Dynamical Mean Field Theory of Spin Glasses and Their Phase Transitions in ac External Fields

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The dynamical mean field theory of spin glasses is investigated in systems with non-zero mean of the infinitely-ranged random exchange interactions under uniform but alternating external magnetic fields. A perturbational expansion with respect to the interaction $u\sigma_j^i$ introduced in a soft-spin version of the model is analyzed by means of the diagrammatic method. Restricting ourselves to its lowest order (single-loop) approximation, we examine relaxational dynamics of the model spin glass and clarify nature of its phase transition in the presence of an ac external field. The results provide a new clue to ac nonlinear measurements on spin glasses.

§ 1. Introduction

There has been much progress recently in study of the mean field theory of spin glasses provided by the Sherrington-Kirkpatrick (SK) model. Parisi's replica symmetry breaking solution in the replica theory is successively related to the existence of multi-valley structure in the free energy of each sample derived from the TAP theory. Thermodynamic properties described by the mean field theory are qualitatively in good agreement with those observed experimentally in various spin glasses. Of most importance among them is the existence of thermodynamic phase transition at a finite temperature. This has been also confirmed recently by the huge computer simulation on three-dimensional Ising spin glasses with nearest-neighbor interactions.

It is then of interest to examine dynamical properties of spin glasses by the same footing, i.e., by the dynamical mean field theory. In particular, the investigation of dynamical instability will further clarify nature of the spin-glass phase transition. Actually much work has been already performed along this line. Among them the work due to Sompolinsky and Zippelius (SZ) is most comprehensive. The purpose of this paper is to extend their dynamical mean field theory, and to examine spin-glass phase transition under an ac external magnetic field from a dynamical point of view.

In the dynamical mean field theory a soft-spin version of the SK model is introduced, and its relaxational dynamics is formulated in terms of a phenomenological Langevin equation. The latter is solved by means of the generating functional method, which is the Lagrangian formulation of classical dynamics introduced by Martin, Siggia and Rose and was applied before to examine critical dynamics in pure systems. For random (spin glass) systems this method is advantageous since direct term-by-term expansions of random averages of static and dynamic quantities can be analyzed. In
the present problem the perturbational expansion with respect to the interaction \( u\sigma_i^4 \) introduced in a soft-spin version of the model is employed. In the mean field limit (i.e., for infinitely-ranged spin glasses), we obtain a set of self-consistent equations for the averaged magnetization, correlation function and generalized response function, which are generated by the averaged generating functional.

In the present paper we extend this final procedure so as to include a case with random exchange interactions with non-zero mean, and as to be able to examine nonlinear responses on ac fields. The latter extension is required to interpret appropriately results of the ac nonlinear measurement which is a powerful experimental method to investigate critical dynamics of spin glasses. We discuss structure of the perturbational expansions of the three functions mentioned above. If all the terms can be evaluated, we can go back to the original Ising SK spin glass by the proper limiting procedure. Under ac fields, however, explicit evaluation is limited only to the lowest order terms with respect to the interaction \( u\sigma_i^4 \). In this sense we have to say that we study dynamical properties of an infinitely-ranged soft spin glass, but some of the results obtained here are thought to be common to real spin glasses. An interesting result among them is the phase-transition behavior under an ac field as briefly reported elsewhere.

This paper is organized as follows: We derive a set of the self-consistent equations for the generalized response function, the correlation function and the magnetization in § 2. Most part of this section is a review of the SZ dynamical mean field theory. Some mathematics not described explicitly in the SZ paper are presented in Appendices. In § 3 the systematic perturbational analysis is presented, and dynamical properties of the soft spin glass under ac fields are examined explicitly in § 4. The corresponding analysis on a ferromagnet is briefly described in Appendix C. In § 5 we discuss the results obtained in connection with ac nonlinear experiments on spin glasses.

\section*{§ 2. Dynamical mean field theory}

The SK Hamiltonian for Ising spins \( S_i = \pm 1 \) is

\[ H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j, \tag{2.1} \]

where \( \langle ij \rangle \) covers all the spin pairs. The exchange interactions \( J_{ij} \) are independent random variables with a Gaussian distribution

\[ P(J_{ij}) = \left( \frac{N}{2\pi f^2} \right)^{1/2} \exp \left[ -\frac{N}{2f^2}(J_{ij} - J_0)^2 \right], \tag{2.2} \]

\( N \) being the total number of spins. In order to examine dynamical properties of spin glasses, we consider a soft-spin version of the SK model defined by

\[ \beta H = \frac{1}{2} \sum_i (r_0 \delta_{ij} - \beta J_{ij}) \sigma_i \sigma_j + \frac{u}{8} \sum_i \sigma_i^4 - \beta \sum_i h_i \sigma_i, \tag{2.3} \]

where \( \beta = 1/T \) \( (k_B = 1) \). The soft spin \( \sigma_i \) is allowed to vary continuously from \(-\infty\) to \(+\infty\). The model with fixed spin length is recovered from Eq. (2.3) in the limit \( r_0 \to -\infty \) and \( u \to +\infty \), such that their ratio remains finite.

To study relaxational dynamics of the model spin glass, the following
Dynamical Mean Field Theory of Spin Glasses

A phenomenological Langevin equation is introduced:
\[ \Gamma_0^{-1} \partial_t \sigma(t) = K_i(\sigma) + \xi_i(t), \quad (2.4) \]
\[ K_i(\sigma) = -\frac{\delta H}{\delta \sigma_i} = -\sum_j (r_{ij} \delta \sigma_j - \beta J_{ij}) \sigma_j - \frac{1}{2} \sigma_i^2 + \beta h_i. \quad (2.5) \]

Here the noises \( \xi_i \) are Gaussian random variables with zero mean and variance
\[ \langle \xi_i(t) \xi_j(t') \rangle = \frac{2}{\Gamma_0} \delta_{ij} \delta(t - t'). \quad (2.6) \]

The physical quantities of particular interest are the correlation function
\[ C_\sigma(t, t') = \langle \sigma_i(t) \sigma_j(t') \rangle \quad (2.7) \]
and the generalized response function
\[ G_\sigma(t, t') = \frac{\partial \langle \sigma_i(t) \rangle}{\partial \delta h_j(t')}, \quad t > t' \quad (2.8) \]
where \( \langle \rangle \) means the average over \( \xi_i \). We note that since we are interested in nonlinear responses against ac fields, both Eqs. (2.7) and (2.8) are functions of \( t \) and \( t' \) (not necessarily of \( t - t' \) alone).

The relaxational dynamics described by Eqs. (2.4) \~ (2.6) is examined by the generating functional method. Let us first define \( Z_{i, \sigma} \) by
\[ Z_{i, \sigma} = \int \mathcal{D} \sigma \exp \left[ \sum \int dt_i(t) \sigma_i(t) \right] \times \prod \delta \left( -\Gamma_0^{-1} \delta \sigma_i + K_i(\sigma) + \xi_i(t) \right) \cdot J(\sigma), \quad (2.9) \]
where \( \mathcal{D} \sigma = \prod \mathcal{D} \sigma_i \). The suffix \( J \) indicates that a set of \( J_\sigma \) is fixed. The average over \( \xi_i \) is carried out by making use of the integral representation of the \( \delta \)-function in Eq. (2.9), thereby the auxiliary variables \( \delta \sigma_i(t) \) are introduced. We obtain
\[ Z_{i, \sigma} = \int \mathcal{D} \sigma \delta \sigma \exp \left[ \sum \int dt_i(t) \sigma_i(t) + i \tilde{\sigma}_i(t) \delta \sigma_i(t) \right] + L(\sigma, \delta \sigma), \quad (2.10) \]
\[ L(\sigma, \delta \sigma) = \sum \int dt \left( i \delta \sigma_i(t) \left[ -\Gamma_0^{-1} \delta \sigma_i(t) + K_i(\sigma) + \Gamma_0^{-1} i \delta \sigma_i(t) \right] - \frac{\Gamma_0}{2} \frac{\delta K_i(\sigma)}{\delta \sigma_i} \right), \quad (2.11) \]
where the conjugate field \( \tilde{\sigma}_i \) against \( i \delta \sigma_i \) has also been introduced. The last term in Eq. (2.11) comes from the functional Jacobian \( J(\sigma) \) as briefly explained in Appendix A. In terms of \( Z_{i, \sigma} \) the site magnetization \( m_i(t) \) and the functions \( C_\sigma(t, t') \) and \( G_\sigma(t, t') \), respectively, are given by
\[ m_i(t) = \frac{1}{Z_0} \frac{\partial Z_{i, \sigma}}{\partial l_i(t)} \bigg|_{t = \tilde{t} = 0}, \quad (2.12) \]
\[ C_\sigma(t, t') = \frac{1}{Z_0} \frac{\partial^2 Z_{i, \sigma}}{\partial l_i(t) \partial l_i(t')} \bigg|_{t = \tilde{t} = 0}, \quad (2.13) \]
\[ G_\sigma(t, t') = \langle i \tilde{\sigma}_j(t') \sigma_i(t) \rangle = \frac{1}{Z_0} \frac{\partial^2 Z_{i, \sigma}}{\partial \tilde{\sigma}_j(t') \partial l_i(t)} \bigg|_{t = \tilde{t} = 0}, \quad (2.14) \]
where \( Z_0 = Z_{i, \sigma}(0, 0) = 1 \). The last equality \( Z_0 = 1 \) is imposed by the definition of Eq. (2.9),
and is ensured by the last term in Eq. (2.11). This term arising from the functional Jacobian also ensures the causality of the response as described in Appendix B.

