Localized Oscillation in a Cellular Pattern

Hidetsugu SAKAGUCHI

Department of Physics, College of General Education
Kyushu University, Fukuoka 810

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We study a coupled system of lattice oscillation and lattice deformation numerically and theoretically. A localized solution is found as an exact solution and it is stable in a certain parameter region.

Recently Nasuno, Sano and Sawada found a target pattern in an experiment of nematic liquid crystal. In their experiment of liquid crystal two dimensional cellular pattern (lattice or grid pattern) is formed by the electro-hydrodynamic convection and the cellular structure exhibits a limit cycle oscillation above a certain critical voltage. Sano et al. investigated further the system and found that the targetlike waves propagate only in a localized region and the grid pattern is deformed in the propagating region of phase waves. In a previous paper we studied a coupled system of lattice oscillation and lattice deformation and found that a targetlike pattern can be generated by a kind of phase instability of the coupled equation. In this paper we show that the same equation has a stable localized solution when the coupling is relatively strong. We first show a result of numerical simulations and then we show that the localized solution can be expressed as an exact solution.

The model equation we study was obtained by Coullet and Iooss as an amplitude equation for the Hopf bifurcation in the cellular pattern. It is written as

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \mu(1 + ic_0)A - (1 + ic_2)|A|^2A + (1 + ic_1) \frac{\partial^2 A}{\partial x^2} + (\xi_1 + i\xi_2) \frac{\partial \phi}{\partial x} A, \\
\frac{\partial \phi}{\partial t} &= \frac{\partial |A|^2}{\partial x} + i\beta \left( \frac{\partial A}{\partial x} A - A \frac{\partial A}{\partial x} \right) + D \frac{\partial^2 \phi}{\partial x^2},
\end{align*}
\]

where \( A \) expresses the complex amplitude of the lattice oscillation, \( \bar{A} \) is the complex conjugate of \( A \), \( \phi \) expresses a phase modulation of the cellular pattern, and \( \mu, c_0, c_1, c_2, \xi_1, \xi_2, \beta \) and \( D \) are real parameters. In this paper we consider only the case \( c_1 = 0 \) and \( D = 1 \). Equation (1) has a uniformly oscillating solution \( A = \sqrt{\mu} \exp(iu(c_0 - c_2)t) \) and \( \phi = \text{const.} \). To study the stability of the uniformly oscillating solution we assume \( A \) to be \( A = \sqrt{\mu}(1 + u(\phi))\exp(iu(c_0 - c_2)t + i\phi) \) and substitute it into Eq. (1) and carry out a systematic expansion in powers of \( \partial/\partial x \). If we neglect the higher order terms than \((\partial/\partial x)^3\), we obtain a coupled phase equation:

\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= \frac{\partial^2 \phi}{\partial x^2} + (\xi_2 - c_2\xi_1) \frac{\partial \phi}{\partial x} + c_3 \left( \frac{\partial \phi}{\partial x} \right)^2, \\
\frac{\partial \phi}{\partial t} &= (1 + \xi_1(1 - \beta c_2)) \frac{\partial^2 \phi}{\partial x^2} - 2\mu \beta \frac{\partial \phi}{\partial x} - \beta \xi_1 \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x}.
\end{align*}
\]
From the Fourier transform of the linearized equation of (2) we can get two linear growth rates corresponding to the two eigenstates with wavenumber $q$ as

$$\lambda = \pm \left\{ 1 + 0.5 \xi_1 (1 - \beta c_2) \right\} q^2 \pm q \sqrt{2 \mu \beta (\xi_2 - c_2 \xi_1) + 0.25 \xi_1^2 (1 - \beta c_2)^2 q^2}.$$

If $1 + \xi_1 (1 - \beta c_2) > 0$ and $2 \mu \beta (\xi_2 - c_2 \xi_1) > 0$, one of the eigenvalues is positive when $q^2 < 2 \mu \beta (\xi_2 - c_2 \xi_1) / (1 + \xi_1 (1 - \beta c_2))$ and the uniformly oscillating solution is unstable for the phase modulation with the small wavenumber $q$. In a previous paper we studied the cases $\mu = 1$, $\xi_1 = 0$, and small $\xi_2$ and $\beta$. In the cases the nonlinear coupled phase equation (2) is a good approximation for Eq. (1). In this paper we study some cases that $\mu$ is small, and $\xi_1$, $\xi_2$ and $\beta$ are not so small.

