DYNAMICS OF PARTICLES AND RIGID BODIES

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DYNAMICS OF PARTICLES AND RIGID BODIES

A SELF-LEARNING APPROACH

Mohammed F. Daqaq

Global Network Associate Professor
Division of Engineering
New York University, Abu Dhabi

Affiliate Associate Professor
Tandon School of Engineering, New York University, NY.

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To Farid, Hana, Shaima,

Rayyan and Laith . . .

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Preface

The first graduate courses on dynamical systems, often referred to as "Intermediate Dynamics", introduce students to the fundamentals necessary to model the motion of particles and rigid bodies. On top of what they learn in introductory undergraduate dynamics courses, intermediate dynamics courses typically introduce the concept of rotating frames to simplify the kinematic description of particles and rigid bodies. Using this concept, they extend the application of Newton's equations and the principles of impulse and momentum to describe three-dimensional motion of rigid bodies, eventually leading to the derivation of Euler's rotational equations. They also present many new concepts including, but not limited to, constraints, generalized coordinates, generalized forces, virtual displacement, the principle of virtual work, Hamilton's extended principle, Lagrange's equations, and Hamilton's canonical equations.

During my tenure at Clemson University, I taught the Intermediate Dynamics course numerous times, and through this I had the opportunity to interact with students hailing from different cultural and academic backgrounds. Some of those students had a very strong background in introductory dynamics, while unfortunately many others had not had the chance to properly learn the fundamentals of dynamics upon which the *Intermediate Dynamics* course was based. Many students in the latter group struggled silently. Due to cultural beliefs and, sometimes, the language barrier, they would nod their heads in agreement every time I asked whether they understood the material I had just explained in class.

After the first midterm examinations, my own delusion of being a good teacher would often be crushed and I would be left to face a hard reality: many of the students, who were constantly nodding their heads, scored very low on the midterm. I made it a habit to bring these students to my office to understand their difficulties and to learn how I could help them improve their understanding. They would often cite their weak background knowledge, and say that the concepts seemed very clear when explained in class. However, when homework time came around, they would struggle to use what they had learned in class to solve the assigned problems. Often, they would try to learn the material from the references provided in the syllabus, but they would have a very difficult time understanding it. They said that many of the textbooks were directed at experts and used difficult language, or that the authors took shortcuts that made it hard for students to follow. They also mentioned that many of the references did not have enough of the solved examples that would help them understand the material on their own.

Ever since I was a graduate student, I knew that dynamics can only be learned properly by solving as many examples as possible. Struggling with problems for hours at a time is what really allowed me to comprehend dynamics and be able to make a career out of it. Based on my experience as a student, a teacher, and my interaction with the students over the years, I came to realize that a modern textbook on intermediate dynamics had become necessary. It is also critical at a time when online courses are becoming common and students often rely on self-study. In my opinion, such a book needs to contain:

- detailed derivations combined with simple, yet comprehensive textual descriptions of these derivations;
- highlights of the important definitions and concepts, serving serve as a constant reminder of the important concepts to be learned in a given chapter;
- a large number of comprehensively solved examples of gradually increasing difficulty;
- methods that help the teacher improve the interactions in the classroom in order to help him/her assess the students' true level of understanding.

While writing this book, I have tried to the best of my ability to adhere to the aforementioned points. I have made it a habit to explain derivations in a very simple, yet comprehensive manner. I have presented highlights emphasizing the important concepts whenever they appear. I have also highlighted the important equations regularly throughout the text. I have solved a large number of examples – 120 to be exact – and presented the solutions in detail along with illustrations. These examples should aid the student in understanding the theoretical concepts at a deep level.

In addition, throughout the book, I have presented a set of exercises, which I call "Flipped classroom exercises". Flipping the classroom is a modern teaching technique which has been shown to be very effective in teaching topics requiring mathematical derivations. In a typical flipped classroom, the instructor will briefly discuss the material for ten minutes, do some of the important derivations, and solve one example. Subsequently, the instructor hands the students one or more problems to solve on their own, but provides some guidelines that help the students by dividing the problem into several sub-problems. The student will then spend the rest of the class trying to solve as many of these problems as possible. The problems are carefully designed to have an increasing difficulty level. At the end of the class, the full solution of the problems is given to the student. This approach allows the instructor to assess the level of each student in the class and allows the student to assess his/her own level of understanding and to pinpoint the exact concepts they need to inquire about.

The book presents a whole chapter on electromechanical modeling of systems involving particles and rigid-body motion. Such topics, which are essential in modern-day systems, have rarely been addressed systematically in a book on intermediate dynamics. Furthermore, in order to establish the critical connection between modeling of dynamical systems, which is the core of this book, and the analysis of their responses, which is usually covered in courses like perturbation theory and non-linear dynamics, I allocated the final chapter of this book to the introduction of some introductory analysis tools. This includes finding equilibrium points and analyzing their stability, establishing the phase-plane of the dynamics, and constructing bifurcation diagrams and basins of attraction of equilibrium points. The examples used in this chapter are mostly a continuation to the models developed in the previous chapters of the book.

