Semiclassical Particles with Arbitrary Spin in the Riemann-Cartan Spacetime

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By means of the Fock-Papapetrou method and the WKB method, the equations of world line and spin precession are derived for elementary particles with arbitrary spin $s$ moving in the Riemann-Cartan spacetime. It is shown that the world line is the geodesics of the metric, and that the spin precession due to torsion depends on the spin $s$. In particular, the intrinsic spin vector is not Fermi-Walker transported along the world line relative either to the nonsymmetric connection or to the ordinary symmetric connection.

§ 1. Introduction

The gauge theory of Poincaré group (Poincaré gauge theory) is built on the Riemann-Cartan spacetime which possesses both curvature and torsion. The latter quantity is coupled to the intrinsic spin of matter fields such as the Dirac field, the Proca field and other higher-spin fields if they exist, thus giving rise to nontrivial and observable effects on the motion of particles with intrinsic spin. Precision experiments of spin precession may then give us a clue to detect the torsion of spacetime.

In a previous paper¹) we have applied the WKB method to the Dirac and Proca fields in the Riemann-Cartan spacetime, and derived the equations of world line and spin precession for massive particles with spin $\frac{1}{2}$ and $1$ in units of $\hbar$. It is shown in particular that the world line of these particles is the geodesics of the metric but that the equation of spin precession due to torsion is different for spin $\frac{1}{2}$ and spin $1$. In particular, if the torsion is nonvanishing, the intrinsic spin vector is not Fermi-Walker transported along the world line relative either to the nonsymmetric connection or to the ordinary symmetric connection defined by the Christoffel symbol. Then we have conjectured that the spin precession due to torsion will depend on the magnitude of the intrinsic spin.

The purpose of this paper is to study the spin precession of higher-spin particles due to torsion. Theory of higher-spin fields was initiated especially by Dirac²) and Fierz,³) and has a long history. The Lagrangian formulation was first attempted by Fierz and Pauli⁴) to study the electromagnetic interaction: The explicit form of the Lagrangian was given only for spin $2$ and $3/2$, however. The Lagrangian for massive fields with spin higher than $2$ was later suggested by Fronsdal⁵) and Chang,⁶) and finally constructed by Singh and Hagen:⁷) The massless limit was investigated by Fronsdal and Fang.⁸) A first-order formalism for boson fields with arbitrary spin was also formulated,⁷) which allows us to introduce the electromagnetic interaction and other gauge couplings unambiguously: Thus, the Poincaré gauge invariant Lagrangian for
higher-spin fields can be derived in an unambiguous manner both for the boson and fermion cases.

The field equations for higher-spin fields in the Riemann-Cartan spacetime can be derived by the variational principle from the Poincaré gauge invariant Lagrangian. Now let us apply the WKB method\textsuperscript{9} to this field equation, and look for semiclassical solutions of the form

\[ \psi(x) = \exp(iS(x)/\hbar)\left(\psi_0(x) + \frac{\hbar}{i}\psi_1 + \cdots\right), \] (1.1)

requiring each power of \( \hbar \) to vanish separately in the field equation, with \( \psi \) representing a higher-spin field. The lowest-order terms of \( \hbar \) give an equation for \( \psi_0 \) from which follows the Hamilton-Jacobi equation for the classical trajectory; on the other hand, the next-lowest ones lead to a coupled equation for \( \psi_0 \) and \( \psi_1 \), from which the equation of spin precession can be derived\textsuperscript{10,11}. It is found that the equation for \( \psi_0 \) is simple and that the world line of semiclassical particles is the geodesics of the metric. However, the coupled equation for \( \psi_0 \) and \( \psi_1 \) is much more complicated, and it seems difficult to arrive at the equation of spin precession by means of the WKB method alone.

We shall therefore employ the results of our preceding paper\textsuperscript{12} in which the Fock-Papapetrou method\textsuperscript{13} is used to obtain the equations of spin and world line for an extended body such as a gyroscope and a wave packet moving in the Riemann-Cartan spacetime. It has been found that these equations involve a quantity \( N^{\alpha\nu} \) which we call the spin-current, and which is defined by integrating the intrinsic-spin tensor density over the whole three-dimensional space.

