The Singularity Structure of a Soliton Solution to the Higher-Dimensional Einstein Equations

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We study a stationary and axisymmetric solution to the higher-dimensional Einstein equations and investigate its singularity structure. The solution consists of two solitons in the four-dimensional part (i.e., the Kerr solution) and \( n \) solitons in the extra dimensions. Naked singularities appear on the symmetry axis (z-axis) and/or at the event horizons of the Kerr solution. In a certain choice of integration constants there are solutions with regular event horizons.

§ 1. Introduction

It seems that higher-dimensional theories are now accepted by many people as a unified theory of all the interactions. When the heterotic string theory breaks down to the four-dimensional theory, it passes through the ten-dimensional theory.\(^1\) It is therefore natural to ask if the ten-dimensional theory allows for black hole solutions like the four-dimensional theory. If such solutions exist, the internal extra dimensions should be compactified at the size of the Planck length far away from black holes, but they might increase in size as they approach the event horizon and affect the structure of space-time of black holes. This may enable us to use black holes to probe the relics of higher-dimensional theories.

We have never had such spherically symmetric solutions corresponding to the Schwarzschild black holes unless the gauge fields are taken into account.\(^2\)\(^,\)\(^3\) This is understandable because of the no hair theorem.\(^4\)\(^\text{-}\)\(^7\) In the case where the dilaton field is coupled to an Abelian gauge field the black hole solutions which correspond to the Reissner-Nordström metric exist.\(^8\)\(^\text{-}\)\(^12\) However, the dilaton charge in these solutions cannot be called a new scalar hair because it is related with the mass and electric charge of black holes. Therefore, the theorem still holds in this case. Stationary and axisymmetric solutions have been also studied.\(^13\)\(^\text{-}\)\(^16\) A few works reveal their space-time structure and show that they cannot be black hole solutions because there are singularities at the event horizon.

In the previous work\(^16\) we considered a stationary and axisymmetric metric in the \((4 + N)\)-dimensional space-time and showed that one of the vacuum Einstein equations can be written as the equation of the non-linear \( \sigma \) model. By means of the inverse scattering method the solution which consists of solitons in the extra dimensions as well as in the four-dimensional part was obtained. We hoped that the spatially

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localized feature of this solution might evade the no hair theorem. The two-soliton solution in the extra dimensions has two integration constants while three in the four-dimensional part which correspond to the mass, angular momentum and NUT parameter of the Kerr-NUT solution. In order to get a compact form of metric the constants in the extra dimensions were fixed to the values related with those in the four-dimensional part. When the NUT parameter is equal to zero the asymptotic behavior was shown to be the same as that of the Kerr metric in the four-dimensional part and some aspects of space-time structure were also pointed out.

As noted in Ref. 16) it is not necessary that the integration constants in the extra dimensions are related with those in the four-dimensional part. Because the space-time structure depends on these constants it should be studied in detail in the alternative choice of them. It is also important to study a general case of \( n \)-soliton solution in the extra dimensions which has \( n \) integration constants. This is the purpose of the present paper. In § 2 we will give the solution in a general form and also the common feature of solution which is independent of the choice of the constants. In § 3 the space-time structure described by the solution which is given in § 2 will be investigated. There are \( n/2 \) naked singularities on the symmetry axis (\( z \)-axis) and/or at the event horizons of the Kerr solution. It will be also shown that there are solutions with regular event horizons in a certain choice of the integration constants.