At first we solved Eq. (1) numerically to find a stable dissipative structure when the above coupled phase instability occurs. The numerical simulation was carried out by the pseudospectral method with 512 modes for the system size $L = 200$. The time step was 0.1 and the periodic boundary condition was assumed. The initial condition was $A(x, 0) = \sqrt{\mu} \exp[i(0.5 \cos(2\pi x/L))]$ and $\phi(x, 0) = 0.5 \sin(2\pi x/L)$. Figure 1 shows the phase $\phi(x, t)$ of oscillation, the amplitude $|A(x, t)|$ and the local wavenumber $Q(x, t) = \partial \phi / \partial x$ at $t = 2000$ for $\mu = 0.05$, $c_0 = c_2 = 0.5$, $\xi_1 = 2.0$, $\xi_2 = 1.05$, $\beta = 2.0$. The amplitude of oscillation is localized in the region where the local wavenumber $Q$ is positive. The phase $\phi$ of oscillation keeps the shape and goes up with a

![Figure 1](https://academic.oup.com/ptp/article-abstract/87/4/1049/1857797)

![Figure 2](https://academic.oup.com/ptp/article-abstract/87/4/1049/1857797)
constant frequency. It means that phase waves are emitted from the center like the target pattern. Thual and Fauve found a similar localized solution numerically in the complex Ginzburg-Landau equation near a subcritical Hopf bifurcation. Figure 2 shows the amplitude $|A(x, t)|$ for the three parameters $\mu = 0.05, 0.1$ and $0.15$ when the other parameters are the same as Fig. 1. The width of the spatial distribution of $|A(x, t)|$ becomes smaller and the peak height becomes larger as $\mu$ is increased.

From the results of the numerical simulations, we assume a form of the solution as

$$A = A_0 \text{sech}(kx) \exp(i\omega t + i\psi),$$

$$\frac{\partial \phi}{\partial x} = B_0 \tanh(kx),$$

$$Q = Q_0 + Q_1 \text{sech}^2(kx),$$

(3)

where $\omega$ is the frequency of the oscillation and $k^{-1}$ is the width of the localized solution and $k$ is assumed to be positive. If we substitute the ansatz (3) into Eq. (1), we find that the localized solution (3) is an exact solution of Eq. (1), if $\omega, k, A_0, B_0, Q_0, Q_1$ satisfy the equations:

$$\omega - \mu c_0 + \xi_2 Q_0 - 2kB_0 = 0,$$

$$\mu + \xi_1 Q_0 + k^2 - B_0^2 = 0,$$

$$A_0^2 + 2k^2 - B_0^2 - \xi_1 Q_1 = 0,$$

$$\beta A_0^2 - 3kB_0 - \xi_2 Q_1 = 0,$$

$$(k^2 + \beta B_0 k)A_0^2 + k^2 Q_1 = 0.$$  

(4a) (4b) (4c) (4d) (4e)

The number of unknown variables is six and the number of equations is five. The last condition is determined by the boundary condition. Namely the periodic boundary condition leads to

$$\int_{-L/2}^{L/2} Q dx = Q_0 L + 2Q_1/k \tanh(kL/2) = \phi(L/2) - \phi(-L/2) = 0,$$

(4f)

where we assumed that the localized solution (3) is a good approximation even for the finite size system, if the width $k^{-1}$ is sufficiently smaller than the system size $L$. The six equations (4a)~(4f) determine $\omega, k, A_0, B_0, Q_0, Q_1$. We can express the variables with the parameters more explicitly. From (4c)~(4e) $z = B_0/k$ is a solution of the equation:

$$\xi_2 \beta z^2 + (c_2 + \xi_2 - 3\xi_1 \beta) z^2 - (2\xi_2 \beta + 3 + 3\xi_1) z - 2(c_2 + \xi_2) = 0.$$  