For wave packets of higher-spin fields, the spin-current \( N^{\alpha\nu} \) can be calculated by applying the lowest-order approximation of the WKB method. Using the resultant expression for \( N^{\alpha\nu} \) in those equations derived by the Fock-Papapetrou method, we can get the equations of world line and spin precession for particles with arbitrary spin \( s \) moving in the Riemann-Cartan spacetime.

The content is arranged as follows. In § 2 we summarize the Fock-Papapetrou method applied to the motion of wave-packets. The Singh-Hagen formulation of massive higher-spin fields is explained in § 3, and the spin-current is calculated in § 4 for wave packets of fermion fields with arbitrary spin by means of the WKB method. In the same way the wave packets of boson fields with arbitrary spin are discussed in § 5. The equation of spin precession is then derived for particles with arbitrary spin in § 6. The final section is devoted to conclusion.

Throughout this paper we use the unit \( c=1 \), while we write the reduced Planck constant \( \hbar \) explicitly. The intrinsic spin is always measured in units of \( \hbar \).

§ 2. The Fock-Papapetrou method applied to the motion of wave packets

The background spacetime is assumed to be a Riemann-Cartan spacetime which possesses (1) a locally Lorentzian metric \( g = \{g_{\mu\nu}(x)\} \), (2) a global tetrad field \( e^\mu_k = \{e^\mu_k(x)\} \) with \( g_{\mu\nu}e^\mu_k e^\nu_l = \eta_{kl} \) and (3) a nonsymmetric affine connection \( \Gamma = \{\Gamma^a_{\mu\nu}\} \).
satisfying the metric condition.*

The tetrad condition is also assumed to be satisfied: Thus, the spin connection $A_{mnv}$ and the affine connection $\Gamma^l_{\mu \nu}$ are expressed by $A_{mnv} = \Delta_{mnv} + K_{mnv}$ and $\Gamma^l_{\mu \nu} = \{_{\mu \nu}^l\} + K^l_{\mu \nu}$, respectively, in terms of the Ricci rotation coefficients $\Delta_{mnv}$ and the Christoffel symbol $\{_{\mu \nu}^l\}$. Here $K_{mnv}$ stands for the contorsion tensor antisymmetric with respect to $m$ and $n$. The torsion tensor $T^l_{\mu \nu} = \Gamma^l_{\mu \nu} - \Gamma^l_{\nu \mu}$ is related to the contorsion tensor by $T^l_{\mu \nu} = K^l_{\mu \nu} - K^l_{\nu \mu}$. We shall denote by $D_{\nu}$ the covariant derivative with respect to both $\{_{\mu \nu}^l\}$ and $A_{mnv}$, while we shall use the symbol $\nabla_{\nu}$ for the one with respect to both $\{_{\mu \nu}^l\}$ and $\Delta_{mnv}$.

Let us consider a wave packet of an arbitrary field in the Riemann-Cartan spacetime, assuming that the following conditions are satisfied:

1. The dimensions of the wave packet are very small compared with the characteristic length of the background Riemann-Cartan spacetime, and the pole-dipole approximation of the Fock-Papapetrou method can be applied.
2. The wavelength is so short compared with the dimensions of the wave packet that the WKB method is applicable.

When these assumptions are satisfied, the wave packet approximately behaves like a particle with intrinsic spin.

The wave packet sweeps a world tube in spacetime. Inside the tube we take an arbitrarily chosen, timelike world line $L$ which will 'represent' the motion of the wave packet as a whole. The coordinates of the points on $L$ will be denoted by $X^\mu(\tau)$ with $\tau$ being the proper time on $L$.

