Jacobi Identity Anomaly in Closed String Field Theory

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The Jacobi identity, which is an indispensable property for gauge invariant closed string field theory, is apparently broken in cases where one of the four legs of the identity has vanishing string-length. This phenomenon contradicts the general proof of the identity valid at the critical dimension. We examine the general proof and clarify the mechanism of this Jacobi identity anomaly.

§ 1. Introduction

The Jacobi identity is one of the most important properties in gauge invariant closed string field theory (SFT) proposed by Kyoto group (see also Ref. 2)). It is a relation among the *-products of the string fields:

\[ (\Phi^{(1)} \Phi^{(2)}) \Phi^{(3)} + (-)^{|\Phi^{(1)}|+|\Phi^{(2)}|} \Phi^{(1)} \Phi^{(2)} \Phi^{(3)} + (-)^{|\Phi^{(1)}|+|\Phi^{(2)}|} \Phi^{(1)} \Phi^{(2)} \Phi^{(3)} = 0, \]

(1.1)

where \(|\Phi^{(r)}|=1(0)\) when the string field \(\Phi^{(r)}\) is Grassmann-odd (even). It just corresponds to the Jacobi identity for the structure constant \(f^{a}_{bc}\) in Yang-Mills theory and is indispensable for the gauge invariance of the SFT action.

However, we have an example which contradicts the Jacobi identity. It arises in relation to the pre-geometrical SFT, the closed SFT described by the purely cubic action \(S_{\text{pre}}=\frac{1}{3} \Phi^{3}\) without the kinetic term. \(S_{\text{pre}}\) is related to the ordinary closed SFT action \(S=\frac{1}{2} \Phi \cdot Q_8 \Phi + \frac{1}{3} \Phi^{3}\) via a field redefinition \(\Phi = \Phi_{0} + \Phi\) using a classical solution \(\Phi_{0}\) of the equation of motion \(\Phi_{0}^{*} \Phi_{0} = 0\) of pre-geometrical SFT. In Ref. 3) a particular form of the classical solution was presented:

\[ \Phi_{0} = -\frac{1}{2} Q_8 \Gamma, \]

(1.2)

where the string functional \(\Gamma\) enjoys the property\(^{*}\)

\[ \Gamma^{*} \Phi = (N_{FP} - a \frac{\partial}{\partial a}) \Phi \]

(1.3)

with \(N_{FP}\) being the ghost number operator. In fact, using the BRST invariance of the 3-string vertex defining the *-product and the property \([N_{FP}, Q_8]=Q_8\), we can show that \(\Phi_{0} (1:3)\) satisfies \(\Phi_{0}^{*} \Phi = \frac{1}{2} Q_8 \Phi\) for any \(\Phi\), and hence \(\Phi_{0}^{*} \Phi_{0} = -\frac{1}{4} (Q_8)^2 \Gamma = 0\) since \((Q_8)^2 = 0\). In Ref. 3) a concrete expression for \(\Gamma\) was presented, and this satisfies Eq. (1.3).

However, Eq. (1.3) is inconsistent with the Jacobi identity (1.1). This is because the Jacobi identity (1.1) with \(\Phi^{(3)} = \Gamma\) implies that \(\partial (\Phi^{(1)} \Phi^{(2)}) = (\partial \Phi^{(1)} \Phi^{(2)})\)

\(^{*}\) Equations (20) and (26) in Ref. 3) are incorrect. See Appendix A for details.
+ $\Phi^{(1)}(\partial \Phi^{(2)})$ for $\partial = N_{FP} - \alpha (\partial / \partial \alpha)$. However, this cannot be true because we have for the 3-string vertex $|V(1, 2, 3)>$

$$\sum_{r=1,3} \left( N_{FP} - \alpha \frac{\partial}{\partial \alpha} \right)^{(r)} |V(1, 2, 3)> = |V> \neq 0,$$  \hspace{1cm} (1.4)

since $|V>$ carries $N_{FP} = 0$, and it is a function of the ratios $\alpha_r / \alpha_s$ multiplied by the delta function $\delta(\sum_r \alpha_r)$. However, the Jacobi identity has been shown to be valid when the space-time dimension is the critical one $D = 26$. The above example clearly seems to contradict the general proof of the identity.

The purpose of this paper is to clarify why and how the Jacobi identity breaks down. The point is that the string functional $\Gamma$ has vanishing string-length $\alpha_s = \epsilon \rightarrow 0$. If we calculate the LHS of Eq. (1.1) with $\Phi^{(3)} = \Gamma$ by keeping $\epsilon$ finite and then take the limit $\epsilon \rightarrow 0$, there is no problem at all. However, in the above calculation, Eqs. (1.3) and (1.4), we take the limit $\epsilon \rightarrow 0$ in one of the *-products first and then calculate the other *-product, and this causes the breakdown of the Jacobi identity. In the context of the general proof of the Jacobi identity, Eqs. (1.3) and (1.4) correspond to Laurent-expanding the LHS of Eq. (1.1) with respect to $\epsilon / \alpha$ ($\alpha = \alpha_1$ or $\alpha_2$). However, this expansion fails when the interaction points of the two 3-string vertices in the identity are close to each other. We can pick up the singularity in the naive Laurent expansion in the general proof to reproduce the anomaly (1.4). We also present the correct treatment of the identity when $|e/\alpha| \ll 1$.

The above example of the Jacobi identity anomaly, however, does not imply that the classical solution $\Psi_6$ (1.2) is meaningless. For $\Psi_6$ (1.2) the singularity of the naive Laurent expansion is milder and does not develop an anomaly. In fact, the Jacobi identity (1.1) with $\Phi^{(3)} = \Psi_6$ requests the BRS invariance of the 3-string vertex, $\Sigma r=1,3 Q_{\psi}^{(r)} |V(1, 2, 3)> = 0$, which is of course satisfied when $D = 26$.

The rest of this paper is organized as follows. In § 2 we recapitulate the general proof of the Jacobi identity presented in Ref. 1). The detailed analysis of the mechanism of the Jacobi identity breaking is presented in § 3. In § 4 we give a correct treatment of the Jacobi identity when $|e/\alpha| \ll 1$. Section 5 is devoted to a summary and discussion. In Appendix A we present the correct expression of the formulae given in Ref. 3). In Appendices B and C we derive various formulae for the Neumann coefficients used in the text.

\section{2. Review of the proof of Jacobi identity}

Before starting the detailed analysis of the Jacobi identity for the counterexample mentioned in § 1, we in this section recapitulate the general proof of the Jacobi identity given in § IV of Ref. 1). In the following, we adopt the formulation where one of the ghost zero-modes, $\pi_c^a$, is omitted from the start (see § V of Ref. 1)). We refer to Refs. 3) and 4) for notations used in this paper.

First, the 3-string vertex $|V(1, 2, 3)>$ for the strings 1, 2 and 3 is given by

$$|V(1, 2, 3)> = [\mu(1, 2, 3)]^2 \mathcal{P}_{123} G_1 V_6(1, 2, 3)>,$$ \hspace{1cm} (2.1)

where $|V_6(1, 2, 3)>$, which is symbolically expressed as $\exp((1/2)\alpha N + i\gamma V \gamma)|0>$
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\[ \mathcal{P}_{123} = \int \frac{d\theta}{2\pi} \exp(i\theta(L_+ - L_-)) \text{on to the subspace invariant under the rigid } \sigma \text{-translation, and the factor } \mu \text{ is given by } \mu(a, b, c) = \exp(-\alpha(a, b, c) \times \sum_{r=a,b,c} |1(\alpha_r)|) \text{ with } \alpha(a, b, c) = \sum_{r=a,b,c} \alpha_r \ln |\alpha_r| .