### § 2. The solution with \( n \) solitons in the extra dimensions

We considered a stationary and axisymmetric metric in the \((4+N)\)-dimensional space-time in Ref. 16)

\[-ds^2=f(d\rho^2+dz^2)+g_{ab}dx^adx^b+h_{ab}dx^adx^b,\]  

(2.1)

where \( f, g_{ab}, \) and \( h_{ab} \) are functions of \( \rho \) and \( z \). For the \( 2\times2 \) matrix \( g=(g_{ab}) \) and \( N\times N \) matrix \( h=(h_{ab}) \), one of the vacuum Einstein equations can be decomposed into the same type of equations

\[(\rho g_{,\rho}g^{-1})_{,\rho}+(\rho g_{,\rho}g^{-1})_{,\rho}=0,\]  

(2.2)

\[(\rho h_{,\rho}h^{-1})_{,\rho}+(\rho h_{,\rho}h^{-1})_{,\rho}=0\]  

(2.3)

with a coordinate condition

\[\det g \cdot \det h = -\rho^2.\]  

(2.4)

We applied the inverse scattering method to both equations (2.2) and (2.3) on the following assumptions: (i) \( h \) is diagonal corresponding to the torus compactification; (ii) \( h \) is two-dimensional for simplicity, which is sufficient to obtain a soliton solution and can easily be extended to any number of dimensions; (iii) the coordinate condition (2.4) is divided between \( g \)- and \( h \)-part as

\[\det g = -\rho^2,\]  

(2.5)

\[\det h = 1,\]  

(2.6)
which ensures the Kerr solution in the four-dimensional part. The pole trajectories for the two-soliton solution which leads to the normal Kerr metric in the four-dimensional part are given by

\[ \mu_1 = z_1 + \sigma - z + \sqrt{(z_1 + \sigma - z)^2 + \rho^2}, \quad (2.7) \]
\[ \mu_2 = z_1 - \sigma - z + \sqrt{(z_1 - \sigma - z)^2 + \rho^2}, \quad (2.8) \]

where \( \sigma \) and \( z_1 \) are the constants which express the positions of the trajectories. These constants also appear in the equations which introduce the standard \((r, \theta)\) coordinate:

\[ \rho = \sqrt{(r - m)^2 - \sigma^2 \sin \theta}, \quad (2.9) \]
\[ z - z_1 = (r - m) \cos \theta. \quad (2.10) \]

As for the extra dimensions, the pole trajectories which lead to an asymptotically flat \( n \)-soliton solution can be written as

\[ \bar{\mu}_k = w_k - z + (-1)^k \sqrt{(w_k - z)^2 + \rho^2}, \quad (k = 1, 2, \ldots, n) \quad (2.11) \]

where the \( w_k \)'s are the \( n \) constants. The number of solitons \( n \) should be even because there must be the same number of pole trajectories which have plus and minus signs in front of the square root in Eq. (2.11) in order to satisfy the asymptotic flatness condition. In Ref. 16 we considered the case of two solitons (strictly speaking, degenerate four solitons) in the extra dimensions and set \( w_1 = Z_1 + \Sigma \) and \( w_2 = Z_1 - \Sigma \). Although these \( Z_1 \) and \( \Sigma \) are not necessarily the same as the constants in Eqs. (2.7) and (2.8), we assumed \( \Sigma = \sigma \) and \( Z_1 = z_1 \) and obtained the solution for the whole metric (2.1) with a compact form, which can be expressed by using only familiar quantities in the Kerr solution.

Now we rewrite our solution in a general form including \( n \) solitons in the extra dimensions:

\[ -ds^2 = Q\omega(D^{-1}dr^2 + d\theta^2) - \omega^{-1}((\Delta - a^2 \sin \theta)d\tau^2 \]
\[ -4m \arcsin^2 \theta d\tau d\phi + [\Delta a^2 \sin^2 \theta - (r^2 + a^2)^2] \sin^2 \theta d\phi^2 \]
\[ + \left( \prod_{k=1}^{n} \bar{\mu}_k \right)/\rho^n (dx^5)^2 + \left( \prod_{k=1}^{n} \bar{\mu}_k \right)/\rho^n q(dx^6)^2, \quad (2.12) \]

where

\[ \omega = r^2 + a^2 \cos^2 \theta, \quad (2.13) \]
\[ \Delta = (r - m)^2 - \sigma^2. \quad (2.14) \]