Putting $\delta x^\mu = x^\mu - X^\mu(\tau)$ with $\delta x^0 = x^0 - X^0(\tau) = 0$, the total spin of the wave packet is defined by

$$J^\mu = \int (\delta x^\nu T^\nu_\mu - \delta x^\mu T^\nu_\nu + S^\mu_\nu d^3 x \tag{2.1}$$

and the spin-current by

$$N^{\mu \nu \lambda} = - U^\nu \int S^{\mu \nu \lambda} d^3 x, \tag{2.2}$$

where $U^\mu$ denotes the four-velocity, $U^\mu = dX^\mu(\tau)/d\tau$, and the integration is carried out over the three-dimensional space with $x^0 =$ constant. Here $T^{\mu \nu}$ and $S^{\mu \nu \sigma}$ are the energy-momentum tensor density and the spin tensor density, respectively, defined by

$$T^{\mu \nu} = \frac{\partial L_M}{\partial \epsilon_{\mu \nu}} e^\mu_h, \quad S^{\mu \nu \sigma} = - 2 \frac{\partial L_M}{\partial A_{\mu \nu \sigma}} \tag{2.3}$$

and $L_M = e L_M$ is the Lagrangian density of the field under consideration. We note that $T^{\mu \nu}$ is not symmetric with respect to $\mu$ and $\nu$: It becomes the canonical energy-momentum tensor in special relativistic limit, when the gravitational coupling is introduced minimally.

Following a previous paper we shall assume that the pole-dipole approximation

*) We shall use the same notation and convention as in Ref. 12). We denote the Minkowski metric by $\eta_{\mu \nu} = \text{diag}(-1, +1, +1, +1)$. The Latin indices and the Greek ones refer to the Lorentz basis $e_\mu$ and the coordinate basis $E_\mu$, respectively.
is valid for $T^{\mu \nu}$, and that the single-pole approximation is applicable to $S^{\mu \nu \rho \sigma}$. Then it can be shown that both the total spin of (2·1) and the spin-current of (2·2) are tensors under general coordinate transformations.

The intrinsic spin tensor $S^{\mu \nu}$ and the rotational spin tensor $L^{\mu \nu}$ are defined by

$$S^{\mu \nu} = N^{\mu \nu \rho \sigma} U_{\rho \sigma}, \quad L^{\mu \nu} = J^{\mu \nu} - S^{\mu \nu}. \quad (2·4)$$

Let us impose the following constraints on $J^{\mu \nu}$ and $S^{\mu \nu}$:

$$J^{\mu \nu} U_{\nu} = 0, \quad S^{\mu \nu} U_{\nu} = 0, \quad (2·5)$$

which ensure that the degrees of freedom of $J^{\mu \nu}$ and $S^{\mu \nu}$ are reduced from six to three. The first constraint on $J^{\mu \nu}$ can be assumed without loss of generality, since it is attained by redefining the world line $X^{\nu}(\tau)$ appropriately, as will be shown in Appendix A. The second one, on the other hand, is postulated on the physical ground that the intrinsic spin tensor should have only spatial components in the instantaneous rest frame. We then define the Pauli-Lubanskii vectors $J^{\mu}$ and $S^{\mu}$ by

$$J^{\mu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} U_{\nu} J_{\rho \sigma}, \quad S^{\mu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} U_{\nu} S_{\rho \sigma}, \quad (2·6)$$

which are equivalent to $J^{\mu \nu}$ and $S^{\mu \nu}$, respectively, owing to (2·5). Furthermore we notice that the relation $L^{\mu \nu} U_{\nu} = 0$ follows from (2·5), which means that $X^{\nu}(\tau)$ coincides with the center of mass of the wave packet in the instantaneous rest frame.

The starting point of the Fock-Papapetrou method is the response equations of the energy-momentum tensor density and the spin tensor density of matter fields in the background Riemann-Cartan spacetime. In the pole-dipole approximation for $T^{\mu \nu}$ and the single-pole approximation for $S^{\mu \nu \rho \sigma}$, we have obtained the following covariant equation of spin, which involves the spin-current $N^{\mu \nu \rho \sigma}$ coupled to the torsion of spacetime:

$$\frac{\partial J^{\mu}}{\partial \tau} = \left( U^{\mu} U^{\nu} \frac{\partial}{\partial \tau} U^{\rho} \frac{\partial}{\partial \tau} U^{\sigma} \right) J^{\nu} - \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} U_{\rho} (2K_{\rho \sigma} N^{\sigma \tau} + K_{\sigma \rho} N^{\tau \nu}). \quad (2·7)$$

This is not an equation of spin precession for $J^{\mu}$, however. This is not surprising, if we remember that the torsion is coupled only with the intrinsic spin, being insensitive to the rotational spin.