The Jacobi identity is diagrammatically expressed in Fig. 1. The first term \( P \) of Fig. 1 with the propagation operator \( \exp((L_+^{(0)} + L_-^{(0)})/\alpha_0) \) inserted at the dotted line (\( T=0 \) in the Jacobi identity), is given by

\[
\int d\theta d\varphi <R(5, 6)| e^{(L_+^{(0)} + L_-^{(0)})/\alpha_0} V(1, 2, 6)| V(3, 4, 5)>
\]

where \( d\alpha (r=5, 6) \) denotes the integration over the zero-modes \((p, a, \tilde{c}_0)\), the \( \theta_\alpha \)-integration has come from the projector \( \mathcal{P}_s \), and the measure factor \( \mathcal{M}_r (\theta_\alpha, T) \) is given by

\[
\mathcal{M}_r = \frac{1}{\langle \alpha_0 \rangle} \text{det}(1 - \widetilde{N}_r(1, 2, 6)\bar{N}_r(3, 4))^{-(D-2)} \exp(2\pi \mu(1, 2, 6) \mu(3, 4, 5))
\]

(2·2)

where \( \gamma_0(\alpha) \) is defined by \( \gamma_0(\alpha) = \tilde{c}_0(\alpha) |\alpha_r| \), and \( \gamma_x \) and \( \gamma_{\rho}\) are expressed in terms of the 4-string Neumann coefficients \( \bar{M}_{nm}^{rs} \) as

\[
\gamma_x = \sum_{\pm} \sum_{n, m \geq 0} \bar{M}_{nm}^{rs(\pm)} \frac{1}{2} a_{\pm}^r \gamma_n^{(\pm)}(r) a_{\pm}^s(\pm)
\]

(2·5)

\[
\gamma_{\rho} = \sum_{\pm} \sum_{n, m \geq 1} \bar{M}_{nm}^{rs(\pm)} \gamma_n^{(\pm)}(r) a_{\pm}^s(\pm) + \frac{1}{2} \sum_{n \geq 1} \sum_{r, s} \bar{M}_{nm}^{rs(\pm)} \gamma_n^{(\pm)}(r) \gamma_0^{(s)}
\]

(2·6)

In Appendix B we summarize the formulae for the Neumann coefficients. They are given in terms of the Mandelstam mapping \( \rho(z) \) for the 4-string configuration:

\[
\rho(z) = \sum_{r=1}^4 \alpha_r \ln(z - Z_r) \text{ with } \sum_{r=1}^4 \alpha_r = 0
\]

(2·7)

In this paper, we fix the projective invariance by taking the gauge,

\[
(Z_1, Z_2, Z_3, Z_4) = (0, 1, \infty, \mu)
\]

(2·8)

\text{We denote the 3-string Neumann coefficients by } \bar{N}_{nm} \text{ and the 4-string ones by } \bar{M}_{nm}^{rs}.
with \( w \) being a complex variable.

The second and third terms, \( Q \) and \( R \), of Fig. 1, are given by the formulae obtained from Eqs. (2·2) and (2·3) by cyclic permutations of \((1, 2, 3)\). The integration angle variables and the quantities \((2·3)\) are generally denoted by \( \theta_M \) and \( M_M \) \((M = P, Q, R)\), respectively, and we define the string-length \( a_M \) of the intermediate string by

\[
(a_P, a_Q, a_R) = (a_1 + a_2, a_2 + a_3, a_3 + a_1).
\]

In particular, we have \( a_P = a_Q = -a_R \).

Then the Jacobi identity, which is the cancellation among the three terms of Fig. 1, is realized owing to the following facts:

i) The \( \theta_M \) integrations in each term of the Jacobi identity (Eq. (2·2) with \( T = 0 \) for \( M = P \)) are along the contours in the \( w \)-plane shown in Fig. 2.

ii) The Neumann coefficients \( M_{MN} \) and \( M_{MN}^\ast \) are smooth functions of \( w \). They are common between the two terms of the Jacobi identity corresponding to the same point in the \( w \)-plane.

iii) The measure factors \( |a_M d\theta_M| M_M \) admit (when \( D = 26 \)) an expression due to Cremmer and Gervais\(^{3,1}\) in terms of the Mandelstam mapping,

\[
M_M = \left| \prod_{r=1}^4 \frac{dZ_r dZ_r^\ast}{dV_{abc} dV_{abc}^\ast} \right| \exp \left( -2 \sum_{s=1}^4 \Re M_{ss}^M \right),
\]

where \( dV_{abc} = dZ_a dZ_b dZ_c (Z_a - Z_b)(Z_b - Z_c)(Z_c - Z_a) \) and \((a, b, c) = (1, 2, 3)\) in the gauge (2·8). Since the “time” variable \( T = \Re \Delta \rho \) (see the next section) is continuously defined across the \( T = 0 \) lines of Fig. 2, Eq. (2·10) implies that \( |a_M d\theta_M| M_M \) are equal between the two corresponding terms of the same \( w \) point.

iv) The ghost prefactor \( G_I G_I \) is also equal (up to sign) between the two terms since \( G_I \) sits at the interaction point. The sign factors can be confirmed to be suitable ones for the cancellation.

§ 3. Why and how does the Jacobi identity break down?

In this section we shall examine the Jacobi identity on the basis of the expression (2·2). We adopt the gauge (2·8), and take the 4-th string with \( r = 4 \) as the string with
an infinitesimal string-length \( \alpha_4 = \epsilon \to 0 \).

First, we need some formulae concerning the Mandelstam mapping. The 4-string configuration has two interaction points \( z_\pm \) on the \( z \)-plane:

\[
\frac{d\rho(z)}{dz} = 0 \Rightarrow z = z_\pm = \frac{(\alpha_1 + \alpha_2)w + \alpha_3 + \epsilon \pm \Delta}{2(\alpha_1 + \alpha_2 + \epsilon)}
\]

with \( \Delta \) given by

\[
\Delta^2 = [\alpha_1(1-w) - \alpha_2 w]^2 + 2[\alpha_1(w-1) + \alpha_2 w] \epsilon + \epsilon^2.
\]

In the example of the breakdown of the Jacobi identity mentioned in § 1, the zero string-length limit was taken first in the vertex to which \( \Gamma \) is attached and then this vertex was connected to the other vertex. This procedure just corresponds to carrying out the naive Laurent expansion with respect to \( \epsilon \) (more precisely \( \epsilon/\alpha_1 \alpha_2 \)) in the present analysis. This expansion gives to order \( \epsilon^2 \)

\[
z_+ = \frac{\alpha_1}{\alpha_1 + \alpha_2} \left( 1 + \frac{\alpha_2}{(\alpha_1 + \alpha_2) A} \epsilon - \frac{\alpha_3 [A^2 + (\alpha_1 + \alpha_2)^2 w(1-w)] \epsilon^2}{(\alpha_1 + \alpha_2)^2 A^3} \right),
\]

\[
z_+ = w \left( 1 - \frac{1 - w}{A} \epsilon + \frac{(1-w)(A^2 + \alpha_1 \alpha_2) \epsilon^2}{(\alpha_1 + \alpha_2)^2 A^3} \right)
\]

with \( A = \alpha_1(1-w) - \alpha_2 w \) (we keep \( \alpha_1 \) and \( \alpha_2 \) fixed and express \( \alpha_3 \) as \( \alpha_3 = -\alpha_1 - \alpha_2 - \epsilon \)).

