# Shape Oscillation of an Encapsulated Bubble in Electric Fields

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## Abstract

We investigate the dynamics of an encapsulated bubble in electric fields theoretically, based on the leaky dielectric model. On the bubble surface, the applied electric field generates a Maxwell stress, in addition to hydrodynamic traction and membrane mechanical stress. Our model also includes the effect of interfacial charge due to the jump of the current and the stretching of the interface. We focus on the axisymmetric deformation of the encapsulated bubble induced by the electric field and carry out our analysis using the Legendre polynomials. In our first example, the encapsulating membrane is modelled as an incompressible interface with bending rigidity. Under the steady uniform electric field, the encapsulated bubble resumes an elongated equilibrium shape, dominated by the second and fourth order shape modes. The deformed shape agrees well with experimental observations reported in the literature. For our second example, we considered a bubble encapsulated with a hyperelastic membrane with bending rigidity, subject to an oscillatory electric field. We show that the bubble can modulate its oscillating frequency and reach a stable shape oscillation at an appreciable amplitude.

**Keywords**: encapsulated bubble; electric field; shape oscillation

## Introduction

Encapsulated bubbles are widely used in various biomedical applications. It is important to understand the dynamics of encapsulated bubbles subject to applied physical forces and a key set of parameters are their natural frequencies. The natural frequencies of shape modes are crucial for controlling membrane stability. An approach capable of controlling the shape oscillation and obtaining the natural frequencies of shape mode directly is needed. In this work, we show that applying an oscillatory electric field offers a noninvasive way for estimating natural frequencies of shape modes when an encapsulated bubble undergoes oscillations.

## Problem formulation

We consider an encapsulated bubble suspended between two conducting plates parallel to each other. Applying temporally constant or varying voltage yields steady or oscillatory electric field. The flow field and electric field are assumed to be axisymmetric with the symmetric axis along the direction of the applied electric field $z$ and through the center of the bubble.

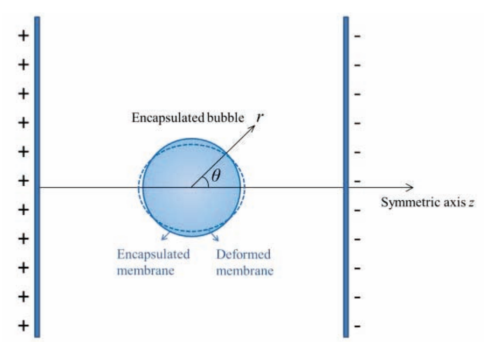


Figure 1. The coordinate of the encapsulated bubble in a spatially uniform electric field.

We establish a polar coordinate system based at the center of the bubble and consider a perturbation to a spherical shape. The positions of the material points at the bubble interface are thereby determined by the radial and zenith directions and expanded in terms of the (associated) Legendre polynomials as:

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$$r(\theta,t) = R(t) + \sum_{k=2}^{\infty} a_k(t) P_k(\cos\theta), \quad \Theta(\theta,t) = \theta + \frac{1}{R(t)} \sum_{k=1}^{\infty} b_k(t) P_k^1(\cos\theta). \tag{1}$$

We assume that the flow field inside the bubble is negligible and consider only the flow field outside the bubble by the incompressible Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0, \quad \rho \frac{\partial \mathbf{u}}{\partial t} + \rho[(\mathbf{u} - w\mathbf{e}_z) \cdot \nabla]\mathbf{u} = -\nabla p + \mu \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}), \tag{2}$$

where $\mathbf{u}$ and $p$ are the velocity and pressure, respectively, and $w$ is the translation velocity, $\rho$ and $\mu$ are the liquid density and viscosity, respectively. The velocity and pressure are decomposed into potential part (with subscript $p$) and viscous pressure correction (with subscript $v$) [1]

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_v, \qquad p = p_p + p_v. \tag{3}$$

We assume that there is no free charge in the bulk, and it only exists on the bubble interface. According to Gauss's law and Faraday's law without induction, the external and internal electric fields $\mathbf{E}_{\mathrm{ex}}$ and $\mathbf{E}_{\mathrm{in}}$ are irrotational and solenoidal. Therefore, the electric fields can be expressed by the electric potentials as

$$\mathbf{E}_{ex} = -\nabla \psi_{ex}, \qquad \mathbf{E}_{in} = -\nabla \psi_{in}, \tag{4}$$

and the electric potentials $\psi_{\mathrm{ex}}$ and $\psi_{\mathrm{in}}$ satisfy the Laplace equations:

$$\nabla^2 \psi_{ex} = 0, \quad \nabla^2 \psi_{in} = 0. \tag{5}$$

Due to the spatially uniform distribution of electric field as $r \to \infty$ and the requirement of regularity at $r = 0$, the solutions to the Laplace equations have the forms of

$$\psi_{ex}(r,\theta,t) = \sum_{k=0}^{\infty} \frac{A_k(t)}{r^{k+1}} P_k(\cos\theta), \qquad \psi_{in}(r,\theta,t) = \sum_{k=0}^{\infty} B_k(t) r^k P_k(\cos\theta), \tag{6}$$

where the coefficients $A_k$ and $B_k$ are determined by the boundary conditions [2,3], including the continuity of the tangential components of the electric fields

$$\mathbf{t} \cdot \nabla \psi_{ex} = \mathbf{t} \cdot \nabla \psi_{in}, \tag{7}$$

and the jump condition for the normal components of the electric displacement vectors

$$-\varepsilon_{ex} \mathbf{n} \cdot \nabla \psi_{ex} + \varepsilon_{in} \mathbf{n} \cdot \nabla \psi_{in} = q / \varepsilon_0, \tag{8}$$

where $\mathbf{t}$ and $\mathbf{n}$ are the unit tangential and normal vectors, respectively. $\varepsilon_0$ is the permittivity of vacuum, $\varepsilon_{\mathrm{ex}}$ and $\varepsilon_{\mathrm{in}}$ are the relative permittivity of the liquid and gas phase, respectively. $q$ is the interfacial charge density, satisfying the transport equation

$$-\kappa_{ex} \mathbf{n} \cdot \nabla \psi_{ex} + \kappa_{in} \mathbf{n} \cdot \nabla \psi_{in} = \frac{\mathrm{d}q}{\mathrm{d}t} + q \nabla_s \cdot \mathbf{u}. \tag{9}$$

The stress balance at the interface includes the pressure difference, the jump of the viscous and the Maxwell stresses, and the membrane stress (for encapsulated bubbles only) or surface tension (for gas bubbles only), in the normal and tangential directions

$$-p_{ex} + \mathbf{n} \cdot \mathbf{T}_{ex} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{M}_{ex} \cdot \mathbf{n} = -p_{in} + \mathbf{n} \cdot \mathbf{M}_{in} \cdot \mathbf{n} + F_n, \tag{10}$$

$$\mathbf{n} \cdot \mathbf{T}_{ex} \cdot \mathbf{t} + \mathbf{n} \cdot \mathbf{M}_{ex} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{M}_{in} \cdot \mathbf{t} + F_t, \tag{11}$$

where $p_{ex}$ and $p_{in}$ are the pressure outside the inside the bubble, respectively. $\mathbf{T}_{ex} = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the viscous stress tensor at the liquid side, while the viscous stress in the bubble is neglected due to the small viscosity of gas. $\mathbf{M}$ (the subscript is omitted temporarily for brevity) is the Maxwell stress

$$\mathbf{M} = \varepsilon \varepsilon_0 \left( \mathbf{EE} - \frac{1}{2} |\mathbf{E}|^2 \mathbf{I} \right). \tag{12}$$

The membrane stress is calculated by [4,5]

$$\mathbf{F} = F_n \mathbf{e}_n + F_t \mathbf{e}_t = -(\mathbf{P} \cdot \nabla) \cdot (\boldsymbol{\tau} + \mathbf{Q}\mathbf{n}), \tag{13}$$

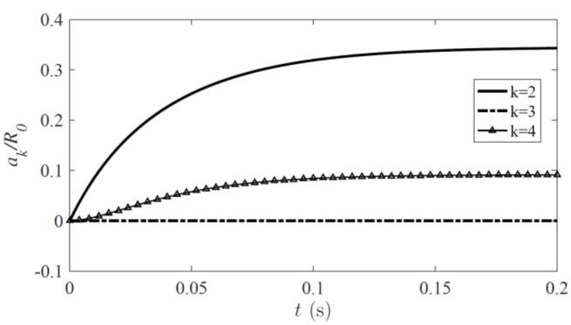
where $\mathbf{P}$ is the tangential projection operator. $\boldsymbol{\tau}$ is the in-plane stress. $\mathbf{Q}$ is the transverse shear tension, associated with the bending moment as

$$\mathbf{Q} = [(\mathbf{P} \cdot \nabla) \cdot \mathbf{m}] \cdot \mathbf{P}. \tag{14}$$

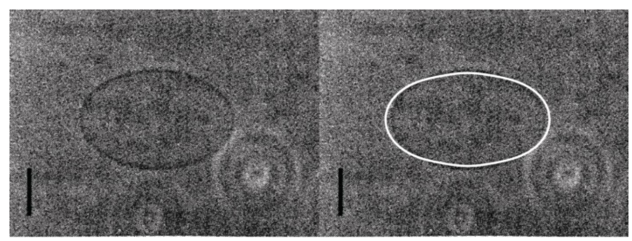
We adopt the neo-Hookean law to connect the in-plane stress and strain, the Love law to connect the bending moment and curvature. Expanding (10) and (11) in terms of the Legendre polynomials, we obtain the dynamic equations for radial oscillation, translation and shape oscillation. The derived equation is referred to [6].