Because of \(Z_0=1\) the average of Eqs. (2.12)~(2.14) with respect to \(\mathcal{J}\) (indicated by \([\ ]\)) is performed only over their numerator. More generally the \(\mathcal{J}\)-average of any noise-averaged (multi-) product of \(\sigma_i(t)\) and/or \(\bar{\sigma}_i(t')\) is generated by the averaged functional \(Z(l, \bar{l})=\langle Z_1(l, \bar{l}) \rangle\). The average is easily done since \(\mathcal{J}\) are involved only in the exponential form \(\exp\{i\mathcal{J} \sigma_i(t) \sigma(t')\}\) in \(Z_1(l, \bar{l})\). This circumstance is the same as in the replica method.\(^3\) The averaged generating functional \(Z(l, \bar{l})\) is given by

\[
Z(l, \bar{l}) = \frac{1}{Z_1 Z_0} \int \prod_{\mathcal{J}} \langle d\mathcal{J} \mathcal{P}(\mathcal{J}) \rangle Z_1(l, \bar{l})
\]

\[
= \int \mathcal{D} \sigma \mathcal{D} \bar{\sigma} \exp \left\{ L_0(\sigma, \bar{\sigma}; l, \bar{l}) + \frac{\beta^2 J^2}{2N} \sum_{ij} \int dt \bar{\sigma}_i(t) \sigma_j(t) \right\}
\]

\[
+ \frac{\beta^2 J^2}{2N} \sum_{ij} \int dt \bar{\sigma}_i(t) \sigma_j(t) \bar{\sigma}_i(t') + i \bar{\sigma}_i(t) \sigma_j(t) \bar{\sigma}_j(t')\sigma_j(t') \right\},
\]

\[
L_0(\sigma, \bar{\sigma}; l, \bar{l}) = \sum_i \int dt \left[ i \bar{\sigma}_i(t) \left\{ -\frac{1}{\tau_0} \partial_t \sigma_i - \nu_0 \sigma_i - \frac{1}{2} u \sigma_i^3 + \beta h_i + \frac{i \bar{\sigma}_i}{\tau_0} \right\}
\]

\[
= -\frac{\nu_0 \delta K_i}{2} + \nu_0 \sigma_i(t) \mp i \bar{\sigma}_i(t) \right\].
\]

(2.15)

The formulation up to here is in fact not restricted to the infinitely-ranged (SK) spin glass. A great advantage of the latter is in the fact that Eq. (2.15) can be further simplified to a one-site problem by introducing auxiliary variables \(Q_a(t, t')\) and \(M_a(t)\), \(a=1 \ldots 4\, \) and \(\beta=1, 2\):

\[
Z(l, \bar{l}) = \int \mathcal{D} Q \mathcal{D} M \exp \left\{ \frac{-2N \beta^2 J^2}{2} \int dt dt' \left\{ Q_1(t, t')Q_2(t, t') + Q_3(t, t')Q_4(t, t') \right\} \right\}
\]

\[
= \frac{1}{Z_0} \int dt M_1(t) M_2(t) + \ln \int \mathcal{D} \sigma \mathcal{D} \bar{\sigma} \exp \left\{ L_0(\sigma, \bar{\sigma}; Q_a, M) \right\},
\]

(2.17)

\[
L_0(\sigma, \bar{\sigma}; Q_a, M) = L_0(\sigma, \bar{\sigma}; l, \bar{l}) + \sum_i \int dt dt' \left\{ Q_1(t, t')i \bar{\sigma}_i(t) i \bar{\sigma}_i(t') \right\}
\]

\[
+ Q_2(t, t') \sigma_i(t) \sigma_i(t') + Q_3(t, t') i \bar{\sigma}_i(t) \sigma_j(t')
\]

\[
+ Q_4(t, t') \sigma_i(t) i \bar{\sigma}_i(t') + \sum_i \int dt \{ M_1(t) i \bar{\sigma}_i(t) + M_2(t) \sigma_j(t) \}.
\]

(2.18)

The integrals over \(Q_a\) and \(M_a\) are evaluated by the steepest descent, which is exact in the thermodynamic limit (\(N \rightarrow \infty\)). The saddle point equations in the leading order with respect to \(N^{-1}\) are:

\[
Q_1^{sp}(t, t') = \frac{\beta^2 J^2}{2N} \sum_i \langle \sigma_i(t) \sigma_i(t') \rangle = \frac{\beta^2 J^2}{2} C_{ii}(t, t'),
\]

(2.19)

\[
Q_2^{sp}(t, t') = \frac{\beta^2 J^2}{2N} \sum_i \langle \sigma_i(t) i \bar{\sigma}_i(t') \rangle = \frac{\beta^2 J^2}{2} G_{ii}(t, t'),
\]

(2.20)

\[
Q_3^{sp}(t, t') = \frac{\beta^2 J^2}{2N} \sum_i \langle i \bar{\sigma}_i(t) \sigma_i(t') \rangle = \frac{\beta^2 J^2}{2} G_{ii}(t', t),
\]

(2.21)
Dynamical Mean Field Theory of Spin Glasses

\[ M_{i}^{\text{sp}}(t) = \frac{\beta J_{0}}{N} \sum_{i} \langle \sigma_{i}(t) \rangle = \beta J_{0} m_{i}(t). \quad (2.22) \]

The last equality in the above equations holds true under a homogeneous field \( h_{i}(t) = h(t) \), and \( C_{ii}, G_{ii} \) and \( m_{i} \) become independent of the site \( i \). It should be noted that we put

\[ Q_{2}^{\text{sp}}(t, t') \propto \langle \delta_{i}(t) \delta_{i}(t') \rangle = 0, \quad (2.23) \]

\[ M_{2}^{\text{sp}}(t) \propto \langle \delta_{i}(t) \rangle = 0. \quad (2.24) \]

The reason for this is briefly mentioned in Appendix B.

In the case under homogeneous fields we thus end up with the following one-site generating functional:

\[ Z_{\text{one}}(l, \bar{l}) = \int \mathcal{D} \sigma \mathcal{D} \bar{\sigma} \exp(L_{\text{one}}(\sigma, \bar{\sigma}; l, \bar{l})), \quad (2.25) \]

\[ L_{\text{one}}(\sigma, \bar{\sigma}; l, \bar{l}) = -\int dt dt' \left\{ \frac{1}{T_{0}} \bar{\sigma}(t-t') + \frac{\beta^{2} J^{2}}{2} C(t, t') \right\} \bar{\sigma}(t) \bar{\sigma}(t') \]

\[ + \int dt \left\{ i \bar{\sigma}(t) \left\{ -\left[ \frac{1}{T_{0}} \bar{\sigma}_{l} + r_{0} \right] \sigma(t) - \frac{1}{2} u \sigma^{2}(t) + \beta h(t) \right\} \right. \]

\[ + \beta J_{0} m(t) + \beta^{2} J^{2} \int dt' G(t, t') \sigma(t') \left\{ -\frac{1}{2} \frac{\delta K}{\delta \sigma} \right\} \right\} \]

\[ + \int dt \left\{ l(t) \sigma(t) + \bar{l}(t) i \bar{\sigma}(t) \right\}. \quad (2.26) \]

The functions \( m(t), C(t, t') \) and \( G(t, t') \) are given by similar functional derivatives to Eqs. (2.12) \( \sim \) (2.14). In Appendix B we evaluate explicitly these functional derivatives and show that the dynamical problem of interest is described by the following modified Langevin equation for \( \sigma(t) \) (which corresponds to the extremum condition of \( L_{\text{one}}(\sigma, \bar{\sigma}; l = \bar{l} = 0) \) with respect to \( i \bar{\sigma}(t) \)):

\[ \int dt' G_{0}^{-1}(t, t') \sigma(t') = \left( \frac{1}{T_{0}} \bar{\sigma}_{l} + r_{0} \right) \sigma(t) - \beta^{2} J^{2} \int dt' G(t, t') \sigma(t') \]

\[ \quad = \beta h(t) + \beta J_{0} m(t) + \phi(t) - \frac{1}{2} u \sigma^{2}(t), \quad (2.27) \]

where \( \phi(t) \) is the modified noise with zero mean and variance

\[ \langle \phi(t) \phi(t') \rangle = \frac{2}{T_{0}} \delta(t-t') + \beta^{2} J^{2} C(t, t') = \Lambda^{(0)}(t, t'). \quad (2.28) \]

As compared with the original Langevin equations (2.4) \( \sim \) (2.6), the average over \( J_{0} \) introduces modifications both in the bare propagator \( (G_{0}(t, t') \) defined by Eq. (2.27)) and in the Gaussian noise (Eq. (2.28)). These equations are self-consistently coupled since the generating functional is evaluated by the steepest descent, i.e., within the mean field theory.
§ 3. Perturbational expansion