In summary, I believe that I was fortunate to learn dynamics at the hands of a passionate teacher who made the topic very enjoyable and the material accessible. Nonetheless, I know that many students will not have such an opportunity. As such, I wrote this book hoping that many students would find it a useful companion in their journey of learning dynamics.

M.F. Daqaq
Abu Dhabi, UAE
November 2017

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Many people I have met in my life contributed, in some way or another, to my personal growth, and consequently to this humble manuscript. To each and everyone of you, I extend my sincerest gratitude. Nevertheless, there are certain individuals who left a long lasting impression on my life and career. I truly believe that, without those individuals, this work would have not been completed, at least not in this stage of my career. In what follows, I would like to acknowledge each one of you.

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Your courses were the best. Your ability to teach dynamics in such a smooth manner has always amazed me. This book is an extension to your teaching style and many examples were taken from your notes. I am so fortunate to have taken your classes. Prof. Ziyad Masoud, without whom I must admit that my Ph.D. research would have stalled. You have always been so dedicated, kind, generous, fair, and supportive. I have learned from you that the human element comes before any career gains. You have sacrificed a lot to make me successful and, for this, I thank you. Last, but not least, my late mentor, Prof. Ali H. Nayfeh from whom I learned most of the things I know today. In addition to his unprecedented technical knowledge, he was the most humble, hard working, professional, and meticulous researcher I have worked with. He taught me that, every minor detail is important and to never give up when it comes to achieving one's dreams. You showed, not only me, but all Palestinians who struggle every day to make ends meet, that dreams can be realized even with little opportunities.

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M.F.D.

Introduction

In the realm of physics and engineering, "dynamics" refers to the study of the motion of material bodies, rigid and elastic, under the influence of their surroundings. A rigid body is a material body that undergoes translational and rotational motions. However, it is assumed that the relative distance between any two points on a rigid body remains constant regardless of the type and magnitude of external forces acting on it. In other words, unlike elastic bodies, a rigid body does not undergo measurable strain levels when subjected to external stresses.

When a rigid body undergoes only translational motion, or when its resistance to rotational motion is negligible, a rigid body is treated as a particle. In such a case, the mass of the rigid body, regardless of its shape, can be reduced to a single point and the body's motion can be fully described by the motion of this point.

This book is concerned *only* with the dynamics of particles and rigid bodies, a topic critical in our understanding of how systems, natural or man made, move under the influence of forces. Such understanding has already enabled us to decipher some of the key forces that sustain the motion of the universe and to develop key enabling technologies that improve our lives, pushing the boundaries of what is considered achievable. Terrestrial, aerial, and waterborne vehicles, energy systems, robots, machinery, bionics, and nano/micro-electromechanical systems are only a few examples of technologies whose evolution would not have been possible without our knowledge of dynamics. Today, dynamics of particles and rigid bodies play an important role in almost every aspect of our daily life, and hence it is imperative for every physicist and engineer to understand its fundamentals.

I.1 Brief History

The word "dynamics" originated from the Greek word *Dynamis*, which means power. Dynamics, as a science, can also be traced back to the Greeks, in particular, to the Greek philosopher Aristotle (384–322 BC; Figure I.1) whose studies on the motion of bodies are the earliest documented evidence of the human curiosity about motion [1]. Similar to other Greek philosophers, Aristotle's understanding of dynamics was influenced by the four classical elements: fire, air, water, and earth. Each of these was thought to have a specific hierarchical position in nature: fire stands at the top, followed by air, then water, then earth. Aristotle argued that when one of these elements is taken out of its place, it naturally wants to return back to it. This explains why a stone thrown in water sinks to the bottom, while a bubble of air inside a fluid rises up to the surface.

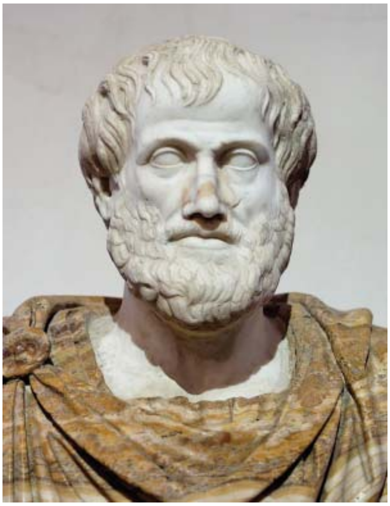

Figure I.1 Aristotle.
Source: https://en.wikipedia.org/wiki/Aristotle.