**Results**

To validate our model about the coupling of electric field and the dynamics of an encapsulated bubble, we focus on the effects of electric stress and membrane stress on the bubble shape. Due to the lack of experimental results for an encapsulated bubble, we compare with the results by Kummrow& Helfrich [7] which are for a vesicle. The vesicle is coated with lipid bilayer and only bears bending rigidity. Accordingly, we modify our membrane model to exlclude hyperelastic stress and only consider the bending moments. The in-plane strain is prescribed to be 1, implying the membrane is tensionless. For this case, the compressibility of gas plays little role. Therefore, the results of a vesicle and an encapsulated bubble can be compared, although the inside of a vesicle is not gas and the compositions of encapsulating membranes are different. According to Kummrow & Helfrich[7], we choose a bubble with initial radius $R_0$ = 23 μm and bending modulus $G_b = 2.47\times10^{-20}$ Nm, and apply a steady electric field $E_0 = 10^4$ Vm$^{-1}$. The external fluid is assumed to be water, with density $\rho$ = 1000 kg m$^{-3}$, viscosity $\mu$ = 0.001 kg(m s)$^{-1}$, relative permittivities $\varepsilon_{\mathrm{ex}}$ = 81 and conductivities $\kappa_{\mathrm{ex}} = 5\times10^{-4}$ S m$^{-1}$. The internal fluid is considered as gas, with relative permittivities $\varepsilon_{\mathrm{in}}$ = 1 and conductivities $\kappa_{\mathrm{in}} = 3\times10^{-15}$ S m$^{-1}$. The ambient pressure is set as $p_\infty = 10^5$ Pa. Figure 2(a) suggests the second and fourth order shape modes develop under the electric field and become stable to be a new shape within 0.2 s. This time scale is close to Kummrow and Helfrich's result. Plotting the shape at the stable state based on the magnitudes of different shape modes and comparing with Kummrow and Helfrich's observation, we find that the shapes agree well with each other (Fig. 2(b)).




(a)

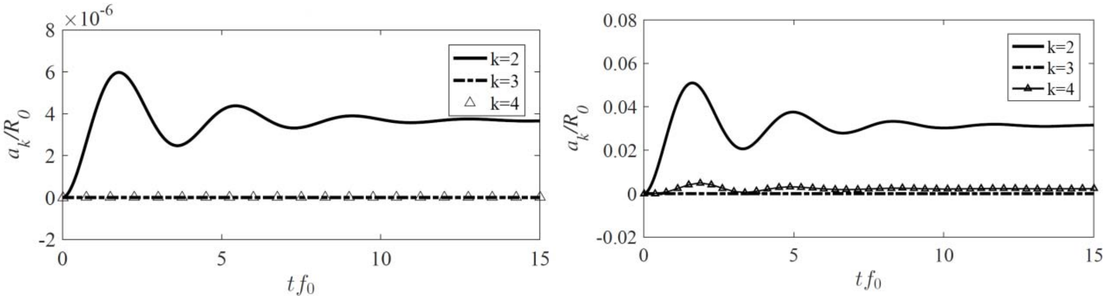


(b)

Figure 2. (a) Time evolutions of shape modes for $k$ = 2, 3 and 4 for an encapsulated bubble. (b) Comparison of the stable shape with the experimental observation by Kummrow & Helfrich[7]. The bar represents 20 μm.

For a general membrane with hyperelasticity and bending rigidity, we add the hyperelastic stress to the above bubble by setting elastic modulus $G_s$ = 0.15 N m$^{-1}$. The deformation of the bubble is much smaller than the bubble whose membrane only has bending rigidity (Fig. 3(a)). It is not easy to obtain the characteristic parameters such as elastic modulus, natural frequency, etc. using steady electric field. Even we increase the electric field, e.g. applying $E_0 = 10^6$ Vm$^{-1}$, the deformation at the stable state is an order of magnitude smaller than the bubble only with bending rigidity (Fig. 3(b)).