The equation of motion (2.27) for $\sigma(t)$ in the frequency representation is written as

$$\sigma(\omega) = \int_{\omega'} G_0(\omega, \omega') \left\{ g(\omega') + \phi(\omega') - \frac{u}{2} \int_{\omega_1} \int_{\omega_2} \sigma(\omega_1) \sigma(\omega_2) \sigma(\omega - \omega_1 - \omega_2) \right\},$$

where

$$(G_0^{-1})_{\omega\omega'} \equiv \tilde{\delta}(\omega - \omega')(\omega_0 - i\tilde{\omega}) - \tilde{\beta}^2 G(\omega, \omega'),$$

$$g(\omega) = \beta h(\omega) + \beta J_0 m(\omega)$$

with the abbreviations $f_\omega = f d\omega/2\pi$, $\tilde{\delta}(\omega) = 2\pi\delta(\omega)$, $\tilde{\omega} = \omega/\Omega_0$ and $\tilde{\beta} = \beta f$. Similarly Eq. (2.28) is written as

$$\langle \phi(\omega) \phi(\omega') \rangle = \frac{2}{I_0} \tilde{\delta}(\omega + \omega') + \tilde{\beta}^2 C(\omega, \omega') \equiv \Lambda^{(0)}(\omega, \omega').$$

To solve Eq. (3.1) iteratively, i.e., perturbatively with respect to $u$, we adopt a diagrammatic method with the following symbols:

- $\longrightarrow : \sigma(\omega)$, $\rightarrow : G_0(\omega, \omega')$, $\mid : g(\omega)$
- $\triangleleft : \phi(\omega)$, $\bullet : -u/2$, $\circ : \langle \phi(\omega) \phi(\omega') \rangle$.

For example, Eq. (3.1) is represented by

$$\boxed{\longrightarrow \equiv \hat{G}(\omega) \tilde{\delta}(\omega - \omega') \equiv [G(\omega, \omega') \text{ with } g(\omega) = 0] \Rightarrow \boxed{\longrightarrow + \longrightarrow \tilde{\Sigma} \Rightarrow \longrightarrow \Rightarrow}}.$$

In the absence of ac fields the time-translational invariance is recovered ($G(\omega, \omega') \propto \tilde{\delta}(\omega - \omega')$ and $C(\omega, \omega') \propto \tilde{\delta}(\omega + \omega')$). If the dc field and $J_0$ are also absent (i.e., $g(\omega) = 0$), we obtain the following set of self-consistent equations:

$$\boxed{\longrightarrow \rightarrow \rightarrow \equiv \hat{G}(\omega) \tilde{\delta}(\omega - \omega') \equiv [G(\omega, \omega') \text{ with } g(\omega) = 0] \Rightarrow \boxed{\longrightarrow + \longrightarrow \tilde{\Sigma} \Rightarrow \longrightarrow \Rightarrow}}.$$
Dynamical Mean Field Theory of Spin Glasses

\[ \hat{\Sigma}(\omega) = 3 \quad + 18 \quad + \ldots, \]  

(3.6)

\[ \Rightarrow \quad \equiv \hat{C}(\omega) \delta(\omega + \omega') = [C(\omega, \omega') \text{ with } g(\omega) = 0] \]

\[ = \hat{G}(\omega) \tilde{\Lambda}(\omega) \hat{G}(-\omega) \delta(\omega + \omega'), \]  

(3.7)

\[ \equiv \tilde{\Lambda}(\omega) = \bigcirc + 6 \quad + \ldots. \]  

(3.8)

In the last equation \( \equiv \) represents an \( n \)-th order vertex with an odd \( n \), such as

\[ \equiv \tilde{\Lambda}(\omega) = \bigcirc + 6 \quad + \ldots. \]  

(3.9)

Now let us introduce the \( g(\omega) \) vertex, or \( m(\omega) \). The latter itself is given by (where \( \rightarrow \) is abbreviated by \( \quad \)

\[ m(\omega) \equiv \quad \equiv \quad \equiv \quad + 3 \quad + \]  

(3.10)

with \( n \geq 3 \). One easily notices that parts of various terms in Eq. (3.10) are summed up to \( m(\omega) \) itself. Actually in terms of \( \hat{G}(\omega) \) of Eq. (3.5) and \( m(\omega) \), Eq. (3.10) is reduced to
Similarly the $g(\omega)$ vertex in the terms contributing to the full propagator $G(\omega, \omega')$ is renormalized to $m(\omega)$, and the Dyson equation for $G(\omega, \omega')$ becomes

$$
\equiv G(\omega, \omega') = \equiv \sum_{\omega_1, \omega_2} \tilde{\Sigma}(\omega_1, \omega_2), \quad (3\cdot11)
$$

$$
\tilde{\Sigma}(\omega, \omega') = 3 + 18 + 36 + 108 + 216 + \cdots + 54 + [\text{other } u^n \text{ terms } (n \geq 3)]. \quad (3\cdot10')
$$
The full correlation function $C(\omega, \omega')$ is given by

$$C(\omega, \omega') = m(\omega) m(\omega') = \int_{\omega_1} \int_{\omega_2} G(\omega, \omega_1) \Lambda(\omega_1, \omega_2) G(\omega', \omega_2) \equiv \tilde{C}(\omega, \omega')$$

(3.13)

We note that the fourth term in Eq. (3.14) is an example of terms which involve the second term of the 3rd order vertex (Eq. (3.9)).

From inspection of the above equations we notice the following way to employ the $m(\omega)$ terms. The rules are to replace (i) each $\tilde{G}(\omega)$ in Eqs. (3.6)~(3.9) by the full propagator $G(\omega, \omega')$ and (ii) each $\tilde{C}(\omega)$ by the full correlation function $C(\omega, \omega')$. In rule (ii), however, the following caution is required. The function $C(\omega, \omega')$ consists of $\tilde{C}(\omega, \omega')$ of Eq. (3.13) and $m(\omega) m(\omega')$. Each $\tilde{C}(\omega)$ has to be replaced by $\tilde{C}(\omega, \omega')$, but not necessarily by $m(\omega) m(\omega')$. Two $\tilde{C}$ attached to one interaction vertex have not to be replaced by $mm$ at the same time, since there remains by this procedure a part of a reducible propagator with the self-energy part of the first term in Eq. (3.12). Also more than one $\tilde{C}$ connecting the two vertex parts in Eq. (3.8) have not to be replaced by $mm$, since this procedure on a term with $n$-th vertex yields one term of $GAG$ with $(n-2)$-th vertex already counted. At the moment we have not yet succeeded to express these rules in certain mathematical formulae. But as for lowest order contributions (up to $O(\omega^3)$), we obtain the following formulae for the full propagator $G(\omega, \omega')$

$$\equiv = + \Sigma$$

(3.15)
\[ \Sigma(\omega, \omega') = \left[ 1 + m(\omega_1) m(\omega_2) \frac{\delta}{\delta \tilde{C}(\omega_1)} \right] \tilde{\Sigma}(\tilde{G}, \tilde{C} \to G, \tilde{C}) \]

\[ = 3 \quad +3 \quad 18 \quad +36 \quad +216 \quad \cdots \cdots \] (3·16)

and for the kernel \( \Lambda(\omega, \omega') \) in Eq. (3·13)

\[ \otimes = \Lambda(\omega, \omega') = \left[ 1 + m(\omega_1) m(\omega_2) \frac{\delta}{\delta \tilde{C}} \right] \tilde{\Lambda}(\tilde{G}, \tilde{C} \to G, \tilde{C}) \] (3·17)

Lastly the self-consistent equation for \( m(\omega) \) is read as

\[ \begin{multline} = \longrightarrow 1 \quad + \quad M_{1}^{\text{IR}} \quad + \quad M_{3}^{\text{IR}} \quad + \quad \cdots \cdots , \end{multline} \] (3·18)

where \( M_{2n+1}^{\text{IR}} \) are the skeleton diagrams, from which \( 2n+1 \) \( m(\omega) \)-branches come out. The first order skeleton diagram is given by

\[ M_{1}^{\text{IR}}(\omega, \omega') = \tilde{\Sigma}(\tilde{G}, \tilde{C} \to G, \tilde{C}) \] (3·19)

All the terms written explicitly in Eq. (3·10'), except for the 1st and 3rd ones, are included in the second term of Eq. (3·18). The third order skeleton diagrams are given by

\[ \begin{multline} \quad M_{3}^{\text{IR}} \quad = \quad \longrightarrow 1 \quad + \quad \longrightarrow \quad + \quad \cdots \cdots . \end{multline} \] (3·20)

Thus we end up with the set of self-consistent equations for \( G(\omega, \omega') \), \( C(\omega, \omega') \) and \( m(\omega) \) given by Eqs. (3·15), (3·13) and (3·18), respectively. For contributions up to \( \nu^2 \), \( \Sigma \), \( \Lambda \) and \( M^{\text{IR}} \) in these equations are given by Eqs. (3·16), (3·17) and (3·19) (and the first term of (3·20)), respectively. We note that there exists another equation (3·2), which relates \( G_0 \) with the full propagator \( G \).