Based on this understanding, Aristotle divided motion into two parts: *natural* to refer to any motion that follows the hierarchy of the classical elements, and *violent* to refer to motion that opposes it. For example, a stone that slides down a hill is performing a natural motion whereas one thrown upwards is undergoing a violent motion.

Aristotle's limited understanding, which was based on the four elements, led him to many fallacious conclusions regarding the motion of physical bodies. He claimed that heavier objects fall faster than lighter ones and that the speed at which an object falls is proportional to its weight. Nevertheless, he correctly surmised that, in a void, all objects, regardless of their weight, fall at the same speed. This, unfortunately however, led him to incorrectly conclude that a void could not exist.

Most of Aristotle's ideas were later demolished by the Alexandrian philosopher Philoponus (490–570 AD) [2]. Philoponus's work led him to conclude that the rate at which objects fall is not proportional to their weight, as claimed by Aristotle. However, he still argued that heavier objects fall faster than lighter ones in a void. Philoponus also deciphered some of Aristotle's work on violent versus natural motion, alluding tangentially to the concepts of the potential and kinetic energy. He stated that, when an object is violently moved, it is subjected to a finite supply of forcing impetus, which makes the body move as long as it lasts. When this forcing impetus is exhausted, the body stops moving. He also indicated that the forcing impetus must be an internal property of the body.

During the Middle Ages, two Muslim Andalusian philosophers namely Avempace (anglicized from the Arabic, Ibn-Bajja) (1085–1138) and Averroes (anglicized from the Arabic, Ibn-Rushd) (1126–1198) played an important role in preserving and expanding upon Philoponus's studies on the motion of bodies [3]. This is clearly evident in their conclusion that Aristotle's work concerning the inverse proportionality between the velocity of a body and the density of the medium it moves through cannot be true. They argued that, if it were true, then a body could move instantaneously in a void.

Contrary to common belief, the understanding of the motion of bodies also evolved during the Middle Ages [4]. In fact, it was in those times that kinematics was transformed from a qualitative philosophy into a mathematical science. The concepts of instantaneous velocity, mean velocity, and acceleration – although without referring to the forces behind them – were introduced and studied by many researchers including the group known as the "Oxford Calculators", notably the scholar William Heytesbury, and the French priest Jean Buridan.

It was not until Galileo (1564–1642) that the forces behind motion were studied in the modern sense [5]. Galileo developed his understanding of forces by studying Archimedes' hydro-static principle. He noted that the same buoyancy forces that cause a body to rest in liquid are responsible for how fast it moves in the liquid. Another important contribution to dynamics was introduced by Huygens (1629–1695), who was the first to allude to the principle of conservation of momentum, through his investigation of the motion of impacting particles.

Sir Isaac Newton (1643–1728; Figure I.2) is, unarguably, the father of vectorial dynamics as we know it today. In his book, the *PhilosophæNaturalis Principia Mathematica*, usually referred to as the *Principia*, he introduced the concepts of force, acceleration, inertia, mass, and linear momentum [6]. He also stated (in the English of the time) his three famous laws, upon which all of our utilization and understanding of vectorial mechanics is based:

- *First Law*: Every body perseveres in its state of rest, or of uniform motion in a right line, unless it be compelled to change that state by forces impressed upon it thereon.
- *Second Law*: The alteration of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.
- *Third Law*: To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

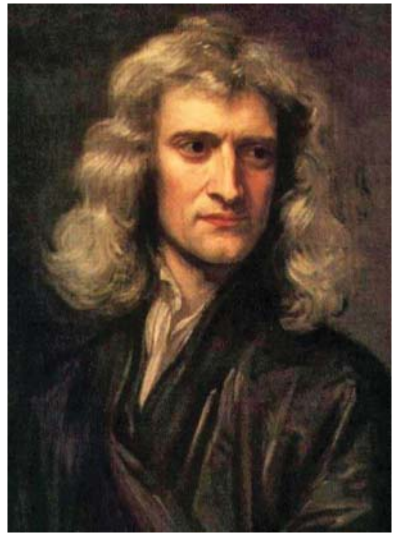

Figure I.2 Isaac Newton.
Source: https://en.wikipedia.org/wiki/Isaac_Newton.

In *The Principia*, Newton also introduced the general law of gravitational attraction used today to study orbital mechanics.

What Newton did not discuss in *The Principia*, was developed by many of his compatriots and successors, such as Daniel Bernoulli (1700–1782), Euler (1707–1783; Figure I.3), D'Alembert (1717–1783), Lagrange (1736–1813), Jacobi (1804–1851), and Hamilton (1805–1865). This work includes, but is not limited to, the treatment of rigid bodies in rotation, and hence the concept of angular momentum, the motion of systems including more than two degrees of freedom, elastic bodies, and the concepts of work and energy. It is also worth noting that many historians argue that credit for the application of Newton's second law along the three Cartesian coordinates is actually due to Euler [1].