In Eqs. (2.12) \sim (2.14) \( m \) and \( a \) are the usual constants in the Kerr metric which satisfy the constraint

\[ \sigma^2 = m^2 - a^2, \quad (2.15) \]

and the NUT parameter has been set as zero. The constant \( q \) is arbitrary for \( n = 4l \) but an even number for \( n = 4l - 2(l = 1, 2, \ldots) \), which keeps positiveness of the metric.
coefficients of the extra dimensions. The four-dimensional part in the solution (2·12) is the same as the Kerr metric except the factor $Q$. This factor is given by

$$Q = \left[ \frac{\rho^{n/2} \prod_{k=1}^{n} (\bar{\mu}_k - \bar{\mu}_0)^2}{\prod_{k=1}^{n} (\bar{\mu}_k^2 + \rho^2) \prod_{k=1}^{n} \bar{\mu}_k^{-2} C^{(n)}} \right]^{q^2}, \quad (2·16)$$

where $C^{(n)}$ is an integration constant determined by the asymptotic flatness condition as

$$C^{(n)} = 2^{n(n-2)/2} \prod_{k=1}^{n} (w_{2k-1} - w_{2i-1})^2 (w_{2k} - w_{2i})^2. \quad (2·17)$$

The space-time structure described by the metric (2·12) depends on the choice of the constants $w_k$ in Eq. (2·11), especially near $\rho=0$ which expresses both the $z$-axis and the event horizons in the Kerr solution. We will investigate it in the following section. However, the asymptotic behaviour at the spatial infinity is not so sensitive to them. In fact, we obtain as $r \to \infty$

$$Q = 1 + O(1/r^2), \quad (2·18)$$

and then

$$-ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{2am\sin^2 \theta}{r} dr d\phi$$

$$+ \left[1 + \frac{q}{r} \sum_{k=1}^{n} (w_{2k} - w_{2k-1})\right] (dx^5)^2 + \left[1 - \frac{q}{r} \sum_{k=1}^{n} (w_{2k} - w_{2k-1})\right] (dx^6)^2. \quad (2·20)$$

Therefore, the four-dimensional part is the same as that of the Kerr metric and the effect of the extra dimensions is of the order of $1/r^2$ in this part. The behavior (2·20) also shows that the scalar charges which are caused by the existence of the extra dimensions are given by $q \sum_{k=1}^{n} (w_{2k} - w_{2k-1})$ and $-q \sum_{k=1}^{n} (w_{2k} - w_{2k-1})$.

§ 3. Singularity structure of the solution

In this section we study the singularity structure of our solution (2·12). To do this, we investigate the zeros or the poles of the metric coefficients which are given by

$$\Delta - a^2 \sin^2 \theta = r^2 - 2mr + a^2 \cos^2 \theta = 0, \quad (3·1)$$

$$\omega = r^2 + a^2 \cos^2 \theta = 0, \quad (3·2)$$

$$\rho = \sqrt{(r-m)^2 - a^2 \sin \theta} = 0. \quad (3·3)$$

The case in which $Q=0$, $\Delta=0$ and $\bar{\mu}_k=0$ is included in the case of Eq. (3·3). Equations (3·1) and (3·2) describe the surfaces of stationary limit and the ring singularity at the origin, respectively, and are the same as those in the Kerr solution.
Therefore, we focus on Eq. (3.3), which expresses both the z-axis and the event horizon in the Kerr solution.

In order to study whether there is a physical singularity at $p=0$ we calculate the curvature invariant defined by

$$I = R^{ABCD}R_{ABCD},$$

where the suffix of the Riemann tensor runs over four and extra dimensions. In the calculation of Eq. (3.4) the angular momentum $a$ is irrelevant because the value of $a$ which is less than $m$ (i.e., $\sigma$ is real) does not affect the nature of singularity. Moreover, the nature of space-time caused by the existence of solitons in the extra dimensions does not depend on the value of $a$ since the space-time does not become spherically symmetric even when $a=0$. Thus, we set $a=0$ for simplicity in calculating Eq. (3.4).