§ 4. Spin-glass phase transition under an ac field

In this section we solve the set of self-consistent equations for the response function \( G(\omega, \omega') \), the correlation function \( C(\omega, \omega') \) and the magnetization \( m(\omega) \) derived in § 3 within the leading order in the interaction \( \nu \), i.e., within the single-loop approximation. Even in this simplest approximated scheme it is rather complicated to obtain solutions explicitly if the nonlinear effect of external ac fields is properly taken into account.
Dynamical Mean Field Theory of Spin Glasses

Within the single-loop approximation Eqs. (3·15), (3·13) and (3·18), respectively, are read as

\[ (G^{-1})_{\omega \omega} = (G_0^{-1})_{\omega \omega} + \frac{3u}{2} \int_{\omega_1} C(\omega_1, \omega - \omega' - \omega_1), \quad (4·1) \]

\[ C(\omega, \omega') = M(\omega, \omega') + \int_{\omega_1} \int_{\omega_2} G(\omega, \omega_1) A^{(0)}(\omega_1, \omega_2) G(\omega', \omega_2), \quad (4·2) \]

\[ \int_{\omega'} (G_0^{-1})_{\omega \omega'} m(\omega') = \tilde{h}(\omega) + \tilde{f}_0 m(\omega) - \frac{3u}{2} \int_{\omega_1} \int_{\omega_2} \tilde{C}(\omega_1, \omega_2) m(\omega - \omega_1 - \omega_2) \]

\[ - \frac{u}{2} \int_{\omega_1} \int_{\omega_2} m(\omega_1) m(\omega_2) (\omega - \omega_1 - \omega_2), \quad (4·3) \]

where \( \tilde{J}_0 = \beta f_0, M(\omega, \omega') = m(\omega)m(\omega'), \tilde{C}(\omega, \omega') = C(\omega, \omega') - M(\omega, \omega'), \) and \( A^{(0)}(\omega, \omega') \) is given by Eq. (3·4). Let us consider a case under the dc and ac fields specified by

\[ \tilde{h}(t) = \beta h(t) = h_0 + 2h_{ac}\cos vt \]

or

\[ \tilde{h}(\omega) = \tilde{h}_0 \delta(\omega) + \tilde{h}_{ac}[\delta(\omega - \nu) + \delta(\omega + \nu)]. \quad (4·4) \]

Then generally there appear components of the magnetization \( m_{nv} \) which are proportional to \( h_{ac}\exp(\nu vt) \) with \( |\nu| \geq |n| \) for any integer \( n \). Correspondingly \( G(\omega, \omega') \) and \( C(\omega, \omega') \) take the following forms:

\[ G(\omega, \omega') = G(\omega) \delta(\omega - \omega') + \delta G(\omega, \omega'), \quad (4·5) \]

\[ \delta G(\omega, \omega') = \sum_{|n| = 1}^{+} \delta G(\omega; n\nu) \delta(\omega - \omega' + n\nu), \quad (4·5') \]

\[ C(\omega, \omega') = C(\omega) \delta(\omega + \omega') + \delta C(\omega, \omega'), \quad (4·6) \]

\[ \delta C(\omega, \omega') = \sum_{|n| = 1}^{+} \delta C(\omega; n\nu) \delta(\omega + \omega' + n\nu). \quad (4·6') \]

The term proportional to \( \delta(\omega - \omega') \) or \( \delta(\omega + \omega') \) represents the time-translationally-invariant (TTI) part of the function \( G(t, t') \) or \( C(t, t') \), in which the nonlinear effect of the ac field is partially involved. The remaining terms \( \delta G \) and \( \delta C \) represent the non-time-translationally-invariant (NTTI) parts induced by the ac field. The substitution of the above equations into Eqs. (4·1)~(4·3) ends up with a set of an infinite number of equations. In the present work we restrict ourselves to the terms up to the order of \( \tilde{h}_{ac}^2(\tilde{h}_{ac}^3) \) in evaluating \( G(\omega, \omega') \) and \( C(\omega, \omega')(m(\omega)) \). Below we further restrict ourselves to the case with \( J_0 = 0 \) and \( \tilde{h}_0 = 0 \), i.e., a pure (soft) spin glass without a static field.25 In Appendix C, a pure ferromagnet \( (J = 0) \) is briefly discussed. More general cases \( (J \neq 0, J_0 \neq 0 \) and \( \tilde{h}_0 \neq 0) \) will be examined elsewhere.

For the pure spin glass with \( \tilde{h}_0 = 0 \), the static magnetization \( m_0 \) does not appear, and the leading order terms in Eqs. (4·5') and (4·6') are \( \delta G(\omega; \pm 2\nu) \) and \( \delta C(\omega; \pm 2\nu) \) which are proportional to \( \tilde{h}_{ac}^2 \). Therefore up to \( O(\tilde{h}_{ac}^2) \) Eq. (4·1) is simplified as

\[ G^{-1}(\omega) = G_0^{-1}(\omega) + \frac{3}{2} u C_0, \quad (4·7) \]
\[ \delta G(\omega, \omega') = -\frac{3}{2} u \Phi(\omega, \omega') \int_{\omega_1} \delta C(\omega_1, \omega - \omega' - \omega_1), \]  

where Eq. (3·2) has been used, and

\[ G_0^{-1}(\omega) = r_0 - i\omega - \beta^2 G(\omega), \]
\[ C_0 = \int_\omega C(\omega) = C(t = 0), \]
\[ \Phi(\omega, \omega') = \frac{G(\omega)G(\omega')}{1 - \beta^2 G(\omega)G(\omega')}. \]

Similarly Eq. (4·2) is reduced to

\[ \{1 - \beta^2 [G(\omega)]^2\} C(\omega) = M(\omega) + \frac{2}{T_0} |G(\omega)|^2, \]
\[ \{1 - \beta^2 G(\omega)G(\omega')\} \delta C(\omega, \omega') = \delta M(\omega, \omega') + G(\omega) A^{(0)}(\omega') \delta G(\omega', -\omega) + G(\omega, \omega') A^{(0)}(\omega') G(\omega'), \]  

where

\[ M(\omega) = |m_\nu|^2 \left[ \tilde{\delta}(\omega - \nu) + \tilde{\delta}(\omega + \nu) \right], \]
\[ \delta M(\omega, \omega') = m_\nu^2 \tilde{\delta}(\omega + \omega' + 2\nu) \tilde{\delta}(\omega + \nu) + m_\nu^2 \tilde{\delta}(\omega + \omega' - 2\nu) \tilde{\delta}(\omega - \nu), \]
\[ A^{(0)}(\omega) = \frac{2}{T_0} + \beta^2 C(\omega). \]

Making use of Eqs. (3·2) and (4·7), we rearrange Eq. (4·3) as

\[ m(\omega) = G(\omega) \left\{ \tilde{h}(\omega) - \frac{3}{2} u \int_{\omega_1} \int_{\omega_2} \delta C(\omega_1, \omega_2) m(\omega - \omega_1 - \omega_2) \right. \]
\[ \left. - \frac{u}{2} \int_{\omega_1} \int_{\omega_2} m(\omega_1) m(\omega_2) m(\omega - \omega_1 - \omega_2) + \beta^2 \int_{\omega_1} \delta G(\omega, \omega_1) m(\omega_1) \right\}, \]  

for \( \omega \neq 0, \pm \nu \), where \( \delta C = \delta C - \delta M \). The integration with a prime in the third term indicates that the contribution \(- (3u/2) \int_\omega m(\omega_1) m(\omega - \omega_1) m(\omega) \) has to be subtracted from the integration, because it is already counted to obtain \( G(\omega) \) instead of \( G_0(\omega) \) in Eq. (4·17) through Eq. (4·7). Since the last three terms in Eq. (4·17) do then not involve terms linearly proportional to \( m(\omega) \), \( G(\omega) \) represents the linear response of the system against a probing field with frequency \( \omega \) in the presence of the ac field of Eq. (4·4).

The TTI parts, Eqs. (4·7) and (4·12), are solved by putting \( C(\omega) \) in the form

\[ C(\omega) = q \tilde{\delta}(\omega) + \frac{1}{2} C_1[\tilde{\delta}(\omega - \nu) + \tilde{\delta}(\omega + \nu)] + C_1(\omega). \]

Substituting this form into Eq. (4·12), we obtain \( C_1(\omega) = 2 \Phi(\omega)/T_0 \) and \( C_v = 2 \tilde{h}_{ac}^2 \Phi(\nu) \), where \( m_\nu = G(\nu) \tilde{h}_{ac} \) is used and \( \Phi(\omega) = \Phi(\omega, -\omega) \). The freezing parameter \( q \) is determined by \( \{1 - \beta^2 G(0)\} q = 0 \), i.e., by \( \beta G(0) = 1 \) in the spin glass phase. To derive an explicit expression for \( G(0) \), it is important to notice that \( C_1(\omega) \) satisfies \( C_1(\omega) = (2/\omega) \times \text{Im} G(\omega) \), which indicates that an apparent fluctuation-dissipation theorem holds between \( C_1(\omega) \) and \( G(\omega) \). Then \( G(0) \) can be related to \( C_0 \) of Eq. (4·10) through the Krammers-
Kronig relation as
\[ G(\omega=0) = \int_0^\infty \frac{2}{\omega} \text{Im} G(\omega) = \int_0^\infty C_1(\omega) = C_0 - q - C_v. \] (4.19)

The value \( C_0 \) is put unity by the following argument.\(^{19}\) From Eq. (4.6) we obtain
\[ C(t=t') = C_0 + \sum_{n>0} \left( \int_0^\infty \delta C(\omega; n\nu) \right) e^{i\nu t}. \] (4.20)

If we introduce the spherical condition in the sense
\[ [C(t=t')]_t = \left( \frac{1}{N} \sum \sigma^2(t) \right)_t = 1, \] (4.21)
where \([ \ ]_t\) means the long time average, we obtain \( C_0 = 1 \). If, on the other hand, the spherical condition is imposed at each instant of time, Eq. (2.4) has to be solved with this restriction. In the present work we use the word “spherical” in the sense of Eq. (4.21).

