambda_j} e_j(t) \otimes e_j(t), \]

with \( \Delta_j \) being given by replacing \( \Delta_0 \) by \( \Delta_{0j} \) in \( \Delta \).

Note that \( S \) is given by Eq. (3.5), which may also be written as

\[ S(t) = \frac{1}{2} \left\{ P(t) \cdot R(t) - p_0 \cdot r_0 + \int_0^t dt' E(t') \cdot R(t') \right\}. \]

When \( B = E = 0 \) and \( \nu \) is constant (=1), wave function (5.10) reduces to the two-dimensional version of squeezed coherent state (II.3.34); in the latter the exponent was given in the form expanded in powers of \( r \).

It is instructive to compute the expectation value of Hamiltonian. First, let \( H_{ho} \) be the Hamiltonian of one-dimensional harmonic oscillator of frequency \( \omega(t) \). Its expectation value with respect to the squeezed state (5.9a) is found to be

\[ \langle H_{ho} \rangle = \frac{1}{2} |\dot{\sigma}|^2 + \frac{1}{2} \omega^2 |\sigma|^2 \]

where \( \sigma = \Delta / \sqrt{2} \), and \( \dot{\sigma} = \Delta^{-1} \dot{\sigma}, \quad \beta = \Delta \beta \). Clearly this can be interpreted as the energy of pulsation of the complex width \( \sigma \). Next, let us consider Hamiltonian (1.1) and evaluate its expectation value with respect to the Gaussian packet (5.10),

\[ \langle H \rangle = \frac{1}{2} \dot{R}^2 - E \cdot R + \frac{1}{2} \nu^2 R^2 + \sum_j \left( \frac{1}{2} |\dot{\sigma}_j|^2 + \frac{1}{2} \omega^2 |\sigma_j|^2 \right), \]

where \( \sigma_j = \Delta_j / \sqrt{2} \). The last term is just the two-dimensional version of (5.13) and comes from the internal motion of the packet, while the rest represents the energy of the center-of-mass motion. Similarly, for the gauge-invariant angular momentum \( \mathcal{L} = -i \mathcal{V} \cdot \Lambda + A(r, t) \) we find

\[ \langle \mathcal{L} \rangle = \mathcal{R} \times \dot{\mathcal{R}} + \mathcal{T} \Omega_L, \]

where

\[ \mathcal{T} = \langle \dot{r}^2 \rangle = \sum_j |\sigma_j|^2, \quad \Omega_L = -\frac{1}{2} B. \]

The first term is associated with the center-of-mass motion and the second term with the internal rotation of angular velocity \( \Omega_L \). Hence we may interpret \( \mathcal{T} \) as the moment of inertia of the packet.
Equation (5·10) illustrates the previous general observation on the motion of a wave packet. How the classical trajectory \( R \) enters into \( |\Psi| \) is obvious from the Ehrenfest theorem, but the fact that \( R \) enters into the complex \( \Psi \) in such a simple fashion would not be obvious without aid of the transformations used above. [One may appreciate this point by trying to obtain Eq. (5·10) from a direct convolution of (5·3) with Feynman kernel (4·8).] If \( \nu = 0 \) the instantaneous frequency of change of the diadic \( \Lambda \) as a whole is \( \omega_c \); this is of course expected from the fact that the energies of Landau levels when \( \omega_c \) is constant are \((n + 1/2)\omega_c\). But our result is not restricted to the case of a constant \( \omega_c \), and shows explicitly how the width and the orientation of an anisotropic Gaussian packet can be controlled by application of a time-dependent magnetic field.

We close this section with the following remark: Given any wave packet \( \phi \) for a time-dependent harmonic oscillator, corresponding wave packet \( \Psi \) for the charged particle can be constructed via Eq. (5·1).