The results above suggest that the effect of a steady electric field on encapsulated bubble dynamics is small. To explore the effect of a temporally oscillatory electric field, we apply $E_0 = E_a\sin(2\pi f_e t)$, where $E_a$ is the amplitude of the oscillatory electric field and $f_e$ is the oscillatory frequency. Since the second order shape mode has a quadratic relation with the electric field[6], if we set the electric frequency equal to half of the natural frequency, the shape oscillation is resonant and thus the amplitude of the shape mode is enlarged. The natural frequency of shape modes for the encapsulated bubble is calculated to be $f_2$ = 45.2 kHz[4]. Accordingly, we set $f_e$ = 22.6 kHz and find that the second order shape mode oscillates continuously with an amplitude twofold larger than the stable magnitude under a steady electric field (Fig. 4).




(a) (b)

Figure 3. Time evolutions of shape modes under a steady electric field with $E_0 = 10^4$ Vm$^{-1}$ (a) and $E_0 = 10^6$ Vm$^{-1}$ (b).

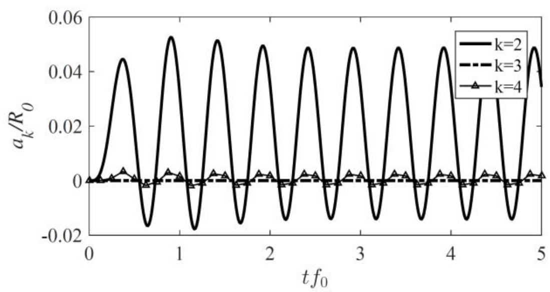
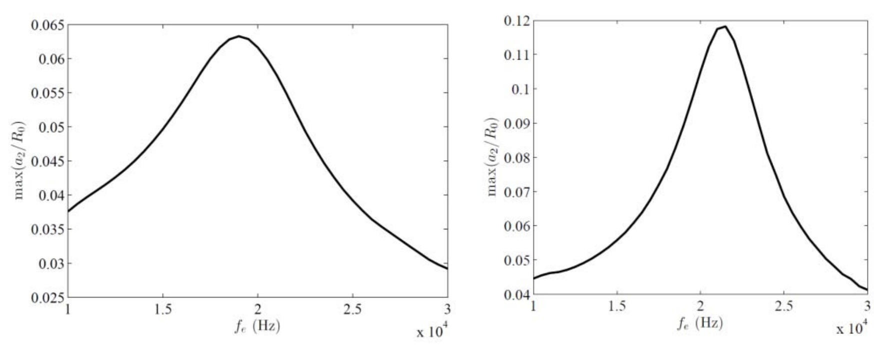



Figure 4. Time evolutions of shape modes under an oscillatory electric field.

To obtain the natural frequency of shape mode for an encapsulated bubble, we can modulate the electric frequency to induce resonance. As an example, we take a gas bubble whose natural frequency of shape mode is calculated to be 42.7 kHz and assign $F_n$ in (10) as the surface tension and $F_t$ in (11) to be zero for a shear-free condition. Sweeping the electric frequency and recording the maximal amplitude of the second order shape mode after the transient state, we find that the maximum of amplitude appears at around 21 kHz (Fig. 5(a)), which is half of its natural frequency. In the same way, we find that the maximum for the encapsulated bubble is at around 20 kHz (Fig. 5(b)), a little smaller than half of the natural frequency we calculated (22.6 kHz). The smaller value is attributed to the significant effect of viscosity for an encapsulated bubble, which makes the resonant frequency smaller than the natural frequency without considering the effect of viscosity. Nevertheless, frequency modulation of an oscillatory electric field is a feasible way to estimate the natural frequencies of shape modes. In addition, the maximum $a_2$ at resonance is more than twice of the stable $a_2$ under steady electric field, making it easier to detect bubble deformation.




(a) (b)

Figure 5. The maximal amplitudes of the second order shape mode under various electric frequencies for a gas bubble (a) and an encapsulated bubble (b).

**Conclusion**

In this paper, we have proposed a leaky dielectric model for investigating the dynamics of an encapsulated bubble in an electric field. The applied electric field is spatially uniform and temporally constant or oscillatory. Outside the bubble, we solve a potential flow field with a viscous pressure correction. The electric field and flow field are coupled through force balance on the bubble surface, including electric, hydrodynamic traction and membrane stresses. Our results for a bubble coated by a membrane with bending rigidity show that it undergoes elongated deformation subject to a steady electric field, whose shape agrees well with the observation by Kummrow & Helfrich[7] with the same parameters. For a membrane both with elasticity and bending rigidity, our results show that the deformation of an encapsulated bubble are not as visible under a steady applied electric field. By modulating the frequency of an oscillatory electric field, on the other hand, we can induce shape mode resonance. Since the deformation is significant, natural frequencies of the shape model can be estimated directly with less error.

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