In the calculation of the various quantities in Eq. (2·2), the \( \rho \)-coordinates of the interaction points are necessary. The \( \epsilon \)-expansion of them yields

\[
\rho(z_+) = \alpha_3 \ln(-Z_3) + \mathcal{D}_o + \epsilon \ln(w_0 - w) - \frac{\alpha_1 \alpha_2}{2(\alpha_1 + \alpha_2) A^2} \epsilon^2,
\]

\[
\rho(z-) = \alpha_3 \ln(-Z_3) + \alpha_1 \ln w + \alpha_2 \ln(w - 1) + \epsilon \ln \left( \frac{w(1-w)}{eA} \right) - \frac{B}{2 A^2} \epsilon^2,
\]

where \( \mathcal{D}_o = \alpha_3 \ln((1-w)/(\alpha_1 + \alpha_2)) + \alpha_2 \ln(-\alpha_2/(\alpha_1 + \alpha_2)) \) and \( B = \alpha_1(1-w)^2 + \alpha_2 w^2 \). We also present the difference of \( \rho(z_+) \) for later use:

\[
\Delta \rho = \rho(z_-) - \rho(z_+)
\]

\[
= - \mathcal{D}_o + \alpha_1 \ln w + \alpha_2 \ln(w-1) + \epsilon \ln \left( \frac{(\alpha_1 + \alpha_2) w(1-w)}{eA^2} \right) - \frac{1}{2(\alpha_1 + \alpha_2)} \epsilon^2.
\]

Equations (3·3)~(3·5) already show the failure of the naive \( \epsilon \)-expansion in the neighborhood of the point \( w = w_0 = \alpha_3/(\alpha_1 + \alpha_2) \) where \( A = 0 \). Note that at \( w = w_0 \) the two interaction points \( z_\pm \) coincide to the lowest order in the \( \epsilon \)-expansion.

We are interested in the breakdown of the Jacobi identity when one of the four legs of the identity (see Fig. 1) is connected to a particular state \( \Gamma \) with vanishing string-length and momentum, i.e.,

\[
\int d^4x \langle \Gamma(4) | \text{Jacobi identity } (1, 2, 3, 4) \rangle_{-4}.
\]

The state \( \vert \Gamma(p, \alpha, \bar{z}_0) \rangle \) has \( N_{FP} = -2 \), anti-hermitian \( \Gamma^* = -\Gamma \), has zero-mode depen-
dence $\delta^0(\rho)\delta(a)\bar{c}_0$ and is subject to the constraint $\mathcal{D}\Gamma = \Gamma$. In order to treat a general $\Gamma$ having this property, we take as $\langle \Gamma \rangle$ of (3·6) the "generating state",

$$\langle j, \lambda, \bar{\lambda} \rangle = \langle 0 \rangle \exp \left( \sum_{n \geq 1} \sum_{a \geq 1} (j_n^{(a)} a_n^{(a)} + \bar{\lambda}_n^{(a)} c_n^{(a)} + \lambda_n^{(a)} \bar{c}_n^{(a)}) \right) \times \delta^0(\rho)\delta(a-e)\bar{c}_0,$$

and keep only those terms which contain one more $\lambda$ than $\bar{\lambda}$.

3.1. The measure factor $\mathcal{M}$

Since $(T, \theta_M)$ is related to $\Delta \rho = \rho(z) - \rho(z_\pm)$ by $\Delta \rho = T + i\alpha_M(\theta_M + \text{const})$, the measure factor $\mathcal{M}_M$ (2·10) in the gauge (2·8) is

$$\mathcal{M}_M = |Z_3|^4 \left| \frac{d \Delta \rho}{dw} \right|^{-2} \exp \left( -2 \sum_{s=1}^{4} \text{Re} \bar{M}_{ss} \right).$$

Substitution of Eq. (3·5) and the $\epsilon$-expansion formula of $\text{Re} \bar{M}_{ss}$, Eq. (B·16) in Appendix B, into Eq. (3·8) yields $\mathcal{M}$ on the boundary line $T = \text{Re} \Delta \rho = 0$:

$$\mathcal{M}_M |_{T = 0} = \left( \frac{\epsilon}{\epsilon} \right)^2 \exp \left\{ -2 \left( \frac{1}{a_1 + a_2} \right) T_0 \right\} \times \left\{ 1 + 2 \left[ \frac{1}{a_1} \ln|w| + \frac{1}{a_2} \ln|1-w| - \left( \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_1 + a_2} \right) \ln|w-w_0| \right] + \text{Re} \left( \frac{(a_1 + a_2)w^2 - 2a_2w - a_1}{2A^2} \right) - \left( \frac{1}{a_1 + a_2} \right) T_0 \right\} \epsilon + O(\epsilon^2)$$

with $T_0 = \text{Re} T_0 = a_1 \ln|a_1| + a_2 \ln|a_2| - (a_1 + a_2) \ln|a_1 + a_2|$.

Of course, we get the same equation (3·9) irrespectively of from which region we approach the boundary line $T = 0$. However, the singularity $A^{-2} \propto (w-w_0)^{-2}$ at $w = w_0$ in Eq. (3·9) is a non-integrable one, and it is one of the origins of the breakdown of the Jacobi identity. The meaning is as follows. Since this singular term in $\mathcal{M}$ is multiplied by $1/\epsilon$, it turns out that we have only to keep the $\epsilon^0$ term of $\Delta \rho$ (3·5) in carrying out the $\theta_M$ integration in (2·2). Note that $\Delta \rho |_{\epsilon = 0} = \mathcal{F}_0 + a_1 \ln w + a_2 \ln(w-1)$ is essentially the Mandelstam mapping of the 3-string configuration and hence the $T = \text{Re} \Delta \rho = 0$ line in the $w$-plane is now given by Fig. 3 (cf. Fig. 2). This $\Delta \rho |_{\epsilon = 0}$ is Taylor expanded with respect to $w$ around $w = w_0$ as

$$\Delta \rho |_{\epsilon = 0} = \frac{(a_1 + a_2)^2}{2a_1 a_2} (w-w_0)^2$$

$$+ O((w-w_0)^3),$$

and it is related to $\theta_M$ ($M = P, Q, R$) in Eq. (2·2) by

$$\Delta \rho |_{\epsilon = 0} = -\alpha_M \left[ \xi_M + i(\theta_M - \theta_M^{(M)}) \right].$$

Fig. 3. The same lines as in Fig. 2 for $\Delta \rho |_{\epsilon = 0}$. Here also the dotted line and the solid one are infinitesimally separated from each other. The two points $X$ and $Y$ (or $U$ and $V$) in Fig. 2 have coalesced at the point $w_0$.  

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where $\xi_M$ is negative, and $\xi_M \to 0$ on the $T=0$ boundary. From Eqs. (3·10) and (3·11), we have

$$\Re\left(\frac{(\alpha_1 + \alpha_2) w^2 - 2\alpha_1 w - \alpha_1}{A^2}\right) = -\frac{3}{2} \Re\left(\frac{1}{\Delta \rho}\right) + O\left(\frac{1}{\sqrt{\Delta \rho}}\right),$$

$$= \frac{3\pi}{2\alpha_M} \delta(\theta_M - \theta_0^{(M)}) + O\left(\frac{1}{\sqrt{\Delta \rho}}\right),$$

(3·12)

where use has been made of the formula $\Im(x + i\delta)^{-1} = -\pi \delta(x) (\delta \to +0)$.

The point is that this $\delta$-function term cannot cancel one another on the LHS of the Jacobi identity since the three integrations $a_M d\theta_M (M = P, Q, R)$ contribute with the same sign: $(3\pi/2) \times (2+1+1) = 6\pi$ (note that the $\theta$ integration has two "interaction points" $\theta_0^{(k)}$ while the $\theta_P$ and $\theta_R$ integrations have only one (see Fig. 3). On the other hand, the $O(1/\sqrt{\Delta \rho})$ terms cancel one another since they are integrable with respect to $a_M d\theta_M \sim -i\Delta \rho$.

For later use, we rewrite Eq. (3·9) in a simpler form:

$$M_M|_{\tau = 0} = \left[\mu(1, 2, 3)\right] \frac{\epsilon}{\epsilon} \left(1 - \frac{3\epsilon}{2} \Re\left(\frac{1}{\Delta \rho}\right)\right) + \cdots,$$

(3·13)

where $\alpha_z = -\alpha_1 - \alpha_2$ in $\mu(1, 2, 3)$, and the dots $\cdots$ denote the integrable $1/\epsilon$ term and the (in general non-integrable) higher order terms.