The response function \( G(\omega) \) is identical to its counterpart in the unperturbed (spherical) limit, Eq. (B.12) in Appendix B, if the parameter \( r_0 \) in the latter is replaced by \( r = r_0 + 3u C_0/2 \). This replacement is physically irrelevant since \( r \) (or \( r_0 \) in the spherical limit) is written in terms of \( C_0(=1) \), \( q \) and \( C_v \) by means of Eq. (4.19). Since \( C(\omega) \) is also identical to its spherical limit, properties derived from the TTI parts of \( G(\omega, \omega') \) and \( C(\omega, \omega') \) within the single-loop approximation are common to the spherical spin glass.\(^{26,27}\) In the latter model, however, the mode-mode coupling effect yielding ac nonlinear magnetizations with higher frequencies \( n\nu \) is absent.

The spin glass phase is specified by \( q \neq 0 \), hence \( G(0) = \tilde{\beta}^{-1} \) as mentioned before and \( r = 2 \tilde{\beta} \). Then \( q \) is determined from Eq. (4.19), \( \tilde{\beta}^{-1} = C_0 - q - C_v \), whose limit in \( q \to 0 \) yields the transition temperature \( T_c \) in the presence of the ac field. With \( C_0 = 1 \), \( T_c \) is given by
\[ T_c = 1 - \frac{\tilde{\nu}^\alpha}{(16 + \tilde{\nu}^2)^{1/2} + 2|\tilde{\nu}|/2|\tilde{\nu}|/2^{1/2} - 1} \]
\[ \simeq 1 - \sqrt{2} \tilde{h}_{ac} \tilde{\nu}^{-1/2}, \quad (|\tilde{\nu}| \ll 1) \] (4.22)
where \( T = T/J, \tilde{h}_{ac} = h_{ac}/J \) and \( \tilde{\nu} = \nu/\beta \Gamma_0 / J \) (note that from the definition of the original equations (2.4) and (2.5), \( \beta \Gamma_0 \) is the ordinary damping constant independent of \( T \)). The above equation is one of the important results in this section, and will be discussed in § 5.

With \( r = 2 \tilde{\beta} \) \( G(\omega) \), which is given by Eq. (B.12) with \( r_0 \to r \), is easily evaluated. Its imaginary part is proportional to \( \omega^{1/2} \) in the limit \( \omega \to 0^{12,18} \), so that the spin relaxation time \( \tau \) defined by
\[ \tau = - \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G^{-1}(\omega) \propto \Phi(\omega \to 0) \] (4.23)
is always diverging in the spin glass phase.

In the paramagnetic phase \( q = 0 \), a key parameter is \( \varepsilon \) which is defined by \( 2\varepsilon = 1 - \tilde{\beta}^2 G^2(0) \approx 2(T - T_c)/T_c \) and represents a distance of the state from \( T_c \) of Eq. (4.22). At high temperatures (or for low frequencies) but still near \( T_c \) in the sense \( 1 \gg \varepsilon \gg |\tilde{\omega}|/\tilde{\beta} \), \( G(\omega) \) is given by
\( G(\omega) \approx G(0) + \frac{1}{2\beta^2} \left( \frac{i\tilde{\omega}}{\varepsilon} - \frac{\tilde{\omega}^2}{4\beta^2\varepsilon^3} \right) \), \hspace{1cm} (4.24)

so that

\[ \Phi(\omega) = \frac{|G(\omega)|^2}{1 - \beta^2|G(\omega)|^2} \approx \frac{\varepsilon}{2\beta^2\varepsilon^3 + \tilde{\omega}^3/\beta^2\varepsilon^3}. \] \hspace{1cm} (4.25)

The above equation tells that the relaxation time \( \tau \) of Eq. (4.23) is diverging as \( \varepsilon^{-1} \), which indicates an instability of the paramagnetic phase at \( T_c \). In the opposite limit \( \varepsilon^2 \ll |\tilde{\omega}|/\beta \), \( G(\omega) \) and \( \Phi(\omega) \) behave as in the spin glass phase, in particular

\[ \Phi(\omega) \approx (2\beta^3\tilde{\omega})^{-1/2}. \] \hspace{1cm} (4.26)

Next we consider the NTTI parts \( \delta G(\omega; \pm 2\nu) \) and \( \delta C(\omega; \pm 2\nu) \). From inspection of Eqs. (4.13)~(4.16) and (4.18), we put \( \delta C(\omega; \pm 2\nu) \) in the form

\[ \delta C(\omega; \pm 2\nu) = \delta C_0^{-}[\delta^2(\omega) + \delta(\omega \pm 2\nu)] + \delta C_{\pm}^{-}\delta(\omega \pm \nu) + \delta \tilde{C}(\omega; \pm 2\nu). \] \hspace{1cm} (4.27)

(\( \delta C_0, \delta C_{\pm} \) and \( \delta \tilde{C} \) correspond to \( \delta \tilde{C}_3, \delta \tilde{C}_4 \) and \( \delta \tilde{C}_4 \) in Ref. 25), respectively.) Substituting Eq. (4.27) into Eq. (4.13) we obtain

\[ \delta C_0^{-} = \frac{\beta^2 G(0)}{1 - \beta^2 G(0)G(\mp 2\nu)} \delta G(\mp 2\nu; \pm 2\nu)q, \] \hspace{1cm} (4.28)

\[ \delta C_{\pm}^{-} = \frac{1}{1 - \beta^2 G(\pm \nu)G(\pm \nu)} m_{\pm \nu}^2, \] \hspace{1cm} (4.29)

and the continuous part \( \delta \tilde{C}(\omega; \pm 2\nu) \) as

\[ \delta \tilde{C}(\omega; \pm 2\nu) = \Phi(\omega, \omega') \left\{ \frac{B_r(\omega')}{G(\omega')} \delta G(\omega; \pm 2\nu) + \frac{B_l(\omega)}{G(\omega)} \delta G(\omega'; \pm 2\nu) \right\}. \] \hspace{1cm} (4.30)

where \( \omega' = -\omega \mp 2\nu \) and \( B_l(\omega) = (2/\Gamma_0) + \beta^2 C_1(\omega) \). On the other hand Eq. (4.8) reduces to

\[ \delta G(\omega; \pm 2\nu) = -\frac{3}{2} u \Phi(\omega, \omega \pm 2\nu) \left\{ 2\delta C_0^{-} + \delta C_{\pm}^{-} + \int_{\omega_1} \delta \tilde{C}(\omega; \pm 2\nu) \right\}. \] \hspace{1cm} (4.31)

Since an algebraic equation for \( \int_{\omega} \delta \tilde{C}(\omega; \pm 2\nu) \) is derived from Eqs. (4.30) and (4.31), the above set of equations are solved easily. The obtained results are substituted into Eq. (4.17) with \( \omega = 3\nu \) and \( \nu \cdot \) (For \( \omega = \nu \), the frequency of the ac field applied, we have to add \( 3u^2/2|m| m_\nu \) in the curly bracket of Eq. (4.17).) Then \( m_{3\nu} \) and \( m_\nu \) are given by

\[ m_{3\nu} = -\frac{3}{2} u \cdot G(3\nu) m_\nu \left\{ \frac{1}{3} + \frac{\tilde{\beta}^2}{A(2\nu)} \left( 1 - \frac{m_\nu^2}{A(2\nu)} \right) \right\}, \] \hspace{1cm} (4.32)

\[ m_\nu = G(\nu) \left\{ \tilde{C}_{ac} - \frac{3}{2} u m_\nu m_{-\nu} \left\{ \frac{\tilde{\beta}^2}{A(2\nu)} \left( 1 - \frac{m_\nu^2}{A(2\nu)} \right) \right\} \right\}, \] \hspace{1cm} (4.33)

where

\[ \tilde{A}(2\nu) = (1 - \beta^2 G^2(\nu)) \left[ 1 + A(2\nu) + \frac{3u\tilde{\beta}^2}{G(2\nu)} \Phi(2\nu, 0) q \right], \] \hspace{1cm} (4.34)

\[ A(\nu) = \frac{3}{2} u \int_{\omega} \Phi(\omega, \omega') \left\{ \frac{B_r(\omega)}{G(\omega')} \Phi(-\omega, \omega') + \frac{B_l(\omega)}{G(\omega)} \Phi(\omega, -\omega') \right\}. \] \hspace{1cm} (4.35)
with $\omega' = -\omega - \Omega$. Within the present perturbational analysis with respect to $\vec{h}_{ac}$, $m_{\pm \nu}$ in the r.h.s. of Eqs. (4·32) and (4·33) are replaced by $G(\pm \nu)\vec{h}_{ac}$.