Figure I.3 Leonhard Euler.
Source: https://en.wikipedia.org/wiki/Leonhard_Euler.

Many also argue that Jacob Bernoulli was the first to introduce the concept of angular momentum, even before *The Prinicipia* was published [8]. He was also the first to develop the equation of motion of an elastic body, after investigating the motion of strings using the balance of forces and moments across a string element.

Euler, on the other hand, was the first to tackle the three-body problem and to introduce the concept of three-body rotations about orthogonal axes, which he reasoned must pass through

Figure I.4 Joseph-Louis Lagrange. Source: http://ro.math.wikia.com/wiki/Teorema_lui_Lagrange.

the center of mass of the rotating body. In 1776, he published his famous equations on the conservation of angular momentum known today as Euler's rotational equations [9].

The concepts of work and energy and the field of analytical dynamics were first alluded to by Leibniz (1647–1716). Subsequently, upon the formalization of the calculus of variation by the likes of d'Alembert, Euler, and John Bernoulli (1667–1748), the principle of virtual work was established by Jacob Bernoulli as a way to understand static equilibrium, and then extended to bodies in motion by d'Alembert.

The principle of least action, which is used to derive the equations of motion of different systems, is due to Maupertuis (1698–1759), who stated that "nature is thrifty in all actions" and applied this concept to mechanics by suggesting the minimization of the kinetic energy of the system to obtain the equations of motion. Along similar lines, Euler suggested applying the principle of least action to the linear momentum in order to obtain the same equations.

In 1788, Lagrange (1736–1813; Figure I.4) introduced his equations of motion for mechanical bodies, leading to what is known today as Lagrangian mechanics [10]. Hamilton in 1833 applied the variational principle to the classical Lagrangian function and obtained Hamilton's variational principle, which later led to the formulation of Hamiltonian mechanics.

I.2 Introductory Concepts

This section presents many of the fundamental concepts used throughout this book. It is assumed that the reader will have been exposed to these concepts in introductory courses on statics, dynamics, vector and matrix algebra, multi-variable calculus, and differential equations. Nonetheless, this section reintroduces some important concepts briefly in order to rejuvenate them in the reader's mind.

I.2.1 Matrices and Linear Algebra

An n by m matrix is an array of objects arranged in n rows and m columns as follows:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}, \tag{I.1}$$

where the a_{ij} are denoted as the matrix entries. A matrix is said to be square when $n = m$. Matrix algebra satisfies the following important properties/identities:

1. Two matrices **A** and **B** are said to be equal if and only if they have the same number of rows and columns and have equal entries; that is, $[a_{ij}] = [b_{ij}]$ for each i and j.
2. The sum of two n by m matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ is equal to another n by m matrix **C** such that $c_{ij} = a_{ij} + b_{ij}$ for each i and j.
3. If a matrix $\mathbf{A} = [a_{ij}]$ is multiplied by a scalar α then $\alpha\mathbf{A} = [\alpha a_{ij}]$.
4. The product of an n by r matrix **A** and an r by m matrix **B** is an n by m matrix **C** whose c_{ij} elements are given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ \dots \ + a_{ir}b_{rj}. \tag{I.2}$$

5. For two matrices **A** and **B** having the same dimensions, $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
6. For three matrices **A**, **B**, and **C**; $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ provided **A** is an n by k matrix and **B** and **C** are k by m matrices.
7. For three matrices **A**, **B**, and **C**; $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ provided **A** is an n by k matrix, **B** is a k by r matrix and **C** is an r by m matrix.

I.2.1.1 Transpose of a matrix

The transpose of an n by m matrix $\mathbf{A} = [a_{ij}]$ is an m by n matrix $\mathbf{A}^T = [a_{ji}]$. The transpose of a matrix satisfies the following properties:

1. $(\mathbf{A}^T)^T = \mathbf{A}$.
2. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.
3. A matrix **A** is said to be symmetric if $\mathbf{A}^T = \mathbf{A}$.
4. A matrix **A** is said to be skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.
5. A matrix **A** is said to be orthogonal if $\mathbf{A}\mathbf{A}^T = \mathcal{I}$, where $\mathcal{I}$ is denoted as the identity matrix

$$\mathcal{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{I.3}$$

I.2.1.2 Inverse of a matrix

A n by n matrix **A** is said to have an inverse $\mathbf{A}^{-1}$ when $\mathbf{A}\mathbf{A}^{-1} = \mathcal{I}$. A matrix that has an inverse is called a non-singular matrix. The following are some of the properties of the matrix inverse:

1. If **A** and **B** are non-singular square matrices then **AB** is also a non-singular matrix and satisfies $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

2. If $\mathbf{A}$ is non-singular then so is $\mathbf{A}^{-1}$.
3. If either $\mathbf{A}$ or $\mathbf{B}$ is singular then $\mathbf{AB}$ and $\mathbf{BA}$ are both singular.
4. For an orthogonal matrix $\mathbf{A}^T = \mathbf{A}^{-1}$.