3.2. The ghost prefactor

The same kind of $\delta$-function singularity arises also from the ghost prefactor $G(z_+)G(z_-)$ multiplying the 4-string vertex (see Eq. (2·2)). Here, $G(z_k)$ represent the ghost pre-factor at the interaction points $z = z_k$ in the $z$-plane, that is, $G(z)$ is the ghost pre-factor of the 3-string vertex to which $<\Gamma(4)>$ is attached and $G(z_k)$ is that of the other 3-string vertex. Specifically, $G(z_+) = G_{126}$ and $G(z_-) = G_{346}$ in Eq. (2·2). The ordering is the same, $G(z_+)G(z_-)$, for $M = P, Q$ or $R$.

In order to analyze the singularity, we have to rewrite $G(z_k)$ in terms of the creation operators alone. Acting the ghost coordinate,

$$G(r)(\sigma_r) = i \sqrt{\pi} \alpha r \pi e^{r(r)}(\sigma_r) = \frac{\partial}{\partial \gamma_0^{(r)}} - \frac{i}{2} \sum_n \sum_{\pm} \frac{1}{n} \gamma_n^{(s)}(r) e^{\pm n \sigma_r},$$

(3·14)

on the $\exp_{EFP}|0\rangle$ part of the 4-string vertex $|V(1-4; w)\rangle$, we get the following expression in terms of the creation oscillators alone:

$$G(r)(\sigma_r) \Rightarrow -\frac{1}{2} \sum_{\pm} \sum_n \left( -\frac{1}{n} \delta^{\pm n} e^{\pm n \sigma_r} + \sum_{m > 0} M_{\pm n} e^{\pm m \sigma_r} \right) \iota_{\pm n}^{(s)}(\sigma_r).$$

(3·15)

Note that this replacement $\Rightarrow$ is not exact. It neglects the terms which arise by acting with $\partial/\partial \gamma_0^{(r)}$ of (3·14) on $\delta(\Sigma_r \gamma_0^{(r)})$ in the 4-string vertex (2·4). However, these terms turn out not to contribute to the Jacobi identity anomaly.

The creation oscillator expression (3·15) (with $\sigma_r = \sigma^{(r)}$) is still not very convenient to study the singularity at $w = w_0$ since the coefficient of $\gamma_n^{(s)}$ is an infinite summation. Fortunately, as shown in Appendix C, (3·15) at the interaction point
admits the following expression:

\[ G(z+)+G(z-)=\frac{1}{2} \sum_{n \geq 1} \sum_{r} J_n^{(\pm)(r)} i r_{-n}^{(\pm)(r)}, \]  (3.16)

\[ G(z+) \Rightarrow \frac{1}{2} \sum_{n \geq 1} \sum_{r} J_n^{(\pm)(r)} i r_{-n}^{(\pm)(r)}, \]  (3.17)

where the coefficient \( J_n^{(\pm)(r)} \) is given by

\[ J_n^{(\pm)(r)} = \int_{-1}^{1} \frac{dz}{\pi i} \frac{1}{z-z_n} e^{-n r(z)}. \]  (3.18)

Since \( G(z+)G(z-)=G(z_+)[G(z_+)+G(z_-)] \), Eqs. (3.16) and (3.17) are sufficient for our purpose.\(^*\)

The \( \delta \)-function singularity arises when \( \gamma^{(\pm)(4)} \) in (3.16) and (3.17) are contracted with \( \bar{c}_n^{(\pm)(4)} \) in the generating state (3.7). Using the \( \epsilon \)-expansion formulae (B.8) \(~(B.10)\) for \( \bar{M}_n^{(\pm)} \) and Eq. (C.9) for \( J_n^{(\pm)(r)} \), we have

\[ \bar{c}_n^{(\pm)(4)} (3.16) = \frac{1}{2} \epsilon \left[ g_n + h_n \left( 1 - \epsilon \left( \frac{1}{4 \Delta^2} \right)^{(\pm)} \right) \right], \]  (3.19)

\[ \bar{c}_n^{(\pm)(4)} (3.17) = -\frac{1}{2} \epsilon^2 h_n \left( \frac{1}{2 \Delta^2} \right)^{(\pm)}, \]  (3.20)

where \( h_n = (1/n)(\delta/\epsilon)^n \) and \( g_n = \sum_{m=1}^{n-1} h_{n-m} h_m \) (\( g_1 = 0 \)), and \( \Psi^{(\pm)} \) for any \( c \)-number \( \Psi \) is defined by \( \Psi^{(\pm)} = (\Psi^{(\mp)})^* = \Psi \). Note that one of the \( \epsilon \) factors multiplying the RHS of Eqs. (3.19) and (3.20) has come from \( i r_{-n}^{(\pm)(4)} = \epsilon c_n^{(\pm)} \). [In Eqs. (3.19) and (3.20), we have used the expansions \( B/2A^2 \sim (4 \Delta^2)^{-1} \) and \( w(w-1)/(w-w_0) \sim (2 \Delta^2)^{-1} \) around \( w = w_0 \).]

3.3. Jacobi identity anomaly

Let us collect the results of the previous subsections to obtain the Jacobi identity anomaly. For this purpose, we have to keep the \( 1/\epsilon \) term and the singular \( \Re(1/\Delta^2) \times \epsilon^0 \) term in the inner-product \( \langle j, \lambda \mid \bar{G}(z_\pm)G(z_-) \mid \bar{V}_m(1-4; w) \rangle \). From the \( \epsilon \)-expansion formulae of the Neumann coefficients and \( J_n^{(\pm)(r)} \) (3.18) given in Appendices B and C respectively, we can convince ourselves that these two kinds of terms have contributions only from

\[ \sum_{n \geq 1} \sum_{r} \bar{c}_n^{(\pm)(4)} (3.17) [G(z_\pm)+G(z_-)] + \bar{c}_n^{(\pm)(4)} G(z_\pm)(3.16) \]

\[ \times \langle j, \lambda \mid \bar{V}_m(1-4; w) \rangle, \]  (3.21)

where \( G(z_-) \) which survive the contraction in (3.21) now consist solely of \( \gamma^{(\pm)(r)} \) with \( r \neq 4 \). Then using Eq. (3.13) and the fact that we can put \( G(z_-) = G(z_+) \) for the first term of (3.21), we have

\(^*\) \( G(z_-) \) has the expression (3.17) with \( z_+ \) in (3.18) replaced by \( z_- \) (see Eq. (C.8)). However, it is not very convenient for estimating the singularity at \( w = w_0 \).
\[ \mathcal{M}_{10} \times \int d^4 \langle j, \lambda, \bar{\lambda} \mid G(z_+)G(z_-) \mid V_M(1-4; w) \rangle \\
= -\frac{\epsilon^2}{2} \sum_{n \geq 1} \left\{ (h_n + g_n) \frac{1}{\epsilon} \frac{3}{4} (h_n + 2g_n) \text{Re} \left( \frac{1}{\Delta \rho} \right) \right\} \bar{\lambda}_n^{(\pm)} G(z_+) \times \exp(K(j, \lambda, \bar{\lambda})) \mu(1, 2, 3) \exp\left( (E_X + E_{FP})(1, 2, 3) \right) |0\rangle_{123} \]
\times (2\pi)^{d+1} \delta\left( \sum_{r=1}^{3} p_r \right) \delta\left( \sum_{r=1}^{3} \bar{\rho}_0(\bar{r}) \right) \delta\left( \sum_{r=1}^{3} \alpha_r + \epsilon \right) + \text{(harmless terms)}, \quad (3.22)\]

where \( K(j, \lambda, \bar{\lambda}) \) is given by
\[ K(j, \lambda, \bar{\lambda}) = \sum_{n, m, n \geq 1} \frac{h_n h_m}{n + m} \left( \frac{1}{2} nmj_{n}^{(\pm)} j_{m}^{(\pm)} + n\lambda_{n}^{(\pm)} \bar{\lambda}_{m}^{(\pm)} \right), \quad (3.23)\]
and it originates from
\[ \int d^4 \langle j, \lambda, \bar{\lambda} \mid \exp\left( (E_X + E_{FP})(1-4) \right) |0\rangle_{1-4} |\rho_{4}=0 \rangle \\
= \exp(K(j, \lambda, \bar{\lambda})) \exp\left( (E_X + E_{FP})(1, 2, 3) \right) |0\rangle_{1, 2, 3} + \cdots, \quad (3.24)\]

where \( a_3 = a_1 - a_2 \) on the RHS, and the dots \( \cdots \) denote the integrable \( \epsilon \) term and (in general non-integrable) higher order terms.