Singular behaviors on the ac nonlinear susceptibilities $\chi(3\nu) \approx m_{3\nu}/\vec{h}_{ac}$ and $\delta\chi(\nu) \approx (m_{\nu} - G(\nu)\vec{h}_{ac})/\vec{h}_{ac}$ are estimated by means of the asymptotic expressions such as Eqs. (4·24) $\sim$ (4·26). The constant $A(\Omega)$ of Eq. (4·35) is proportional to $u|\vec{\beta}\vec{\Omega}|^{-1/2}$ ($u|\vec{\beta}\vec{\varepsilon}$) in the critical region $\varepsilon^2 \ll (\nu|\Omega|$. Also we have to notice that in the spin glass phase $\tilde{A}(2\nu)$ of Eq. (4·34) is proportional to the last term with $q$ when $q^2 \gg |\vec{\nu}|$. Combining these results we obtain $\chi(3\nu) \approx \delta\chi(\nu)$ and

i) above $T_c$ ($1 \gg \varepsilon^2 \gg |\vec{\nu}|$)

$$\chi(3\nu) \approx -\varepsilon^{-1}, \quad (4\cdot36a)$$

ii) in the close vicinity of $T_c$ ($|\vec{\nu}| \gg \varepsilon^2$ or $q^2$)

$$\chi(3\nu) \approx -|\vec{\nu}|^{-1/2}, \quad (4\cdot36b)$$

iii) below $T_c$ ($q^2 \gg |\vec{\nu}|$)

$$\chi(3\nu) \approx -q^{-1}. \quad (4\cdot36c)$$

It is noted that the above results come out after cancellation of some diverging terms in $\tilde{A}(2\nu)$ and $\Phi(n\nu, mw)$ in Eqs. (4·32) and (4·33), and they are proportional to $O(u^0)$. What we have done is that although we restrict ourselves to the single-loop approximation scheme, we solve the set of self-consistent equations in the scheme, Eqs. (4·1) $\sim$ (4·3), by specifying the parameter $u$ no longer small. If we assume that $u$ is the smallest parameter in Eq. (4·32), on the other hand, we obtain $\chi(3\nu) \approx -u\varepsilon^{-2}$ which does not agree with the $\varepsilon^{-1}$-dependence of the static nonlinear susceptibility.27

§ 5. Discussion

We have reexamined the dynamical mean field theory of spin glasses in order to apply it to ac nonlinear phenomena. Within the single-loop approximation with respect to the interaction $u\alpha^4$ introduced in the soft spin version of the SK spin glass model we have explicitly investigated relaxational dynamics of spins near the phase transition point. For comparison the corresponding argument for a ferromagnet is presented in Appendix C.

One of the most important results is that the transition temperature $T_c$ is lowered by an ac external field of frequency $\nu$ as described by Eq. (4·22) for the spin glass and by Eq. (C·7) for the ferromagnet. It is emphasized here that, even in the presence of the ac field, $T_c$ is well-defined by the instability of the paramagnetic phase (e.g., divergence of $\tau$ of Eq. (4·23)) as well as by an occurrence of a static order parameter $q$ or $m_0$ below $T_c$. But the responses to the ac field of frequency $\nu(>0)$ itself are not divergent at $T_c$. In the spin glass the nonlinear susceptibility $\chi(3\nu)$ is bounded proportionally to $-\nu^{-1/2}$ (Eq. (4·36b)). The nonlinear responses of the ferromagnet are summarized on Table I. The instability mentioned above can be observed in principle by additional probing field with infinitesimal amplitude in the static limit.

The fact that the shift of $T_c$ vanishes in the limit $\nu \to \infty$ is simply because spins cannot follow such rapidly changing field, i.e., they do not feel the field. For the ferromagnet we
can see from Eq. (C·3) that $2|m_v|^2$ induced by the ac field plays the same role as $m_0^2$, namely, the effectively frozen (to the ac field) component of spins which do not respond even to the mutual exchange field. This explains the shift of $T_c$ given by Eq. (C·7). For the spin glass the corresponding frozen (non-responding) component of spins is represented by $C_\nu$ in Eq. (4·19). The different behavior of $C_\nu$ from that of $2|m_v|^2$ in the ferromagnet comes out from two origins. One is that in the critical region $G(\omega)-G(0)$ is proportional to $\omega^{1/2}(\omega)$ for the spin glass (ferromagnet). The other is that the generalized susceptibility conjugate to the freezing parameter $q$ is not $G(\omega)$ but essentially $\Phi(\omega)-\omega$ of Eq. (4·11). These explain the shift of $T_c$ given by Eq. (4·22).

The phase diagram specified by Eq. (4·22) is shown in Fig. 1. In the limit $\nu \rightarrow 0$ the spin glass phase disappears if the amplitude of the field remains finite. This corresponds to vanishing of the phase transition in the spherical spin glass under a static field.\(^{26},^{27}\) We know, on the other hand, that the spin glass phase of the SK Ising spin glass is not broken by a static field below the AT line\(^{28}\) drawn by the chain line in Fig. 1. To clarify the relation between the AT line and the static limit of the ac nonlinear effects investigated in the present work we have to go beyond the single-loop approximation, at least up to $O(u^2)$ where a critical line corresponding to the AT line can be reproduced under a static field.\(^{19}\) Such higher order analysis is also required to confirm that the new critical line given by Eq. (4·22) is not restricted to the spherical spin glass. Since, however, the critical line is concave the higher order effects in $u$ are shown to be not crucial so long as $\tilde{h}_{ac}$ is sufficiently small. We therefore expect that the new critical line predicted in this work is common to the SK Ising spin glass and even to real spin glasses. In fact $\chi(3\nu)$ of $(Ti_{1-x}V_x)O_3$ measured by Chikazawa et al.\(^{24}\) exhibits the higher peak at the lower temperature under the measuring field with the lower frequency. This behavior agrees qualitatively with the results obtained in the present work.

Finally we note that the present analyses differs from those analyses such as by Togashi and Suzuki\(^{17}\) in the following respect. In the latter the system is considered to be driven by an ac field along the free energy surface which is determined without the ac field. In particular below $T_{c0}=T_c(h_{ac}=0)$, even in its very vicinity, the system is supposed to oscillate around one of the minima of the symmetry-broken free energy. In the present work, on the other hand, we consider that the ac field with finite amplitude hinders occurrence of the ordered state, and that the (replica) symmetry breaking in the free energy surface (and so a finite $q$) appears only below $T_c(h_{ac})<T_{c0}$. For quantitative arguments on the present results we have to specify an explicit value of $T_c$ which has not yet been well examined. As for dynamical
properties near and below $T_c$ (or $T_{co}$) some extrinsic effects not included in our basic
equations, such as the domain effect as in ferromagnets, might be crucial and complicate
the problem. Further investigations are certainly required to clarify dynamical nature,
including ac nonlinear properties, of the spin glass transition.

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Appendix A

--- Functional Jacobian $J[\sigma]$ ---

The functional Jacobian $J[\sigma]$ in Eq. (2·9) is formally expressed as

$$J[\sigma]=\det\left[\frac{\delta}{\delta \sigma_v(t)} [\Gamma_0^{-1} \partial_t \sigma(t) - K(\sigma(t)) - \xi(t)]\right],$$

where vector notations $\sigma(t)\equiv(\sigma_1(t), \sigma_2(t), \cdots, \sigma_N(t))^T$, etc., are introduced. To be more
explicit we discretize the $t$-space by $\Delta t$ and integrate the Langevin equation (2·4)
between the interval $t_\mu \leq t \leq t_{\mu+1}(=t_\mu+\Delta t)$ as,

$$\Gamma_0^{-1}[\sigma(t_{\mu+1})-\sigma(t_\mu)]=\Delta t[aK\{\sigma(t_{\mu+1})\}+bK\{\sigma(t_\mu)\}]+\eta_\mu,$$

where

$$\eta_\mu=\int_{t_\mu}^{t_{\mu+1}} dt \xi(t),$$

and $a+b=1$ with $a, b>0$. Then Eq. (A·1) is interpreted as (note $\delta \xi/\delta \sigma=0$)

$$J[\sigma]=\det\left[\Gamma^{-1}_0(\sigma_\mu-\sigma_{\mu-1})-\Delta t(aK_\mu+bK_{\mu-1})\right]$$

$$=\det[\Gamma_0^{-1}\tilde{D}-\Delta t\tilde{M}\tilde{A}]$$

$$=\det[\Gamma_0^{-1}\tilde{D}]\cdot\det[\tilde{I}-\Gamma_0\Delta t\tilde{A}],$$

where we have introduced the abbreviations $\sigma_\mu=\sigma(t_\mu)$, $K_\mu=K\{\sigma(t_\mu)\}$, and the following
matrix representations:

$$(\tilde{D})_{\mu\nu}=(\delta_{\mu,\nu}-\delta_{\mu-1,\nu})1,$$

$$(\tilde{M})_{\mu\nu}=(a\delta_{\mu,\nu}+b\delta_{\mu-1,\nu})1,$$

$$(\tilde{A})_{\mu\nu}=\frac{\delta K_\nu}{\delta \sigma_\nu},$$

(A·5)
and $\tilde{\Theta} \equiv \tilde{D}^{-1}\tilde{M}, 1$ being the $N \times N$ unit matrix. The factor $\text{det} | I_0^{-1} \tilde{D} |$ in the last expression of Eq. (A·4) is a constant $C$ independent of $\sigma$. The other factor in it is formally evaluated as

$$
\text{det} [ I - I_0 \Delta t \tilde{\Theta} \cdot \tilde{A}] = \exp \{ \text{Tr} \ln ( I - I_0 \Delta t \tilde{\Theta} \cdot \tilde{A}) \} = \exp \left\{ - \sum_{k=1}^{\infty} \frac{1}{k} (I_0 \Delta t)^k \text{Tr} (\tilde{\Theta} \tilde{A})^k \right\}. \quad (A·6)
$$