The world line equation derived by the Fock-Papapetrou method will be discussed in Appendix D.

§ 3. Higher-spin fields in the Minkowski spacetime

Fields with half-integer spin

Consider a spin-$s$ fermion field described by the Dirac-Rarita-Schwinger symmetric tensor-spinor $\psi^{(n)}_{t_{1} \cdots t_{n}}$ of rank $n=s-1/2(n=0, 1, 2, \cdots)$ satisfying the spinor-trace condition,\footnote{This condition implies the tracelessness with respect to Lorentz indices.}$\gamma^{i} \psi^{(n)}_{t_{1} \cdots t_{n}} = 0$. The background spacetime is assumed to be flat, and so Latin letters are used also for coordinate indices.

The free field satisfies the following equations:
and

\[(3\cdot2)\]

Fierz and Pauli⁴ noticed that it is impossible to construct a Lagrangian that will yield (3·1) and (3·2) by using only \(\phi^{(n)}_{t_1...t_n}\) and that additional fields (the so-called auxiliary fields) have to be introduced. The Lagrangian formulation for the spin-s fermion field was later given by Singh and Hagen.⁷ Their Lagrangian takes a complicated form involving many auxiliary fields \(\psi^{(n-1)}, \psi^{(n-2)}, ..., \psi^{(0)}\) and \(\chi^{(n-2)}, ..., \chi^{(0)}\), which are also Dirac-Rarita-Schwinger tensor-spinors, and it may be written as

\[L_M = \frac{1}{2}\overline{\phi^{(n)}_{t_1...t_n}}(i\hbar \gamma^k \partial_k - m)\phi^{(n)}_{t_1...t_n} + \text{c.c.} + \text{(terms involving the auxiliary fields)},\]  

(3·3)

abbreviating all the terms involving the auxiliary fields. Here c.c. means the complex conjugate of the preceding term. The terms of the auxiliary fields are chosen so that all the auxiliary fields automatically vanish for the free field case.

If we collectively denote by \(\Psi\) the set of \(\phi^{(n)}_{t_1...t_n}\) and all the auxiliary fields, then (3·3) can be represented as

\[L_M = \frac{i\hbar}{2} \overline{\psi} Y^k \partial_k \psi - (\partial_k \overline{\psi} Y^k \psi) - m \overline{\psi} Y^k \psi,\]  

(3·4)

in terms of the appropriate constant matrices \(Y^k\) and \(Y\) satisfying \(\Gamma^0 Y^k \Gamma^0 = Y^k\) and \(\Gamma^0 Y^k \Gamma^0 = Y\), where \(\overline{\psi}\) means \(\psi^\dagger \Gamma^0\). Here the matrix \(\Gamma^0\) operates on each spinor index of \(\psi\) just like \(\gamma^0\). The Euler-Lagrange equation for the free field can then be expressed concisely by

\[(i\hbar Y^k \partial_k - m Y) \Psi = 0.\]  

(3·5)

We remember that the matrices \(Y^k\) and \(Y\) are determined by requiring that all the components of \(\Psi\) corresponding to the auxiliary fields vanish. Accordingly, (3·5) reproduces (3·1) and (3·2).

Since the auxiliary fields are also assumed to be symmetric tensor-spinors satisfying the spinor-trace condition, the Euler-Lagrange equation obtained by varying the action with respect to \(\Psi\) should also consist of symmetric terms which satisfy the spinor-trace condition.