When $a=0$ the terms in Eq. (3.4) are all positive and there are no cancellations among them. Therefore, we can show that there is a physical singularity at $p=0$ by showing that one of the terms, say, $(g^{\rho \rho}g^{zz}R_{\rho \rho \rho \rho})^2$ diverges. This quantity can be written as

$$(g^{\rho \rho}g^{zz}R_{\rho \rho \rho \rho})^2 = Q^{-2}(R_{\rho \rho \rho \rho}^\text{Kerr})^2 - \frac{1}{2} \frac{\Gamma}{\omega} [(\ln Q)_{,\rho \rho} + (\ln Q)_{,zz}],$$

(3.5)

where $\Gamma = (r-m)^2 - \sigma^2 \cos^2 \theta$ and $R_{\rho \rho \rho \rho}^\text{Kerr}$ is a component of the Riemann tensor of the Kerr metric. As $R_{\rho \rho \rho \rho}^\text{Kerr}$, $\Gamma$ and $\omega$ are all finite at $p=0$, we evaluate $Q$ and the quantity inside the square bracket on the right-hand side of Eq. (3.5), which is given by

$$(\ln Q)_{,\rho \rho} + (\ln Q)_{,zz} = -\frac{1}{2}[(\ln h_{11})_{,\rho}^2 + (\ln h_{11})_{,z}^2]$$

$$= -\frac{1}{2} \left( \sum_{k=1}^{n} \frac{\rho^2}{\mu_k^2 + \rho^2} - \frac{\rho^2}{\nu^2} \right) + \left( \sum_{k=1}^{n} \frac{\tilde{\mu}_k}{\mu_k^2 + \rho^2} \right)^2. $$

(3.6)

In getting Eq. (3.6) we have used the Einstein equations for $Q$ and the equations which the pole trajectories satisfy:

$$\tilde{\mu}_{k,\rho} = 2\rho \tilde{\mu}_k/(\tilde{\mu}_k^2 + \rho^2), \quad \tilde{\mu}_{k,z} = -2\tilde{\mu}_k/(\tilde{\mu}_k^2 + \rho^2). $$

(3.7)

In the neighbourhood of $p=0$ the pole trajectories behave as

$$\tilde{\mu}_k = w_k z + (-1)^k |w_k - z| + (-1)^k \frac{\rho^2}{2 |w_k - z|} + O(\rho^4).$$

(3.8)

This behavior depends on the position in the z-coordinate. There are two types of behavior according to the value of $k$ and $l$: $\tilde{\mu}_k \sim O(\rho^2)$ and $\tilde{\mu}_k \sim O(\rho^0)$. If we assume that $l$ pole trajectories behave as $\sim O(\rho^2)$ and $(n-l)$ pole trajectories as $\sim O(\rho^0)$, then we obtain

$$Q \sim \rho^{(q^2/2)(n-2l)^2},$$

$$q^{-2} (n-2l)^2 + \text{const}.$$
Table I. The behavior of μ's in the neighborhood of \( \rho=0 \). 0 and 1 are the abbreviations of \( O(\rho^2) \) and \( O(\rho^0) \), respectively.

| \( \mu_1 \) | 0 | 1 | 1 | 1 |
| \( \mu_2 \) | 1 | 1 | 0 | 0 | 0 |
| \( \mu_3 \) | 0 | 0 | 0 | 1 |
| \( \mu_4 \) | 1 | 1 | 1 | 0 |

When \( l \) is not equal to \( n/2 \), \( Q \rightarrow 0 \) and \( (\ln Q)_{,\rho} + (\ln Q)_{,z} \rightarrow \infty \) as \( \rho \rightarrow 0 \). Then, \( (g^{\rho \rho}g^{zz}R_{\rho\rho\rho\rho})^2 \rightarrow \infty \) as \( \rho \rightarrow 0 \) and there is a physical singularity at \( \rho=0 \).