From the matrix structures given by Eq. (A·5) it is easily checked that $\tilde{\Theta}$ is given by

$$
(\tilde{\Theta})_{\mu\nu} = \begin{cases} 1, & (\mu > \nu) \\ a1, & (\mu = \nu) \\ 0, & (\mu < \nu) \end{cases}
$$

and so $\tilde{\Theta} \cdot \tilde{A}$ is retarded (i.e., it is a similar triangular matrix with respect to the $t$-space). Therefore the magnitude of $( (\tilde{\Theta} \cdot \tilde{A})^k )_{\mu\nu}$ is of the order of unity and so the expansion in Eq. (A·6) terminates at $k=1$ in the limit $\Delta t \to 0$. Thus we obtain finally

$$
F(\sigma) = C \exp \{ - I_0 \Delta t \text{Tr} (\tilde{\Theta} \cdot \tilde{A}) \}
$$

where $\text{Tr}'$ indicates the trace in the site-space. With $a=1/2$, Eq. (A·7) is identical to the corresponding term in Eqs. (2·10) and (2·11). It is noted that within the present argument the constant $a$ originates from Eq. (A·2), namely, how we discretize the $t$-space. Its value can be arbitrary so long as $0 < a < 1$, but the same value has to be used when we specify the equal-time response function as in Eq. (B·14) in Appendix B.

Now we present a brief argument that $a=b=1/2$ has to be taken in Eq. (A·2). Let us consider the following conditional probability function,

$$
P(\sigma_{\mu+1}, t_{\mu+1} | \sigma_{\mu}, t_{\mu}) = \mathcal{N}_{\Delta t} \langle \delta ( I_0^{-1} [\sigma_{\mu+1} - \sigma_{\mu}] - \Delta t (aK_{\mu+1} + bK_{\mu}) - \eta_{\mu} ) \rangle_\eta
$$

$$
= \mathcal{N}_{\Delta t} \exp \left\{ - \frac{I_0 \Delta t}{4} \left[ \frac{\sigma_{\mu+1} - \sigma_{\mu}}{I_0 \Delta t} - aK_{\mu+1} + bK_{\mu} \right]^2 \right\}, \quad (A·8)
$$

where we have used the fact that $\eta_{\mu}$ is a Gaussian random variable with variance $\langle \eta_{\mu} \cdot \eta_{\mu}^T \rangle = 2I_0^{-1} \Delta t \delta_{\mu\nu} 1$, which is easily seen from Eqs. (A·3) and (2·6). Equation (A·8) is valid for the linear order in $\Delta t$. The normalization constant $\mathcal{N}_{\Delta t}$ is fixed by the condition

$$
\int d^N \sigma_{\mu+1} P(\sigma_{\mu+1}, t_{\mu+1} | \sigma_{\mu}, t_{\mu}) = 1.
$$

We impose that the conditional probability function bears the Markov property, i.e., it satisfies the Chapman-Kolmogorov equation

$$
P(\sigma_{\mu+2}, t_{\mu+2} | \sigma_{\mu}, t_{\mu}) = \int d^N \sigma_{\mu+1} P(\sigma_{\mu+2}, t_{\mu+2} | \sigma_{\mu+1}, t_{\mu+1}) P(\sigma_{\mu+1}, t_{\mu+1} | \sigma_{\mu}, t_{\mu}). \quad (A·9)
$$

The substitution of Eq. (A·8) into the r.h.s. of Eq. (A·9) yields $P(\sigma_{\mu+2}, t_{\mu+2} | \sigma_{\mu}, t_{\mu})$. The condition $a=b=1/2$ follows from the requirement that this $P(\sigma_{\mu+2}, t_{\mu+2} | \sigma_{\mu}, t_{\mu})$ has to be identical to Eq. (A·8) with the replacements $\Delta t \to 2\Delta t, \mu+1 \to \mu+2$. Note that the functional Jacobian $F(\sigma)$ is also obtained from Eq. (A·9) by
More precisely, \( Z_I \) is evaluated as \(^{22}\)

\[
Z_I = \int \prod \mu d^n \sigma_{n+1} P(\sigma_{n+1}, t_{n+1} | \sigma_n, t_n)
\]

\[
= \int \mathcal{D} \sigma \exp \left\{ -a \Gamma_0 \int dt \ Tr \left( \frac{\delta K}{\delta \sigma} \right) - \frac{\Gamma_0}{4} \int dt | \Gamma_0^{-1} \partial_t \sigma - K(\sigma) |^2 \right\},
\]

which is essentially identical with Eqs. (2.10) and (2.11) with \( \ell = \bar{\ell} = 0 \).

**Appendix B**
---

**Evaluation of Generating Functional**

By the Fourier transformations such as

\[
\sigma(\omega) = \sigma_0 = \int dt \ e^{i\omega t} \sigma(t), \quad \xi_0 = \int dt \ e^{-i\omega t} \partial_t \sigma(t),
\]

\( L_{\text{one}}(\sigma, \bar{\sigma}; \ell, \bar{\ell}) \) of Eq. (2.26) is rewritten as

\[
L_{\text{one}}(\sigma, \bar{\sigma}; \ell, \bar{\ell}) = L^{(0)}(\ell, \bar{\ell}) + L_{\text{int}},
\]

where

\[
L^{(0)}(\ell, \bar{\ell}) = \int_0^\ell \int_0^{\bar{\ell}} \left\{ \frac{1}{2} \xi_0 A^{(0)}_{\omega \omega'} \xi_0 - \xi_0 (G_0^{-1})_{\omega \omega'} \sigma_0 + \int_0^\ell \int_0^{\bar{\ell}} \xi_0 + l_0 \sigma_0 \right\},
\]

and the interaction part \( L_{\text{int}} \) is given by

\[
L_{\text{int}} = \int dt \left\{ -\frac{\omega}{2} \xi(t) \sigma^2(t) + \frac{3}{4} \mu \Gamma_0 \sigma^2(t) \right\}.
\]

For other notations in Eq. (B.2) see § 3 in the text.

The unperturbed part of the generating functional \( Z^{(0)}(\ell, \bar{\ell}) \) is defined and evaluated as

\[
Z^{(0)}(\ell, \bar{\ell}) = \int \mathcal{D} \sigma \mathcal{D} \bar{\sigma} \exp(L^{(0)}(\ell, \bar{\ell})) = C \bar{Z}^{(0)}(\ell, \bar{\ell}),
\]

\[
\bar{Z}^{(0)}(\ell, \bar{\ell}) = \exp \left( \int_0^\ell \int_0^{\bar{\ell}} \left\{ \frac{1}{2} l_0 \xi_0 A^{(0)}(\omega) \xi_0 + l_0 G_0(\omega, \omega') \right\} \right),
\]

where \( C \) is a constant independent of \( \ell \) and \( \bar{\ell} \), and

\[
\Xi_{\omega \omega'} = \int_0^\ell \int_0^{\bar{\ell}} G_0(\omega, \omega') A^{(0)}(\omega_1, \omega_2) G_0(\omega', \omega_2).
\]

Note that \( \bar{Z}^{(0)}(\ell = \bar{\ell} = 0) = 1 \). Various averages such as Eqs. (2.12) \~ (2.14) in the unperturbed state are evaluated by the functional derivative of \( \bar{Z}^{(0)}(\ell, \bar{\ell}) \), or equivalently from \( \exp(L^{(0)}(\ell = \bar{\ell} = 0)) \) which is a Gaussian form in \( \sigma \) and \( \bar{\sigma} \):

\[
\langle \sigma_0 \rangle^{(0)} = m^{(0)}(0) = \int_0^\ell G^{(0)}(\omega, \omega') g(\omega'),
\]