The Jacobi identity anomaly is obtained by carrying out the \( \theta_M \) integration of (3.22) and summing over \( M = P, Q, R \):
\[ \int d^4 \langle j, \lambda, \bar{\lambda} \mid \text{Jacobi identity} \rangle_{1-4} \]
\[ = \sum_{M} a_M \int_{-\pi}^{\pi} \frac{d\theta_M}{2\pi} \times (3.22) \]
\[ = -\frac{\epsilon^2}{2} \sum_{n \geq 1} \left\{ (h_n + g_n) \frac{1}{\epsilon} \frac{3}{4} (h_n + 2g_n) \frac{4\pi}{2\pi} \right\} \bar{\lambda}_n^{(\pm)} \times \delta\left( \sum_{r=1}^{3} \alpha_r + \epsilon \right) \exp(K(j, \lambda, \bar{\lambda})) \mid V(1, 2, 3) \rangle / \delta\left( \sum_{r=1}^{3} \alpha_r \right) \]
\[ = -\frac{\epsilon^2}{2} \sum_{n \geq 1} \frac{1}{2} (h_n - g_n) \bar{\lambda}_n^{(\pm)} \exp(K(j, \lambda, \bar{\lambda})) \mid V(1, 2, 3) \rangle, \quad (3.25)\]

where use has been made of \( \text{Re}(1/\Delta \rho) = -(1/\alpha_M) \delta(\theta - \theta^{(M)}) \) (cf. the paragraph below Eq. (3.12)). The meaning of \( \mid V(1, 2, 3) \rangle / \delta(\sum \alpha_r) \) would be evident. Note that there is a contribution to the anomaly also from the \( \epsilon \) term of (3.22) since we have \( \sum_{M=1}^{M} = -2\epsilon \) (see Eq. (2.9)). One might think it strange that the \( \epsilon \) term in Eq. (3.22) cannot cancel completely in the Jacobi identity in spite of the fact that it is non-singular. To understand this matter, let us recall how the sign difference necessary for the cancellation in the Jacobi identity is supplied when \( a_3 = \epsilon \) is finite (see § IV of Ref. 1)). First, the cancellation between the \( P \) and \( Q \) integrations and that between \( P \) and \( R \) in Fig. 2 are owing to the sign difference of \( a_M \). Second, the cancellation between the \( Q \) and \( R \) integrations in Fig. 2(a) and that between \( P \) itself in Fig. 2(b)
are realized by the sign difference due to the fact that \( G(z_\pm) \) in one term corresponds to \( G(z_\pm) \) in the other. However, the latter type of cancellation is lost for the \( 1/\epsilon \) term in Eq. (3.22) since the present calculation is based on the \( \epsilon \)-expansion (3.3) which does not respect the double-valuedness of \( z_\pm \) (cf., the description at the end of § 4).

Now comparing Eq. (3.25) with Eq. (A.1) in Appendix A (which is the corrected version of Eq. (26) of Ref. 3), we see that Eq. (3.25) correctly reproduces the Jacobi identity anomalies Eqs. (1.4) and (A.3).

3.4. Another example

In the examples of the Jacobi identity anomaly discussed above and in Appendix A, the linear operators \( N_{P\rho} - a(\partial/\partial a) \) and \( U^{(\pm)} \) (see Appendix A) both contain \( a(\partial/\partial a) \) originating from \( a/\epsilon \) which existed in the operators but has disappeared after averaging over \( \epsilon = \pm 0.3 \). One might wonder whether the Jacobi identity breaks down whenever the corresponding linear operator contains \( a(\partial/\partial a) \) and hence is the next-to-leading order term in the \( \epsilon \)-expansion. In this subsection we present an example where this is not the case.

Consider \( \Lambda^{\mu
u} \) given by

\[
\Lambda^{\mu
u}(p, a, \bar{c}_0) = \bar{c}_0 \left( a e^{(\pm)} c^{(\mp)} + a e^{(-)} c^{(\pm)} \right) |0> \times \left( -\frac{\partial}{\partial p_\nu} \delta^0(p) \right) \frac{1}{2} \left[ \delta(a - 0) + \delta(a + 0) \right].
\]

This \( \Lambda^{\mu
u} \) is interpretable as a stringy local gauge transformation parameter which effects the coordinate transformation \( \delta x^\rho = \delta_{\mu} x^\nu \). The linear operator \( W^{\mu
u} \) corresponding to \( \Lambda^{\mu
u} \),

\[
\Lambda^{\mu
u} \Phi = W^{\mu\nu} \Phi,
\]

is given by

\[
W^{\mu\nu} = \mathcal{D} x^{\mu
u} + \frac{1}{2} \eta^{\mu\nu} \sum \chi U^{(\pm)},
\]

where

\[
\mathcal{D} x^{\mu\nu} = -i \int_{-\pi}^{\pi} d\sigma \left\{ X^\nu(\sigma), P^\mu(\sigma) \right\}
\]

\[
= \frac{1}{2} \left\{ p_\mu, \frac{\partial}{\partial p_\nu} \right\} + \frac{1}{2} \sum \chi \sum_{n=0}^{\infty} \frac{1}{n} \left( a_n^{(\pm)} a_n^{(\mp)} \right) + \left. \alpha n^{(\pm)} a_n^{(\pm)} \right|,
\]

and \( U^{(\pm)} \) is defined by Eq. (A.2).

This \( W^{\mu\nu} \) contains \( a(\partial/\partial a) \). Nonetheless, it annihilates the 3-string vertex,

\[
\sum_{r=1}^{3} W^{\mu\nu(r)} V(1, 2, 3) = 0,
\]

where we have used

\[
\sum_{r=1}^{3} \mathcal{D} x^{\mu\nu(r)} V(1, 2, 3) = \frac{1}{2} \eta^{\mu\nu} V,
\]
and Eq. (A·3). (A proof of Eq. (3·31) is found in Refs. 6 and 7.) Therefore, the Jacobi identity is not violated for $\Lambda^\mu$.

We may understand this example by deriving a formula like Eq. (3·25). There are two points that are different from the previous case. i) Since the $p$-dependence of $\Lambda^\mu(p, a, \bar{c}, \bar{d})$ is given by $(\partial/\partial p^\nu)\delta^\lambda(p)$, we have to take terms proportional to $p^\nu$, and ii) we have to keep only those terms which have odd powers of the source $j_n^{(\pm)}$. In order to fulfill these two requirements, we have to prepare from $|\text{Jacobi identity}>$ the term $\alpha_n^{(\pm)} p_+^\nu$ whose coefficient is of the form $a + b\text{Re}(1/\partial p)$. From the $\varepsilon$-expansion formula of the Neumann coefficients given in Appendix B, we see that the only candidate term comes from $(1/2)\sum_n \sum_{n \geq 1} \tilde{M}^{44}_n d_n^{(\pm)} p_4$ in $E_K$ (2·5) (cf. Eq. (B·8)). Namely, in the present case we have to multiply the RHS of Eq. (3·22) by $\sum_n j_n^{(\pm)} h_1[1-(\varepsilon/4)\text{Re}(1/\partial p)]$. Repeating the calculation in Eq. (3·25), the anomaly formula applicable to the case of $\Lambda^\mu$ reads

$$\int d^4 \langle j, \lambda, \bar{\lambda}; \nu | \text{Jacobi identity} >_{\pm=4}$$

$$= -e^2 \frac{3}{2} \left( \sum_{n \geq 1} \sum_{m \geq 1} g_n h_{m j_n^{(\pm)}} \right) \exp(K(j, \lambda, \bar{\lambda})) V(1, 2, 3)$$

where $\langle j, \lambda, \bar{\lambda}; \nu >$ is given by Eq. (3·7) except its $p$-dependence which is now $(\partial/\partial p^\nu)\delta^\lambda(p)$, and we have kept only those terms which contain odd number of $j_n^{(\pm)}$. The Jacobi identity anomaly for $\Lambda^\mu$ corresponds to the term $j_1^{(\pm)} \bar{\lambda}_1^{(-)} + j_1^{(-)} \bar{\lambda}_1^{(\pm)}$ in Eq. (3·32), which, however, is missing.