I.2.1.3 Determinant

The determinant of a 2 by 2 matrix $\mathbf{A}$ is denoted as $|\mathbf{A}|$ and is given by $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$. The determinant of a 3 by 3 matrix $|\mathbf{A}|$ is given by

$$|\mathbf{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{I.4}$$

In general, the determinant of any n by n matrix is given by

$$|\mathbf{A}| = (-1)^{i+j} a_{ij} \mathcal{Q}_{ij} \tag{I.5}$$

where $\mathcal{Q}_{ij}$ are called the minors of $|\mathbf{A}|$ and are obtained by finding the determinant of the matrix formed by deleting the ith row and jth column of $|\mathbf{A}|$. Note that a matrix is singular if its determinant is zero. The determinant satisfies the following properties:

1. $|\alpha\mathbf{A}| = \alpha|\mathbf{A}|$.
2. If $\mathbf{A}$ has a zero row or column, then $|\mathbf{A}| = 0$.
3. If $\mathbf{A}$ has a repeated row (or column), or has one row (or column) that is a constant multiplication of another row (or column), then $|\mathbf{A}| = 0$.
4. If $\mathbf{A}$ is an n by n matrix and $\mathbf{u}$ is an n by 1 vector, then $\mathbf{Au} = 0$ if and only if $\mathbf{u} = 0$ (trivial solution) or $|\mathbf{A}| = 0$ (nontrivial solution).

I.2.1.4 Eigenvalues and eigenvectors

A scalar λ, real or complex, is said to be a right eigenvalue of the matrix $\mathbf{A}$ if and only if it satisfies the following equation:

$$\mathbf{Au} = \lambda\mathbf{u}, \tag{I.6}$$

where $\mathbf{u}$ is a non-zero n by 1 vector known as the eigenvector of $\mathbf{A}$. An easy way to find the eigenvalues of a matrix $\mathbf{A}$ is by manipulating the previous equation such that

$$(\mathbf{A} - \lambda\mathcal{I})\mathbf{u} = 0. \tag{I.7}$$

For the previous equation to have non-trivial solutions, $\mathbf{u} \neq 0$, then by property 4 of the determinants of a matrix, we can write

$$|\mathbf{A} - \lambda\mathcal{I}| = 0. \tag{I.8}$$

For each distinct eigenvalue λ_i of the matrix $\mathbf{A}$, there exists an eigenvector $\mathbf{u}_i$. The eigenvectors $\mathbf{u}_i$ of the matrix $\mathbf{A}$ are said to be linearly independent if they satisfy

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \ldots + \alpha_n\mathbf{u}_n = 0, \tag{I.9}$$

only when all of the $\alpha_i = 0$, where α_i are scalars.

I.2.1.5 Diagonalization

A matrix $\mathbf{A} = [a_{ij}]$ is said to be diagonal if $a_{ij} = 0$ when $i \neq j$. A non-diagonal n by n matrix can be diagonalized if there exists an n by n matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. In such a case, we say that $\mathbf{P}$ diagonalizes $\mathbf{A}$.

The condition that guarantees that existence of a matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ is that $\mathbf{A}$ has n linearly independent eigenvectors, $\mathbf{u}_i$. In such a case, $\mathbf{P}$ can be formed by using the eigenvectors of $\mathbf{A}$ as the columns of $\mathbf{P}$. The resulting diagonal matrix, $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ will have the eigenvalues of $\mathbf{A}$ as its diagonal elements.

The following are some properties associated with diagonalization:

1. Any n by n matrix with n distinct eigenvalues is always diagonalizable because the presence of n distinct eigenvalues guarantees the existence of n linearly-independent eigenvectors.
2. Any n by n real symmetric matrix is diagonalizable even when it does not possess n distinct eigenvalues. By construction, real symmetric matrices have n linearly-independent eigenvectors.
3. Any n by n real symmetric matrix $\mathbf{A}$ is diagonalizable by an orthogonal matrix, $\mathbf{P}$. This implies that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^T\mathbf{AP}$ for any real symmetric matrix $\mathbf{A}$.

I.2.2 Vectors

Vectors are used in physics to describe quantities that have both magnitude and direction. In a Cartesian coordinate, a vector is described by three components, each of which indicates its magnitude along the unit vectors ($\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$) forming the Cartesian coordinate system. For instance, as shown in Figure I.5, the vector $\mathbf{u}$ can be expressed as

$$\mathbf{u} = u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + u_z\hat{\mathbf{k}}. \tag{I.10}$$

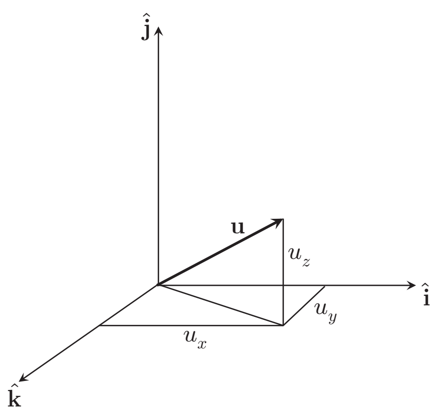

Figure I.5 Projection of a vector into its different components along the unit vectors of a Cartesian coordinate system.