To understand what the condition \( l \neq n/2 \) means, let us give order among the constants \( w_k \): \( w_1 < w_2 < \cdots < w_n \). In the region \( z < w_1 \), all the \( (w_k - z) \)'s in Eq. (3.8) are negative for any \( k \), so we have \( l = n/2 \). In the region \( w_1 < z < w_2 \), only \( (w_1 - z) \) changes its sign, so \( l = n/2 - 1 \). We have again \( l = n/2 \) in the region \( w_2 < z < w_3 \) since \( (w_2 - z) \) also changes its sign this time. We show these facts in Table I. Therefore, we find that there are physical singularities in the regions between \( w_1 \) and \( w_2 \), \( w_3 \) and \( w_4 \), and so on. As for in the regions below \( w_1 \), above \( w_n \) and between \( w_2 \) and \( w_3 \), \( w_4 \) and \( w_5 \), and so on, \( Q \rightarrow 1 \) and \( (\ln Q)_{,\rho} + (\ln Q)_{,z} \rightarrow \text{const.} \) as \( \rho \rightarrow 0 \), so \( (g^{\rho \rho}g^{zz}R_{\rho\rho\rho\rho})^2 \) is finite at \( \rho=0 \). We can easily see that all other components of the curvature invariant (3.4) stay finite in these regions.

The equation \( \rho=0 \) expresses both the \( z \)-axis and the surfaces which correspond to the event horizons in the Kerr solution as we already mentioned. We find that there are \( n/2 \) naked-singularity regions specified by the coordinate \( w_k \)'s on the \( z \)-axis (Fig. 1). By arranging the magnitudes of \( w_k \)'s properly, we can consider several cases that the distributions of singularity regions appear differently. If the singular regions on the \( z \)-axis exist inside the surfaces naked singularities appear on the surfaces (Fig. 2). Therefore, the surfaces are not event horizons at all. However, if we choose the constants \( w_k \) so as to set a regular region on the \( z \)-axis inside the surfaces we can get regular event horizons and naked singularities outside the event horizons (Fig. 3). Here we should comment on the geometrical structure of the singular regions. Although the regions seem to be string-like on the \( z \)-axis and surface-like on the event horizon, they are point-like and ring-like singularities geometrically. This is because \( g_{zz} \rightarrow 0 \) as \( \rho \rightarrow 0 \). Note that when \( w_1 = w_2, w_3 = w_4 \), and so on, there are no singularities but the solution reduces to a trivial one, which is the Kerr solution with a constant metric in the extra dimensions.

We have studied the singularity structure of our solution which includes \( n \) solitons in the extra dimensions and shown that naked singularities are inevitable. Although there is the solution which has regular event horizons, no
hair theorem still holds in this case because the naked singularities exist outside the event horizons. Last of all we discuss our assumption on the coordinate condition (2.4) to see whether it causes the existence of singularities. Let us consider more general conditions than Eqs. (2.5) and (2.6):

\[ \text{det} g = -\left(\frac{\rho}{\phi}\right)^2, \]
\[ \text{det} h = \phi^2. \]

where \(\phi\) is the function of \(\rho\) and \(z\) satisfying a differential equation

\[ (\rho \phi, \phi^{-1})_\rho + (\rho \phi, \phi^{-1})_z = 0. \]

The non-trivial solution of Eq. (3.13) which leads to an asymptotically flat soliton solution is given by

\[ \phi = \exp\left[ c/\rho^2 + (z - z_0)^2 \right], \]

where \(z_0\) and \(c\) are constants. This solution brings us to the same situation as in Koikawa and Shiraishi's case: There is a ring-type singularity or a singularity on the \(z\)-axis at \(\rho = 0\). Therefore the general conditions (3.11) and (3.12) only add a new type of singularity caused by \(\phi\) to the naked singularities shown before.

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