\[
\langle \sigma_0 \xi_0 \rangle^{(0)} = G^{(0)}(\omega, \omega') = G_0^{(0)}(\omega, \omega'),
\]
\[
\langle \sigma_\omega \sigma_{\omega'} \rangle^{(0)} = C^{(0)}(\omega, \omega') = \Sigma^{(0)}(\omega, \omega') + m^{(0)}(\omega)m^{(0)}(\omega'),
\]
(B·9)
\[
\langle \zeta_\omega \rangle^{(0)} = \langle \zeta_\omega \zeta_{\omega'} \rangle^{(0)} = 0,
\]
(B·10)
where \(\Sigma^{(0)}\) is given by Eq. (B·6) with \(G_0\) replaced by \(G^{(0)}_0\). Equation (B·8) is self-consistent since \(G_0\) is a function of \(G\) through Eq. (3·2). In the unperturbed state the two propagators are identical (Eq. (B·8)), and are solved as
\[
G^{(0)}(\omega, \omega') = G^{(0)}(\omega, \omega) \delta(\omega - \omega'),
\]
(B·11)
\[
G^{(0)}(\omega) = \frac{1}{2\tilde{\beta}^2} \{ r_0 - i\tilde{\omega} - [(r_0 - i\tilde{\omega})^2 - 4\tilde{\beta}^2]^{1/2} \}.
\]
(B·12)
This \(G^{(0)}(\omega)\) has a branch cut between \(\tilde{\omega} = -i(r_0 \pm 2\tilde{\beta})\) and is proportional to \(|\tilde{\omega}|^{-1}\) in the limit \(|\tilde{\omega}| \to \infty\). Therefore if \(r_0 > 2\tilde{\beta}\) is imposed, the response function is given by
\[
G^{(0)}(t, t') = G^{(0)}(t-t')
\]
\[
= \theta(t-t') \cdot \Gamma_0 \int_0^{4\tilde{\beta}} \frac{dz}{2\pi \tilde{\beta}^2} (4\tilde{\beta}z - z^2)^{1/2} \exp\{-\Gamma_0(z + r_0 - 2\tilde{\beta})(t-t')\},
\]
(B·13)
where \(\theta(t-t')\) is the step function, which describes the causality of the response. Corresponding to the prefactor of the functional Jacobian (see Appendix A), we put
\[
G^{(0)}(t = t') = \frac{1}{2} G^{(0)}(t \to t' + 0^+) = \frac{\Gamma_0}{2}.
\]
(B·14)
In order to explain typical features of the perturbative expansion with respect to \(L_{\text{int}}\), let us consider the first order correction to \(m(t)\). Making use of the fact that \(\exp(L^{(0)}(l = \overline{l} = 0))\) is a Gaussian weight, we obtain
\[
\delta < \sigma(t)^{(1)} > \propto \frac{\mu}{2} \int dt' \int d\sigma d\overline{\sigma} [ - G^{(0)}_0(t, t') \sigma(t')
\]
\[
- 6C^{(0)}(t, t')G^{(0)}_0(t', t')\sigma(t') + 3\Gamma_0 C^{(0)}(t, t')\sigma(t')] \exp(L^{(0)}(l = \overline{l} = 0)).
\]
(B·15)
Because of Eq. (B·14) the last two terms in Eq. (B·15) cancel exactly with each other. This type of cancellation occurs in every order of the expansion. In other words, in perturbational analysis \(L_{\text{int}}\) of Eq. (B·3) has to be reread as
\[
L_{\text{int}} = -\frac{\mu}{2} \int dt' G^{(0)}_0(t, t')\sigma(t'),
\]
(B·16)
where \(t\) indicates any other external or interaction vertex, with which \(\sigma(t)\) associates. This exactly corresponds to the role of the last term in r.h.s. of Eq. (2·27) or (3·1).

By means of the causality of the unperturbed response function, Eq. (B·13), as well as the cancellation between terms with \(G^{(0)}_0(t, t)\) and those from the functional Jacobian \(J[\sigma]\), the causality of the true response function \(G(t, t')\) is ensured. Also we can see Eqs. (2·23) and (2·24) hold true, which tells that they are in fact one of the self-consistent solutions for the entire generating functional of Eq. (2·18). If we apply the above perturbational argument on \(Z(j, \overline{l})\), i.e., before averaging over \(J_{\overline{\sigma}}\), we still obtain \(\langle \overline{\sigma}_i(t) \rangle = 0\) and \(\langle \overline{\sigma}_i(t) \overline{\sigma}_j(t') \rangle = 0\), because the structures of \(L[\sigma, \overline{\sigma}]\) of Eq. (2·11) and \(L^{(0)}(l, \overline{l}) + L_{\text{int}}\) in Eq.
(B·1) are essentially identical, although the response function (or propagator) depends also on sites in the former case. This provides another reasoning for Eqs. (2·23) and (2·24).

**Appendix C**

--- Relaxational Dynamics of Ferromagnets ---

Here we solve Eqs. (4·1)–(4·3) in the text for a ferromagnetic case ($J = 0$). In this case the self-consistent conditions originated from the average over $J_{ij}$ disappear so that $G_0(\omega)$ is simply given by $(\{\omega - i\omega\})^{-1}$, and $C(\omega, \omega')$ is written in terms of $G(\omega, \omega')$ and $m(\omega)m(\omega')$. The self-consistency is required in determining $G(\omega, \omega')$ and $m(\omega)$. In the present case, Eq. (A·7) is reduced to

$$G^{-1}(\omega) + G^{-2}(\omega)\sum_{\pm} \delta G(\omega; \pm \nu)G^{-1}(\omega \pm \nu) = G_0^{-1}(\omega) + \frac{3}{2} uC_0,$$

where

$$\delta G(\omega; \pm \nu) = -3uG(\omega)G(\omega \pm \nu) \left\{ m_0m_{\pm \nu} + \int d\omega \, G(\omega_1) \delta G(-\omega_1 \mp \nu; \pm \nu) \right\}$$

$$+ \int d\omega \, \delta G(\omega_1; \pm \nu)G(-\omega_1 \mp \nu)$$

$$= -3uG(\omega)G(\omega \pm \nu)D_{\pm \nu}m_0m_{\pm \nu}$$

with $D_{\pm \nu} = 1 + 6u\int d\omega |G(\omega)|^2 G(\omega \pm \nu)$. Because of the second term in the l.h.s. of Eq. (C·1) and the corresponding modification in $C(\omega, \omega')$, $G(0)$ is now related to the equal time correlation $C_0$ as

$$G(0) = C_0 - m_0^2 - 2|m_\nu|^2 + 18u^2m_0^2 \left| \frac{m_\nu}{D_\nu} \right|^2 F(\nu),$$

where

$$F(\nu) = \int |G(\omega)|^2 \left\{ \frac{1}{\omega} \text{Im} \Psi(\omega, \nu) - \frac{2}{I_0} [ |G(\omega + \nu)|^2 + |G(\omega - \nu)|^2 ] \right\}$$

with $\Psi(\omega, \nu) = G(\omega + \nu) + G(\omega - \nu)$. From Eq. (4·3), on the other hand, the linear response magnetization $m(\omega)$ with $\omega \neq 0, \pm \nu$ is given by

$$m(\omega) = G_f(\omega)\tilde{h}(\omega) + O(m_0\tilde{h}^2, \tilde{h}^3),$$

while the uniform magnetization $m_0$ by

$$G_f^{-1}(0)m_0 = \tilde{h}_0 + um_0^3 + O(\tilde{h}_0^4).$$

The above $G_f(\omega)$ involves the uniform part of $J_{ij}$, i.e., $J_0$, and is given by

$$G_f^{-1}(\omega) = G^{-1}(\omega) - \tilde{J}_0 + 9u^2m_0^2 \left| \frac{m_\nu}{D_\nu} \right|^2 \Psi(\omega, \nu).$$

We see from the above equations that the phase transition temperature $T_c$ under $h_0 = 0$ is given by
Table I. Asymptotic behavior of the ac magnetizations of ferromagnets above $T_c$ (see $e \gg |\tilde{v}|$), in the close vicinity of $T_c$ ($|\tilde{v}| \gg e$ or $m_0$) and below $T_c$ ($m_0^2 \gg |\tilde{v}|$), where $e = (\beta G(0))^{-1} - f_0 = T - T_c$, $m_0^2 = um_0^2/\beta$ and $\tilde{v} = v/\beta I_0$.

<table>
<thead>
<tr>
<th>$m_y / h_{ac}$</th>
<th>above $T_c$</th>
<th>near $T_c$</th>
<th>below $T_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon^{-1}$</td>
<td>$i\tilde{v}^{-1}$</td>
<td>$\beta m_0^2$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$i\frac{u}{\beta}m_0 \tilde{v}^{-3}$</td>
<td>$-(\beta^2 m_0^{-5})$</td>
<td></td>
</tr>
<tr>
<td>$-\frac{u}{\beta} \varepsilon^{-3}$</td>
<td>$-\frac{u}{\beta} \tilde{v}^{-4}$</td>
<td>$-(\beta^3 m_0^{-8})$</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{T_c}{f_0} = 1 - 2|m_y|^2 = 1 - \frac{2h_{ac}}{\tilde{v}^2}, \tag{C.7}
\]

where $\tilde{v} = v/\beta I_0$ and $m_y \equiv i\hbar_{ac}/\tilde{v}$ at $T = T_c$ has been used. The asymptotic behavior of the ac nonlinear magnetizations is listed on Table I.

Finally we make a comment on the structure of the present mean field theory. By definition represented by Eqs. (2.19)~(2.21), $G(t, t')$ and $C(t, t')$ in the theory are all site-diagonal and only their leading order with respect to $N^{-1}$ are kept. Therefore $G(\omega)$ or $C(\omega)$ obtained above is not singular at $T_c$. Usually, on the other hand, the ferromagnetic instability is described by such quantities as $N^{-1}\sum G(t, t')$ and $N^{-1}\sum C(t, t')$. To analyze them we have to take into account quantities which are smaller by the order of $N^{-1}$ than those investigated here. In the present theory the ferromagnetic instability is described only through the magnetization, which is the first derivative of the generating functional.

References

Dynamical Mean Field Theory of Spin Glasses