§ 6. Aharonov-Bohm-Berry-like phase

Several years ago Berry\(^3\) performed an interesting gedanken experiment with a charged particle confined to a box situated outside a magnetic flux. The box was slowly transported around a closed circuit \( \mathcal{C} \) under the condition that the box was never penetrated by the magnetic flux. He showed that the wave function of the charged particle acquires a phase factor \( \exp(2\pi i\Phi) \) in addition to the usual dynamical phase factor, where \( \Phi \) is the magnetic flux enclosed by \( \mathcal{C} \) in units of fluxons (i.e., \( hc/q \)) appropriate for the charge \( q \) of the particle. If the energy-eigenvalues \( \varepsilon_n \) associated with the box are commensurate with each other, the time \( T \) to complete the round trip may be so chosen that \( \varepsilon_n T/2\pi \) are integers for all \( n \). In this case dynamical phase factors are all equal to unity, and an arbitrary wave packet \( \phi \) will satisfy

\[
\phi(r, T) = \phi(r, 0)\exp(2\pi i\Phi). \tag{6·1}
\]

In this gedanken experiment the Aharonov-Bohm (AB) phase thus emerges as Berry’s phase. [For a review of Berry’s phase, see Refs. 8) and 9).]

Let us consider an analogous situation for our system defined by Eq. (1·1). Let the external force \( E(t) \) be so controlled that the center \( R(t) \) of the packet moves once around a closed circuit \( \mathcal{C} \) in time \( T \). We assume that

\[
R(T) = R(0), \quad \dot{R}(T) = \dot{R}(0). \tag{6·2}
\]

(For example, if \( \mathcal{C} \) is to be a circle of radius \( r_0 \) described by \( R = r_0 \cos \Omega t + n \times r_0 \sin \Omega t \), \( \Omega = 2\pi/T \), in a constant magnetic field, then the required force is a rotating electric field: \( E = [\nu^2 - \Omega^2 (\omega_c + \omega_r)]R \).) Although Berry’s result could be obtained very simply by use of gauge transformation, such a trick will not work in the present situation where the system is immersed in the magnetic field. Therefore it is not immediately obvious whether a result like (6·1) holds or not without examining relevant solution (5·1) of Eq. (1·1). If both \( \omega \) and \( \omega_r \) are constant and are commensurate with each other, then we can choose \( T \) such that both \( \omega T \) and \( \omega_r T \) are integral multiples of \( 2\pi \).
Since \( \psi \) in Eq. (5·1) is a wave packet for the two-dimensional harmonic oscillator of frequency \( \omega \), we then have

\[
\psi(x, T) = \psi(x, 0) .
\]  
(6·3)

The above condition on \( T \) also implies

\[
e_{\psi}(T) = e_{\psi}(0) .
\]  
(6·4)

We now suppose that there exists \( T \) satisfying Eqs. (6·2)~(6·4) and that the condition \( B(T) = B(0) \) holds, even when \( \omega \) and \( \omega_L \) depend on time. It follows from Eq. (5·1) that

\[
\Psi(r, T) = \Psi(r, 0) \exp[i S(T)],
\]  
(6·5)

where

\[
S(T) = S_0 + S_1,
\]  
(6·6a)

\[
S_0 = \int_0^T dt \left[ \frac{1}{2} \dot{R}^2(t) - \frac{1}{2} \nu^2(t) R^2(t) + A(R(t), t) \cdot \dot{R}(t) \right],
\]  
(6·6b)

\[
S_1 = \int_0^T dt E(t) \cdot \dot{R}(t).
\]  
(6·6c)

Under condition (6·2) we have the virial theorem \( S_1 = -2 \, S_0 \) [cf. Eq. (5·12)]. Consequently

\[
S(T) = -S_0 .
\]  
(6·7)

If the size of \( C \) is of order of \( r_0 \), then the first two terms of \( S_0 \) are of order of \( r_0^2/T \) and \( \nu^2 r_0^2 T \), respectively. If \( \nu T \ll 1 \ll \omega_L T \), therefore, it follows that

\[
S(T) \simeq -\int_0^T A(R(t), t) \cdot \dot{R}(t) dt .
\]  
(6·8)

In particular, when \( B \) is constant, this reduces to

\[
S(T) \simeq -\int_C A(r) \cdot d\mathbf{r} = -2\pi \tilde{\Phi} .
\]  
(6·9)