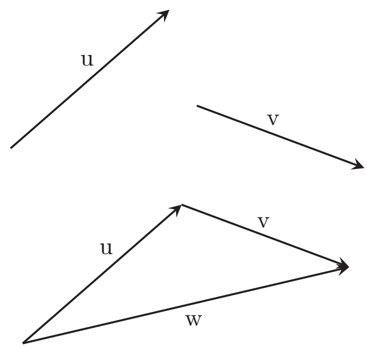

Figure I.6 Graphical representation of the sum of two vectors.

The magnitude of the vector is defined as

$$\text{Magnitude of } \mathbf{u} = \|\mathbf{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2}. \tag{I.11}$$

As shown in Figure I.6, the sum of two vectors, **u** and **v**, is another vector **w** whose magnitude and direction depends on the magnitude and direction of **u** and **v**; that is:

$$\begin{aligned} \mathbf{w} = \mathbf{u} + \mathbf{v} &= (u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + u_z\hat{\mathbf{k}}) + (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) \\ &= (u_x + v_x)\hat{\mathbf{i}} + (u_y + v_y)\hat{\mathbf{j}} + (u_z + v_z)\hat{\mathbf{k}}. \end{aligned} \tag{I.12}$$

The summation of vectors is commutative; that is, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ and associative; that is, $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Multiplication involving vectors can be classified as follows:

1. *Multiplication by a scalar quantity,* α: In this case the product is defined by

$$\mathbf{v} = \alpha\mathbf{u} = \alpha u_x\hat{\mathbf{i}} + \alpha u_y\hat{\mathbf{j}} + \alpha u_z\hat{\mathbf{k}}, \tag{I.13}$$

   in which case $\|\mathbf{v}\| = \alpha\|\mathbf{u}\|$.
2. *The dot product*: The dot product of two vectors **u** and **v** is a scalar defined as

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta, \tag{I.14}$$

   where θ is the angle between **u** and **v**. We say that two vectors are orthogonal when $\mathbf{u}\cdot\mathbf{v} = 0$. In general, the dot product can be expressed as

$$\mathbf{u}\cdot\mathbf{v} = u_xv_x + u_yv_y + u_zv_z. \tag{I.15}$$

   It follows that $\mathbf{u}\cdot\mathbf{u} = \|u\|^2$. The dot product satisfies the following properties:

   (a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}\cdot\mathbf{w}) + (\mathbf{v}\cdot\mathbf{w})$
(c) $\mathbf{u}\cdot\mathbf{u} = 0$ only when $\mathbf{u} = 0$.

3. *The cross product*: The cross product of two vectors **u** and **v** is another vector **w** which is perpendicular to both **u** and **v** and is defined as

$$\mathbf{w} = \mathbf{u}\times\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta\hat{n}, \tag{I.16}$$

where θ is the angle between **u** and **v** and $\hat{n}$ is the vector perpendicular to both **u** and **v**. We say that two vectors are parallel when $\mathbf{u}\times\mathbf{v} = 0$. In terms of the Cartesian unit vectors, the cross product can be written as

$$\mathbf{u}\times\mathbf{v} = (u_y v_z - u_z v_y)\hat{\mathbf{i}} + (u_z v_x - u_x v_z)\hat{\mathbf{j}} + (u_x v_y - u_y v_x)\hat{\mathbf{k}}, \tag{I.17}$$

or

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \tag{I.18}$$

The cross product satisfies the following properties:

(a) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$
(b) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
(c) $\alpha(\mathbf{u}\times\mathbf{v}) = \alpha\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\alpha\mathbf{v}$
(d) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}$
(e) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{v}(\mathbf{u}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{u}\cdot\mathbf{v})$.
(f) The cross product of two vectors can also be expressed as the product of a skew-symmetric matrix and a vector as following:

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

(g) $(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{z}) = (\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{z}) - (\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{z})$
(h) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{z}) = (\mathbf{u}\cdot(\mathbf{v}\times\mathbf{z}))\mathbf{w} - (\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}))\mathbf{z}$

I.2.3 Vector Calculus

I.2.3.1 Gradient

Consider a scalar function $\Phi(x, y, z)$. The gradient of Φ is a vector defined as

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{k}}, \tag{I.19}$$

where $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$ is known as the *del operator*. For a surface $\Phi(x, y, z) = 0$, the gradient of the surface evaluated at a point $P_0 : (x_0, y_0, z_0)$ represents the vector normal to the surface, $\Phi = 0$ at P_0.