§ 4. Correct treatment near $w = w_0$

As seen from Eqs. (3·3)~(3·5), the radius of convergence of the naive $\varepsilon$-expansion is $|w - w_0| \approx 1/|\varepsilon|$, and the use of this naive $\varepsilon$-expansion around $w = w_0$ has resulted in the breakdown of the Jacobi identity. Then, what is the correct treatment of the region near $w = w_0$ when $|\varepsilon| |\alpha_1, 2| < 1$? It is as follows.

Let us consider the region $|w - w_0| < 1/|\varepsilon| |\alpha_1, 2|$ and introduce a new variable $\nu(= O(\varepsilon^0))$ by

$$A \equiv (a_1 + a_2)(w_0 - w) = -2 \left( \frac{a_1 a_2 \varepsilon}{a_1 + a_2} \right)^{1/2} \nu .$$

Then $z_\pm$ may be expanded as a series in $\sqrt{\varepsilon}$ which is valid in a region including $\nu = 0$ ($w = w_0$):

$$z_\pm = w_0 \left[ 1 + \sqrt{s} \left( v \pm \sqrt{1 + \nu^2} \right) \xi - \frac{1}{2} (1 - s) \left( 1 - \frac{\nu}{\sqrt{1 + \nu^2}} \right) \xi^2 + O(\xi^3) \right] ,$$

where we have defined $s = a_2/a_1$ and $\xi = \sqrt{\varepsilon} / (a_1 + a_2)$. The difference of the $p$-coordinates of the two interaction points $z_\pm$ is calculated from Eq. (4·2) to be given by

$$\Delta p_\equiv p(z_+) - p(z_-) = \varepsilon \left( \ln \left( \frac{v + \sqrt{1 + \nu^2}}{v - \sqrt{1 + \nu^2}} \right) + 2v \sqrt{1 + \nu^2} \right) + O(\xi^3) .$$
In Fig. 4, we depict the $T=\text{Re}\Delta\rho = 0$ line on the $v$-plane for $\Delta\rho$ given by Eq. (4·3). The measure factor $M$ (3·8) in this approximation is

$$M = \frac{e}{4\epsilon^2} \frac{1}{|1 + v^2|} \left| \frac{v + \sqrt{1 + v^2}}{v - \sqrt{1 + v^2}} \right| \exp \left\{ -2 \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_1 + \alpha_2} \right) T_0 \right.$$

$$+ 2\text{Re}(v\sqrt{1 + v^2 - v^2}) + O(\epsilon^{1/2}) \}.$$

(4·4)

Note that (4·4) coincides with (3·9) to the lowest order in $\epsilon$ when $|v| \to \infty$.

One might be afraid that $M$ (4·4) is singular at the points $v = \pm i$ (where the two interaction points coincide) and that this leads to the Jacobi identity anomaly. In order to show that this is not the case, we express $v$ on the $T=0$ line as $v = i\cos \phi$ (although this parametrization covers only a part of the $T=0$ line, i.e., the line $\text{Re}v = 0$ ($|\text{Im}v| \leq 1$), the result is the same even when we approach $v = i$ along the other two $T=0$ lines). Then we have

$$M = -\frac{e}{4\epsilon^2} \frac{1}{\sin^2 \phi} \exp(2\cos^2 \phi) \sim -\frac{e}{4\epsilon^2} \frac{1}{\phi^2}$$

near $\phi = 0$ ($v = i$). On the other hand, $\Delta\rho$ (4·3) is expressed in terms of $\phi$ as

$$\Delta\rho = i\epsilon(\sin 2\phi - 2\phi) \sim -\frac{4i\epsilon}{3} \phi^3,$$

(4·6)

and hence $\theta_M \sim (\epsilon/\alpha_M)(4/3)\phi^3$. Each term of the Jacobi identity is given by the integration $\int d\theta_M M(\cdots) \sim \int d(\phi^3) (\phi^3)^2 (\cdots)$, which is non-singular at $\phi = 0$ ($v = i$).* Moreover the cancellation along the $\text{Re}v = 0$ line connecting $v = \pm i$ in Fig. 4 is correctly realized since we have $G(z_1)_{\nu=\nu+0} = G(z_1)_{\nu=\nu-0}$ ($-1 < y < 1$) due to the double-valuedness of $\sqrt{1 + v^2}$ in Eq. (4·2) (cf. the description below (3·25)). Therefore, there is no fear of the Jacobi identity anomaly in the region $|w - w_0| \leq \sqrt{|\epsilon/\alpha_{1,2}|}$.

In summary we have to use two different kinds of approximations in discussing the Jacobi identity when $|\epsilon/\alpha_{1,2}| \ll 1$: the naive $\epsilon$-expansion of § 3 in the region $|w - w_0| \geq \sqrt{|\epsilon/\alpha_{1,2}|}$, and the approximation of this section in the other region $|w - w_0| \leq \sqrt{|\epsilon/\alpha_{1,2}|}$.

§ 5. Summary and discussion

In this paper we have traced the origin of the Jacobi identity anomaly which arises in relation to pre-geometrical SFT. The anomaly is due to the naive Laurent expansion with respect to $\epsilon/\alpha_{1,2}$, which starts with $(\epsilon/\alpha_{1,2})^{-1}$ and diverges when the interaction points of the two 3-string vertices coincide. We have also presented a

*) Neumann coefficients $M_{\alpha_M}$ are also non-singular at $v = \pm i$. 

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way of correctly treating the Jacobi identity when $|e/a| \ll 1$.

Fortunately, the Jacobi identity anomaly discussed in this paper does not do direct harm to the classical solution $\Psi_0 = -(1/2)Q_\alpha \Gamma$ of pre-geometrical SFT since this $\Psi_0$ is free from anomaly. However, every string functional generated from $|j, \lambda, \lambda\rangle$ (3.7) can be a parameter $\Lambda$ of the gauge transformation $\delta \Phi = (Q_\alpha \Lambda + )2 \Phi \Lambda$ in closed SFT (the $Q_\alpha \Lambda$ term is missing in pre-geometrical SFT), and the Jacobi identity anomaly means that the gauge invariance is violated if we first take the limit $a \to 0$ of $\Lambda$ in $\Phi \Lambda$ and then calculate the variation of the SFT action. [Of course, gauge invariance holds if we calculate the variation of the action before taking the limit $a \to 0$ for $\Lambda$.] It is desirable to “improve” the expression of the linear operators such as $N_{FP} - \alpha(\partial / \partial a)$ and $U^{(\pm)}$ to automatically incorporate the treatment of § 4. [This seems of course very difficult since the linear operators have to know the presence of another 3-string vertex before they operate on it.]

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Appendix A

— A Formula for the *-Product —

In this Appendix, we give the master formula for obtaining $\Gamma$ which satisfies Eq. (1.3). This formula was erroneously presented in Ref. 3) as Eq. (26).