Note that \( \tilde{\Phi} \) is the flux enclosed by the orbit \( C \) of the center of the packet; in the above argument no condition was imposed on the size of the packet, which may be comparable or even larger than the size of \( C \). The present phenomenon, therefore, is not an AB effect. Indeed the above phase has opposite sign to that of Eq. (6·1). But it may be regarded as a kind of Berry's phase, because being independent of \( T \) it is geometrical though not topological. Another point to note is the following. Unlike Berry's gedanken experiment, where the agent which moved the box was not represented in the Hamiltonian, we have explicitly included the agent in the form of the external force (i.e., \( E(t) \)). Moreover we did not introduce such artifice as a box, instead of which the magnetic field kept our wave packet from spreading indefinitely. The presence of the external force is essential in producing the phase in Eq. (6·5), as is obvious from the relationship \( S(T) = S_1/2 \). On the other hand, a wave packet can
move along a closed circuit $C$ satisfying (6·2) autonomously without aid of an external force. An example of such $C$ is a cyclotron orbit or more generally epitrochoidal orbit (2·30) in constant magnetic field. But $S(T)$ vanishes in such a case.

§ 7. Aharonov-Bohm phase and cyclotron motion

In connection with the remark made in the last paragraph of § 6, one might wonder if Berry's gedanken experiment can be dynamically realizable without introduction of an artificial box and an unspecified agent to move the box. Indeed it can be, as we shall now show.

A simple example is the system described by Eq. (1·1), with $\nu=E=0$, augmented by a localized magnetic flux at $r=0$ in the $n$-direction. We denote by $A_0(r)$ the vector potential for this flux line, and by $\Psi(r, t)$ the wave function of the system. The condition $\nu=E=0$ is not essential but is imposed just for brevity. We shall also suppose for simplicity that $B$ is constant, and denote the vector potential (1·1c) by $A(r)$. The Schrödinger equation to be solved is

$$i\frac{\partial}{\partial t} \Psi(r, t) = -\frac{1}{2} \left[ \nabla_r - iA_0(r) - iA(r) \right]^2 \Psi(r, t). \tag{7·1}$$

We assume the initial condition

$$\Psi(r, 0) = f(r-r_0) \exp \left\{ ip_0 \cdot (r-r_0) + i \int_{r_0}^r A_0(r') \cdot dr' \right\}, \tag{7·2}$$

where $f$ is given by Eq. (5·5) with $A_0=\Delta_0 \ll r_0 = |r_0|$; this ensures that $f(r-r_0)$ is well localized in a simply connected region around $r=r_0$ off the flux line. Under this condition $\Psi(r, 0)$ is single-valued to a good approximation and represents an initial packet of velocity $v_0 = p_0 - A(r_0)$. We assume that $\Psi(r, t)$ also remains well localized around $r-R(t)$ off the flux line, where $R$ is the classical trajectory (2·25) with $\nu=E=0$; this assumption shall be later justified self-consistently. Then we can perform the gauge transformation

$$\tilde{\Psi}(r, t) = \Psi(r, t) \exp \left[ i \int_{R(t)} A_0(r') \cdot dr' \right] \tag{7·3}$$

with the result that $\tilde{\Psi}(r, t)$ is again single-valued to a good approximation, and Eq. (7·1) is converted into

$$i\frac{\partial}{\partial t} \tilde{\Psi}(r, t) = \left\{ -\frac{1}{2} \left[ \nabla_r - iA(r) \right]^2 + \phi_0(t) \right\} \tilde{\Psi}(r, t), \tag{7·4a}$$

where

$$\phi_0(t) \equiv -A_0(R(t)) \cdot \dot{R}(t) \tag{7·4b}$$

is the scalar potential induced by the time-dependent gauge transformation. [Compare Mondragon and Berry\(^{10}\) for a related observation.] With a further transformation
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\[ \Psi(r, t) = \Psi(r, t) \exp \left\{ -i \int_0^t dt' \phi_0(t') \right\}, \quad (7.5) \]

we arrive at

\[ i \frac{\partial}{\partial t} \Psi(r, t) = -\frac{1}{2} \left[ \nabla r - i A(r) \right]^2 \Psi(r, t), \quad (7.6a) \]

\[ \Psi(r, 0) = f(r - r_0) \exp \left[ ip_0 \cdot (r - r_0) \right]. \quad (7.6b) \]