I.2.3.2 Divergence

The divergence of a vector field $\mathbf{u}(x, y, z) = f(x, y, z)\hat{\mathbf{i}} + g(x, y, z)\hat{\mathbf{j}} + h(x, y, z)\hat{\mathbf{k}}$ is a scalar quantity defined as

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \cdot (f(x, y, z)\hat{\mathbf{i}} + g(x, y, z)\hat{\mathbf{j}} + h(x, y, z)\hat{\mathbf{k}}), \\ \nabla \cdot \mathbf{u} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}. \end{aligned} \tag{I.20}$$

In general, the divergence of the vector field evaluated at a given point is a measure of the net flow into or out of that point in the field.

I.2.3.3 Curl

The curl of a vector field $\mathbf{u}(x, y, z) = f(x, y, z)\hat{\mathbf{i}} + g(x, y, z)\hat{\mathbf{j}} + h(x, y, z)\hat{\mathbf{k}}$ is a vectorial quantity defined as

$$\begin{aligned} \nabla \times \mathbf{u} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \times (f(x, y, z)\hat{\mathbf{i}} + g(x, y, z)\hat{\mathbf{j}} + h(x, y, z)\hat{\mathbf{k}}), \\ \nabla \times \mathbf{u} &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}} \end{aligned} \tag{I.21}$$

The curl at a given point in the vector field is a measure of the degree of rotation of the flow field around that point.

The gradient, divergence, and curl satisfy the following identities:

1. $\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$.
2. $\nabla(\Phi\Psi) = \Psi\nabla\Phi + \Phi\nabla\Psi$.
3. $\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})$.
4. $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$.
5. $\nabla \times (\nabla\Phi) = 0$. The curl of the gradient of a scalar field is zero. This is because ∇ is parallel to $\nabla\Phi$.
6. $\nabla \cdot (\nabla \times \mathbf{u}) = 0$. The divergence of the curl of a vector field is zero. This is because ∇ is perpendicular to $\nabla \times \mathbf{u}$.
7. $\nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) = \nabla^2\mathbf{u}$.
8. $\nabla \cdot (\Phi\nabla\Psi) = \Phi\nabla^2\Psi + \nabla\Phi \cdot \nabla\Psi$.

I.2.3.4 Gauss divergence theorem

The Gauss divergence theorem states that the flow in or out (flux) of a closed surface σ is equal to the divergence of the vector field $\mathbf{u}$ away from the points in the volume V enclosed by σ; that is

$$\iint_\sigma \mathbf{u} \cdot \mathrm{d}\mathbf{A} = \iiint_V \nabla \cdot \mathbf{u} \, \mathrm{d}V, \tag{I.22}$$

where, as shown in Figure I.7, $\mathrm{d}\mathbf{A}$ is an element normal to the surface σ.

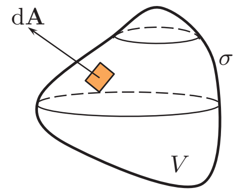

Figure I.7 Graphical representation of the Gauss divergence theorem.

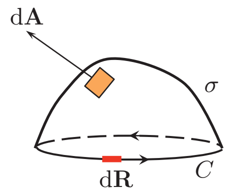

Figure I.8 Graphical representation of Stokes' theorem.

I.2.3.5 Stokes' theorem

Stokes theorem states that the circulation of the vector field $\mathbf{u}$ around a closed loop C is equal to the flux of the curl of the vector field through a surface bounded by the curve; that is

$$\oint_C \mathbf{u} \cdot \mathrm{d}\mathbf{R} = \iint_\sigma (\nabla \times \mathbf{u}) \cdot \mathrm{d}\mathbf{A}, \tag{I.23}$$

where, as shown in Figure I.8, $\mathrm{d}\mathbf{A}$ is an area element normal to the surface σ and $\mathrm{d}\mathbf{R}$ is a differential length along the closed loop C.

I.2.4 Basic Quantities used in Dynamics

The following are some basic definitions of the scalar and vectorial quantities that are commonly used to describe the motion of dynamical systems.

- Mass (symbol: m, M, SI unit: kg): a scalar quantity used to measure the resistance of a body to movement. A larger mass implies larger resistance to motion.
- Time (symbol: t, SI unit: s: a scalar quantity that is used to measure the irreversible succession of events in a dynamical system.
- Position (symbol: $\mathbf{r}$, SI unit: m): a vectorial quantity used to locate the position of a point in space with respect to another point.
- Distance (symbol: s, SI unit: m): a scalar quantity used to measure the magnitude of a vector $\mathbf{r}$ connecting two points in space; $s = \|\mathbf{r}\|$.
- Velocity (symbol: $\mathbf{v}$, SI unit: m/s): a vectorial quantity used to measure the time rate of change of a position vector; $\mathbf{v} = \dot{\mathbf{r}}$.