Let $|j, \lambda, \lambda\rangle$ be given by Eq. (3.7). Then the master formula reads

$$
\frac{1}{2} \left( \lim_{\varepsilon \to 0} \lim_{\varepsilon \to 0} \right) \int d1, |j, \lambda, \lambda\rangle V(1, 2, 3)\rangle
$$

$$
= \frac{\theta^2}{2} \mathcal{P}_2 \mathcal{P}_3 \sum_{n \geq 2} g_{n} A_{n}^{(2)} \left( N_{2}^{(3)} - a_{n} \frac{\partial}{\partial a_{n}} \right) + \sum_{n \geq 1} h_{n} A_{n}^{(3)} U^{(\pm)\langle 3)} \right) |R(2, 3)\rangle \times \exp K(j, \lambda, \lambda) ,
$$

(A.1)

where the operator $U^{(\pm)}$ is

$$
U^{(\pm)} = -\frac{1}{2} \left\{ a, \frac{\partial}{\partial a} \right\} + \sum_{n \geq 0} c_{n}^{(\pm)} c_{n}^{(\mp)} - c_{n}^{(\pm)} c_{n}^{(\mp)} ,
$$

(A.2)

and $|R(2, 3)\rangle = (\bar{\phi}_{2}^{(2)} - \bar{\phi}_{2}^{(3)}) |r(2, 3)\rangle$ with $|r(2, 3)\rangle$ given by Eq. (27) of Ref. 3). On the RHS of Eq. (A.1) we have kept only those terms which contain one more $\lambda$ than $\lambda$. A desired $\Gamma$ which satisfies Eq. (1.3) is obtained by taking a linear combination of states which cancels the unwanted term $U^{(\pm)}$. Although the RHS of Eq. (1.3) differs from that of Eq. (20) in Ref. 3) by the term +1, the argument given in Ref. 3) applies straightforwardly to the corrected $\Gamma$.

Following the technique of Refs. 6) and 7), we can show that $U^{(\pm)}$ satisfies

$$
\sum_{r=1}^{3} U^{(\pm)\langle r} |V(1, 2, 3)\rangle = -\frac{1}{2} \sum_{r} \left\{ a_{r}, \frac{\partial}{\partial a_{r}} \right\} |V\rangle = -\frac{1}{2} |V\rangle .
$$

(A.3)
Finally, let us check the consistency of the formula (A·1). The operator $\mathcal{O}$ which appear on the RHS of Eq. (A·1) should share the property

$$\left(\mathcal{O}^{(a)} + \mathcal{O}^{(b)}\right)|R(2, 3)\rangle = 0,$$

(A·4)

since i) $|V(1, 2, 3)\rangle = |V(1, 3, 2)\rangle$ and hence $\mathcal{O}^{(a)}|R(2, 3)\rangle = \mathcal{O}^{(b)}|R(3, 2)\rangle$, and ii) $|R(3, 2)\rangle = -|R(2, 3)\rangle$. We can easily see that the two operators $N_{FP} - a(\partial/\partial a)$ and $U^{(a)}$ in fact satisfy Eq. (A·4). However, neither $N_{FP} + 1 - a(\partial/\partial a)$ in Ref. 3) nor $N_{FP} + 1 - \text{sgn}(a) - a(\partial/\partial a)$ in Ref. 8) satisfies Eq. (A·4).

---

**Appendix B**

---

**Neumann Coefficients**

The Neumann coefficients $\tilde{M}^{rs}_{nm}$ are defined by the following expansion:

$$\ln(z - \bar{z}) = -\delta_{rs}\left\{\theta(\xi - \bar{\xi})\sum_{n\geq 1} \frac{1}{n} e^{n(\xi - \bar{\xi})} - \xi\right\} + \theta(\bar{\xi} - \xi)\left\{\sum_{n\geq 1} \frac{1}{n} e^{n(\bar{\xi} - \xi)} - \bar{\xi}\right\} + \sum_{n, m \geq 0} \tilde{M}^{rs}_{nm} e^{n\rho + m\bar{\rho}}.$$

(B·1)

In Eq. (B·1), the variable $z$ and another variable $\xi = \xi + i\sigma$ ($\xi < 0, -\pi \leq \sigma \leq \pi$) defined on the strip of the $r$-th string are related through the Mandelstam mapping Eq. (2·7) and

$$\rho(z) = \alpha_0\xi + \rho_0^{(r)}$$

(B·2)

where $\rho_0^{(r)} = \alpha_0^{(r)} + i\beta_0$ is determined by the condition that $(\xi, \sigma) = (0, \sigma^{(r)})$ at the interaction point $z = z_0^{(r)}$ of the $r$-th string satisfying $(dp/dz)(z_0^{(r)}) = 0$, viz.,

$$\rho(z_0^{(r)}) = \rho_0^{(r)} + i\alpha_0\sigma^{(r)}.$$

(B·3)

We list the explicit expression of $\tilde{M}^{rs}_{nm}$ derived from Eq. (B·1):

$$\tilde{M}^{rs}_{nm} = \begin{cases} \ln(Z_r - Z_s), & (r \neq s) \\ -\frac{B}{2\pi} \int \frac{dz}{Z_r - Z_s} + \frac{1}{\alpha_0} \rho_0^{(r)}, & (r = s) \end{cases}$$

(B·4)

$$\tilde{M}^{rs}_{nm} = \tilde{M}^{rs}_{nm} = \frac{1}{n} \int_{z_r}^{z_s} 2\pi i \frac{1}{z - Z_s} e^{-n\xi/(2z)},$$

(B·5)

$$\tilde{M}^{rs}_{nm} = \frac{\alpha_r}{n} + \frac{\alpha_s}{m} \left(\sum_{l=1}^{4} \alpha_l \tilde{M}^{rs}_{nl} \tilde{M}^{ns}_{ll}\right)^{-1} \sum_{l=1}^{4} \alpha_l \tilde{M}^{rs}_{nl} \tilde{M}^{ns}_{ll},$$

(B·6)

where $\xi(z) = (1/\alpha_0) \left(\sum_{l=1}^{4} \alpha_l \ln(z - Z_s) - \rho_0^{(r)}\right)$. In particular, the $\epsilon$-expansion of $\xi(z)$ is given by

$$\xi(z) = \frac{1}{\epsilon} \left\{a_1\ln\left(\frac{z}{w}\right) + \alpha_2\ln\left(\frac{z - 1}{w - 1}\right) + \sigma_3^{(4)}\right\} + \ln(z - w) - \ln\left(\frac{w(w - 1)\epsilon}{eA}\right) + \frac{B}{2A} \epsilon + O(\epsilon^2).$$

(B·7)
Using these formulae we can calculate the $\varepsilon$-expansion of $\tilde{M}^{rs}_{nm}$ used in the text. In the following, i) we omit the phase factor $\exp(-in\theta_r - im\theta_s)$ with $\theta_r = \sigma^{(r)} + \text{const}$ multiplying $\tilde{M}^{rs}_{nm}$ since it can be absorbed into the projector $\mathcal{P}$ in the vertex, and ii) the indices $n$, $m$ and $r$, $s$ of the Neumann coefficients should be understood to be in the range $n, m \geq 1$ and $r, s = 1, 2, 3$ unless written explicitly.

First, we give the three with explicit expressions:

$$
\tilde{M}^{44}_{00} = \frac{h_n}{n} \left( 1 - \frac{B}{2A^2} \varepsilon + O(\varepsilon^2) \right),
$$

$$
\tilde{M}^{44}_{00} = \frac{h_n}{n} F_r(w) \varepsilon \left( 1 + \left[ \left( \frac{2}{n} - 3 \right) \frac{B}{2A^2} - \frac{n-1}{n} F_r(w) \right] \varepsilon + O(\varepsilon^2) \right),
$$

$$
\tilde{M}^{44}_{nm} = \frac{h_nh_m}{n+m} + O(\varepsilon^2),
$$

where

$$
h_n = \frac{1}{n!} \left( \frac{n}{\varepsilon} \right)^n,
$$

$$
(F_1, F_2, F_3) = \frac{1}{A} (w-1, w, 0).
$$

Next, we present other three:

$$
\tilde{M}^{rs}_{nm} = \tilde{N}^{rs}_{nm} + [\varepsilon] + O(\varepsilon^2), \quad (n, m \geq 0, n + m \neq 0)
$$

$$
\tilde{M}^{rs}_{nm} = [\varepsilon] + O(\varepsilon^2),
$$

$$
\tilde{M}^{rs}_{00} = [\varepsilon^3] + [\varepsilon] + O(\varepsilon^6),
$$

where i) $\tilde{N}^{rs}_{nm}$ in Eq. (B.13) is the 3-string Neumann coefficient in the gauge $(Z_1, Z_2, Z_3) = (0, 1, \infty)$ (see (3.11) of Ref. 9)) with $\alpha_0 = -\alpha_1 - \alpha_2$, and ii) $[\varepsilon^n]$ represents a term which is proportional to $\varepsilon^n$ and does not contain the $(w - w_0)^{-2}$ singularity (on the other hand, the $O(\varepsilon^n)$ part contains in general the $(w - w_0)^{-2}$ singularity).