(5a)

Equations (5a) and (4e) yield

$$A_0^2 = k^2(z^2 - 2)/(1 + \xi_1(1 + z_0)).$$

(5b)

This equation shows that the peak height is proportional to $k$ or the reciprocal of the width of the localized structure. Substitution of (5b), (4e), (4f) into (4b) yields an equation for $k$:
where \( \tanh(kL/2) \approx 1 \) is assumed. The other variables can be easily expressed with \( k \) and \( z \) and the parameters. For the parameters of Fig. 1, i.e., \( \mu = 0.05, c_0 = c_2 = 0.5, \xi_1 = 2.0, \xi_2 = 1.05, \beta = 2.0 \) and \( L = 200 \), one solution of (5a) and (5b) is \( z = -0.756 \) and \( k = 0.0855 \) and the other variables are determined to be \( A_0 = 0.666, B_0 = -0.0646, Q_0 = -0.0266, Q_1 = 0.227 \) and \( \omega = 0.0419 \). Figure 3 shows the localized solution (3) with the above theoretical values. Good agreement between Figs. 1 and 3 means that the localized solution (3) is attained as a stable solution when we start from the phase modulated initial state.

The localized solution (3) is unstable when \( \mu + \xi_1 Q_0 > 0 \), because the localized solution (3) approaches the nonoscillating state \( A = 0 \) as \( |x| \) is increased, and the nonoscillating state \( A = 0 \) is unstable when \( \mu + \xi_1 Q_0 > 0 \). From Eqs. (4b) and (5a) the critical value of \( z \) is \( -1 \) and the threshold of the instability is \( \beta_c = (3 + 3\xi_1 - c_0 - \xi_2)/(3\xi_1 - \xi_2) \). When \( \beta < \beta_c \), the localized solution (3) is unstable. We carried out a numerical simulation for \( \beta = 1.3, \mu = 0.15, c_0 = c_2 = 0.5, \xi_1 = 2.0, \xi_2 = 1.05 \). For the parameters the localized solution is unstable, since \( \beta_c = 1.505 \).

The initial condition was the same as Fig. 1, i.e., \( A(x, 0) = \sqrt{\mu} \exp[i(0.5 \cos (Lx/2\pi))] \) and \( \phi(x, 0) = 0.5 \sin(Lx/2\pi) \). Figure 4 shows a snap shot of \( |A(x, t)| \) at \( t = 3700 \) and the mean frequency \( \langle \partial \phi/\partial t \rangle \), i.e., the average value of \( \partial \phi/\partial t \) between \( t = 2000 \) and \( t = 4000 \). The localized structure is seen at the center but the amplitude \( |A| \) does not approach \( |A| = 0 \) as \( |x| \) is increased. The outer region exhibits a small amplitude oscillation.
with a different frequency. At the boundary points between the two regions which are mutually entrained with different frequencies, the complex amplitude $A(x, t)$ goes through the phase singularity point $A = 0$ periodically. It is characteristic of desynchronization in a continuum medium. Figure 4 shows that $|A(x, t)|$ is nearly zero at the boundary points and at $t = 3700$.

There are six pairs of solutions for $z$ and $k$ in Eqs. (5a) and (5c). But some solutions are complex number and some violate the inequality $k > 0$ or $A_0^2 > 0$ and some violate the stability condition $\mu + \xi_1 Q_0 < 0$. Only a few solutions are candidates for the stable localized solution (3). For example, when $\mu$ is increased for $c_0 = c_2 = 0.5$, $\xi_1 = 2.0$, $\xi_2 = 1.05$, $\beta = 2.0$ and $L = 200$, $k$ becomes complex number for $z = -0.756$ at $\mu = 0.226$ and then the localized solution (3) disappears.

To summarize we found a localized solution (3) as an exact solution of Eq. (1) and the solution is stable in a certain parameter region. It may explain the localized target pattern found in the experiment of nematic liquid crystal.

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