This problem has been solved in § 5 with result (5.10); \( \Psi(r, t) \) is peaked around \( r = R(t) \), and its width pulsates between \( \Delta_0 \) and \( \omega/R_0 \). Moreover \( R(t) \) traces the circle \( |r| = r_0 \) if we choose \( p_0 = -B \times r_0/2 \). Therefore the assumption on \( \Psi(r, t) \) is satisfied if we choose \( \omega \) so that \( \max(\Delta_0, \omega/c) \ll r_0 \). Taking \( T = 2\pi/|\omega| \) we can ensure condition (6.2). But the wave function (5.10) reverses sign, which is obvious also from the presence of zero point term \( \omega/c \) in Landau energy eigenvalues.

We therefore find from Eqs. (7.2), (7.3) and (7.5) that the wave packet acquires the AB phase thus,

\[ \Psi(r, T) = \Psi(r, 0) \exp \left[ 2\pi i \left( \Phi_0 - \frac{1}{2} \right) \right], \quad (7.7a) \]

where

\[ 2\pi \Phi_0 = - \int_0^T dt \phi_0(t) = \int_C A_0(r) \cdot dr \quad (7.7b) \]

with \( C \) being the closed circuit traced by the center of the wave packet. In contrast to Berry’s gedanken experiment, however, no condition of adiabaticity has been used. Adiabaticity in this case would mean that \( \omega c T \gg 1 \), which is impossible because the equation of motion of the packet dictates the choice \( T = 2\pi/\omega c \). It is to be remembered that result (7.7) is approximate because we have treated the wave packet as if it were completely localized in a simply connected region off the flux line. The effect of incomplete localization will modify (7.7b) by a small amount of order of \( |f(2r_0)|^2 \). This expectation, which comes from a recent study of such an effect in a related simple system, is yet to be confirmed.

§ 8. Comments

In § 1.5 we briefly mentioned a possible application of our method to non-linear Schrödinger equations. Explicit examples have been worked out recently concerning the response of charged solitons to external electromagnetic fields in the non-relativistic Chern-Simons field theory as well as in the standard nonlinear Schrödinger field theory in 1+2 dimensions; transformations (3·1), (3·4) and (3·7) go through, where \( \phi \) is a solution in the absence of electromagnetic field. Although details of soliton motion depend on the model and the type of the soliton considered, its generic feature is the same as the motion of a wave packet described in § 5. In 1+d dimensions, external potentials in the equation
\[ i \frac{\partial}{\partial t} \Psi(r, t) = \left\{ -\frac{1}{2} \nabla^2 - E(t) \cdot r + \frac{1}{2} \nu^2(t) r^2 + g |\Psi(r, t)|^{4/3} \right\} \Psi(r, t), \tag{8\cdot1} \]

where \( g \) is an arbitrary constant, can be eliminated by transformations to appropriate EGF and CF without modifying \( g \).

After publication of Refs. 1) and 2) the present author has become aware of some related works. Fujikawa and Ui\(^{15}\) formulated a theory which is form-invariant under time-dependent rotations in the context of nuclear physics. As has been noted in § I.2.2, Hill and Wheeler\(^{16}\) applied a time-dependent scale transformation to quantum mechanics in the context of a particle confined by a moving wall. This Hill-Wheeler problem has since been treated in more details both in quantum mechanics and in quantum field theory.\(^{17)\cdot19)\) [In the context of relativistic quantum field theory it is often called the moving mirror problem.\(^{20)\}]\) Some particular scale transformation have been discussed also for a particle in the inverse-square potential\(^{21}\) and for a charged particle interacting with a magnetic monopole\(^{22}\) or a magnetic vortex.\(^{23}\)

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