In $E_x (2-5)$, $\tilde{M}^{rs}_{00}$ ($r, s = 1-4$) appears only through its real part and it is multiplied by a conserved quantity $\alpha_0^{(k)}(r) = p_r/2$. Defining $\tilde{M}^{rs}_{00} = \text{Re} \tilde{M}^{rs}_{00} - (\delta_{r3} + \delta_{s3}) \ln|Z_3|$, for which $\sum_{r,s} \tilde{M}^{rs}_{00} p_r \cdot p_s = \sum_{r} \text{Re} \tilde{M}^{rs}_{00} p_r \cdot p_s$, we have

$$
\tilde{M}^{rs}_{00} = \begin{cases} 
0, & (r \neq s) \\
\frac{1}{\alpha_r} T_0 + \frac{\varepsilon}{\alpha_r} \left( \ln |w-w_0| - G_r(w) \right) \\
-\frac{\alpha_1 \alpha_2 \varepsilon^2}{2(\alpha_1 + \alpha_2) \alpha_r} \text{Re} \left( \frac{1}{A^2} \right) + O(\varepsilon^3), & (r = s)
\end{cases}
$$

$$
\tilde{M}^{r4}_{00} = G_r(w),
$$

$$
\tilde{M}^{44}_{00} = \ln \left| \frac{w(w-1)\varepsilon}{\varepsilon A} \right| - \text{Re} \left( \frac{B}{2A^2} \right) \varepsilon + O(\varepsilon^2),
$$

where $\alpha_3 = -(\alpha_1 + \alpha_2 + \varepsilon)$ in Eq. (B.16), and $G_r(w)$ is defined by
Equation (B·16) is valid when $T=0$. The $e^0$ part of Eq. (B·16) is related to $\bar{N}^{sf}_{oo}$ of the 3-string vertex, Eq. (3·11) of Ref. 9), through $\bar{M}^{sf}_{oo}|_{e=0} = \bar{N}^{sf}_{oo} - (\delta_{s3} + \delta_{s3}) T_0/\alpha_s$ and hence is essentially the same one.

**Appendix C**

--- Formulae for the Ghost Prefactor ---

In this appendix, we prove two formulae for obtaining Eqs. (3·16) and (3·17), and present the $\epsilon$-expansion of $J_n^{(\pm)(1)}$ (3·18). First, we shall derive the formula for Eq. (3·16).

i) Taking the limit $\tilde{z} \to z$ ($\xi_s < \tilde{\xi}_s < 0$) in Eq. (B·1) with $s=r$, we get

$$-\ln\left( -\frac{1}{\alpha_r} \frac{d\rho(z)}{dz} \right) = \sum_{n,m \geq 0} \bar{M}^{rr}_{nm} e^{(n+m)\xi(z)} + \xi_r(z) - i\pi. \quad \text{(C·1)}$$

Since $d\rho(z)/dz = \text{const} \times (z - z_t) / (z - z_t)$, Eq. (C·1) is rewritten as

$$\sum_{n} \ln(z - z_t) - \sum_{\tilde{z}} \ln(z - \tilde{z}) = \sum_{n,m \geq 0} \bar{M}^{rr}_{nm} e^{(n+m)\xi(z)} + \xi_r(z) + \text{const.} \quad \text{(C·2)}$$

ii) Putting $\tilde{z} \to z_0^{(s)}$ ($\xi_s < \tilde{\xi}_s < 0$) in Eq. (B·1), we get

$$\ln(z - z_0^{(s)}) = -\delta_{rs} \left( \sum_{n \geq 1} \frac{1}{n} e^{n\xi(z)} - i\sigma_1^{(s)} + i\pi \right) + \sum_{n,m \geq 0} \bar{M}^{rr}_{nm} e^{n\xi + i\sigma_1^{(s)}}. \quad \text{(C·3)}$$

iii) Taking the limit $\tilde{z} \to Z_s$ ($\tilde{\xi}_s \to -\infty$) in Eq. (B·1) and summing over $s=1-4$, we get

$$\sum_{t} \ln(z - Z_t) - \xi_r(z) = \sum_{n,m \geq 0} \bar{M}^{rt}_{nm} e^{n\xi(z)} . \quad \text{(C·4)}$$

Let $s_\pm$ be any two strings whose interaction point is $z_\pm$, respectively (in the region $P$, for example, $s_+ = 1$ or 2, and $s_- = 3$ or 4). Then, since $\sum_{n} \ln(z - z_\pm) = \sum_{s=s_+} \ln(z - z_0^{(s)})$, Eqs. (C·2)~(C·4) give

$$\sum_{s=s_+} \left( -\delta_{rs} \sum_{n \geq 1} \frac{1}{n} e^{n\xi(z)} - i\sigma_1(z) + i\pi \right) + \sum_{n,m \geq 0} \bar{M}^{rr}_{nm} e^{n\xi + i\sigma_1^{(s)}}$$

$$= \sum_{t} \sum_{n,m \geq 0} \bar{M}^{rt}_{nm} e^{n\xi(z)} - \sum_{n,m \geq 0} \bar{M}^{rt}_{nm} e^{(n+m)\xi(z)} + \text{const.} \quad \text{(C·5)}$$

Comparing the coefficient of $e^{n\xi}$ ($n \geq 1$), we finally obtain the formula for Eq. (3·16):

$$\sum_{s-s_+} \left( -\delta_{rs} \sum_{n \geq 1} \frac{1}{n} e^{n\xi(z)} + \sum_{m \geq 0} \bar{M}^{r}_{nm} e^{m\xi(z)} \right) = -\delta_{r3} \frac{n}{2}\pi i \int_{e^{-\eta}}^{e^{\eta}} \frac{dz}{2\pi i} e^{-(n+1)\xi(z)} \times (C·3) = -\frac{n}{2}\pi i \int_{e^{-\eta}}^{e^{\eta}} \frac{dz}{2\pi i} e^{-n\xi(z)} \times (C·3) , \quad \text{(C·6)}$$

The other formula for Eq. (3·17) is derived from

$$\int_{e^{-\eta}}^{e^{\eta}} \frac{d(e^{\xi(z)})}{2\pi i} e^{-(n+1)\xi(z)} \times (C·3) = -\frac{1}{n} \int_{e^{-\eta}}^{e^{\eta}} \frac{dz}{2\pi i} \left( \frac{dz}{dz} e^{-n\xi(z)} \right) \times (C·3) , \quad \text{(C·7)}$$

through integration by parts.
\[
- \delta_{rs} \frac{1}{n} e^{-in\sigma_{13}} + \sum_{m=0}^{\infty} \tilde{M}_{nm} e^{in\sigma_{13}} = \frac{1}{n} \int_{2\pi i} \frac{dz}{z - z_0} e^{-n\tau(z)}. \tag{C\cdot8}
\]

Finally we present the \(\epsilon\)-expansion of \(J_n^{(+)r}(T)\) (3\cdot18).

\[
J_n^{(+)r} = \frac{h_n}{n} \frac{w(w-1)}{(w-w_0)A} \epsilon + O(\epsilon^2), \tag{C\cdot9}
\]

\[
J_n^{(+)r} = \sum_{z=1}^{3} \tilde{N}_{n0}^{rs} - \sum_{m=0}^{n} \tilde{N}_{n-m,m}^{rr} + [\epsilon] + O(\epsilon^2), \quad (r = 1, 2, 3) \tag{C\cdot10}
\]

where the \([\epsilon]\) term in Eq. (C\cdot10) contains only integrable singularities, \(A^{-1}\) and \(\ln A\).

References