- Speed (symbol: $\dot{s}$, SI unit: m/s): a scalar quantity used to measure the magnitude of the velocity vector; $\dot{s} = \|\mathbf{v}\|$.
- Acceleration (symbol: $\mathbf{a}$, SI unit: m/s^2): a vectorial quantity used to measure the time rate of change of the velocity vector; $\mathbf{a} = \dot{\mathbf{v}}$.
- Gravitational acceleration (symbol: $\mathbf{g}$, SI unit: m/s^2): a vectorial quantity that measures the acceleration of a free-falling body in the earth's atmosphere. The value of $\mathbf{g}$ differs ever so slightly at different latitudes. Nonetheless, a value of 9.81 m/s^2 is usually used by scientists.
- Angle (symbol: none specific (θ commonly used), SI unit: rad): a vectorial quantity that measures the amount of turn between two straight lines that have a common end point; that is, it is the orientation of one vector with respect to the other.
- Angular velocity (symbol: ω, SI unit: rad/s): a vectorial quantity that measures the time rate of change of the angle between two vectors; $\omega = \dot{\theta}$.
- Angular acceleration (symbol: α, SI unit: rad/s^2): a vectorial quantity that measures the time rate of change of the angular velocity; $\alpha = \dot{\omega}$.
- Force (symbol: $\mathbf{F}$, SI unit: Newton N): a vectorial quantity that measures the action of one body on another. According to Newton's second law of dynamics applied to a body of constant mass, the force acting on a body can be related to its ensuing acceleration via $\mathbf{F} = m\mathbf{a}$.
- Weight (symbol: $\mathbf{W}$, SI unit: N): the force exerted by the gravitational field on a given body; $\mathbf{W}=m\mathbf{g}$.
- Moment (symbol: $\mathbf{M}$, SI unit: N.m): a vectorial quantity that measures the turning effect of a force about a point. The moment is defined as $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ where $\mathbf{r}$ is the position vector from the point about which the moment is measured to the point at which the force is exerted.
- Mass moment of inertia (symbol: I, SI unit: kg.m^2): a scalar quantity that measures a body's resistance to rotation about a given axis.

I.3 Book Organization

The study of bodies in motion under the influence of their surroundings can be divided into three different parts:

- Part 1 concerns modeling, a process through which the equations used to describe the motion of the system are obtained. For particles and rigid bodies, the motion is sufficiently described by one or more, linear or non-linear, ordinary differential equations.
- Part 2 concerns the analysis, a process through which the equations of motion are analyzed to understand the influence of the different design parameters on the motion of the system.
- Part 3 concerns controlling the motion, a process through which special controllers are designed to force the dynamical system to behave in a desired manner.

This book focuses on Part 1 but also presents some of the introductory analysis tools commonly used in Part 2. Specifically, this book is organized as follows:

1. Chapter 1 presents the concepts of inertial and rotating frames and their utilization to study the kinematics of particles.
2. Chapter 2 presents Newton's laws of dynamics and their implementation to obtain the equations of motion of particles. This is commonly referred to as the "vectorial" approach to dynamics.

3. Chapter 3 implements the vectorial approach to obtain the equations of motion of rigid bodies. Both planar and non-planar motion are considered. In the process, the concepts of center of mass, moment of inertia, and principal axis of rotation are presented and Euler's rotational equations are derived.
4. Chapter 4 classifies the types of constraints to which a dynamical system can be subjected and discusses the difference between an actual and a virtual displacement.
5. Chapter 5 presents the analytical approach to dynamics. In this chapter, Lagrange's equations for conservative and non-conservative systems as well as Hamilton's principle are derived and applied to model the motion of particles.
6. Chapter 6 extends the analytical approach to model the motion of rigid bodies. Both planar and non-planar motion are discussed.
7. Chapter 7 presents the concepts of the impulse and the linear and angular momenta, their conservation and utilization to derive the equations of motion of particles and rigid bodies.
8. Chapter 8 studies the motion of charged bodies in electrostatic and electromagnetic fields. To this end, the Coulomb force, Lorentz force, Maxwell's equations, Ampere's law, Gauss's law, and Faraday's law of induction are presented and used in conjunction with Newton's second law and Lagrange's equation to obtain the equations of motion for charged bodies.
9. Chapter 9 presents some of the most-widely utilized tools to analyze the behavior of dynamical systems. This includes finding equilibrium solutions of ordinary differential equations and assessing their stability, establishing the phase-space representation of the dynamics, and constructing bifurcation diagrams and basins of attractions for equilibrium solutions. Examples of analyzing the motion of particles and rigid bodies using these tools are presented to establish the critical connection between modeling and analysis of dynamical systems.

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