For the plane wave solution of

\[\Psi = A \exp \left( \frac{i}{\hbar} P_k x^k \right),\]  

(3·6)

the amplitude \(A\) satisfies the coupled linear equation,

\[(Y^k P_k + m Y) A = 0.\]  

(3·7)

Since all the amplitudes for the auxiliary fields vanish, (3·7) leads to
Fields with integer spin

Let us now turn to the boson field with integer spin $s$ described by a symmetric, traceless tensor $\phi^{(s)}_{l_1\cdots l_s}$ of rank $s$. The free field satisfies the Klein-Gordon equation

$$\left[\partial^2 - \left(\frac{m}{\hbar}\right)^2\right] \phi^{(s)}_{l_1\cdots l_s} = 0$$

and

$$P^{l_1} A^{(n)}_{l_1\cdots l_n} = 0.$$  

Singh and Hagen\textsuperscript{7} has given the Lagrangian formulation by introducing besides $\phi^{(s)}$, a set of auxiliary tensor fields $\phi^{(s-2)}$, $\phi^{(s-3)}$, and $\phi^{(0)}$. Their Lagrangian reads

$$L = -\hbar \left[ \frac{1}{2} h_{l_1\cdots l_k} h^{l_1\cdots l_k} + (s-1) \partial^m \phi^{(s)} \partial_m \phi^{(s)} h^{l_1\cdots l_k} - \left(\frac{m}{\hbar}\right)^2 \phi^{(0)} \partial^{l_1\cdots l_s} \phi^{(0)} + \text{(terms involving the auxiliary fields)} \right],$$

where all the terms involving the auxiliary fields are abbreviated and $h_{l_1\cdots l_k}$ is defined by

$$h_{l_1\cdots l_k} = \partial_{l_1} \phi^{(s)}_{l_1\cdots l_k} - \partial_{l_k} \phi^{(s)}_{l_1\cdots l_k}. \tag{3.13}$$

The additional terms of the auxiliary fields in $L_M$ are determined by requiring that all the auxiliary fields vanish for the free field case. They have further constructed a first-order formalism, introducing another set of symmetric, traceless tensor fields $H^{(p)}_{l_1\cdots l_p}$ ($p=s, s-2, \cdots, s-1, 0$) and $H$. The field equations for $H^{(p)}$ and $H$ are coupled linear equations and can be solved with respect to them: If we use those expressions for $H^{(p)}$ and $H$ in the Lagrangian in the first-order formalism, we get (3.12). In this sense the Lagrangian (3.12) is unique and unambiguous.

Let $\Phi$ represent the $\phi^{(s)}_{l_1\cdots l_s}$ and all the auxiliary fields collectively, then (3.12) can be rewritten as:

$$L_M = \hbar^2 \left[ -\partial_{l_1} Z^{km} \partial_{m} \Phi + \frac{i}{2} \left(\frac{m}{\hbar}\right) (\Phi^{\dagger} Z^{k} \partial_{l_1} \Phi - \partial_{l_1} \Phi^{\dagger} Z^{k} \Phi) - \left(\frac{m}{\hbar}\right)^2 \Phi^{\dagger} Z \Phi \right], \tag{3.14}$$

in terms of the appropriate, constant matrices satisfying $Z^{km} = Z^{mk}$, $Z^{k} = Z^{k}$ and $Z^{\dagger} = Z$. Then the Euler-Lagrange equation for the free field takes the form

$$\left( Z^{km} \partial_{m} \Phi + i \left(\frac{m}{\hbar}\right) Z^{k} \partial_{l_1} \Phi - \left(\frac{m}{\hbar}\right)^2 Z \Phi \right) = 0, \tag{3.15}$$

The auxiliary fields are also symmetric and traceless, and therefore the Euler-Lagrange equation should consist of symmetric, traceless terms. Though (3.15) is a very complicated form involving many auxiliary fields, all the auxiliary fields are
vanishing due to this field equation. Accordingly, (3·15) reproduces (3·10) and (3·11).

For the plane wave solution of the form

$$\Phi = B \exp \left( \frac{i}{\hbar} P_k x^k \right),$$  \hspace{1cm} (3·16)

the amplitude $B$ satisfies the coupled linear equation,

$$(Z^{km} P_k P_m + m Z^k P_k + m^2 Z) B = 0,$$  \hspace{1cm} (3·17)

from which follow that all the amplitudes for the auxiliary fields vanish, and that

$$\eta^{km} P_k P_m + m^2 = 0$$  \hspace{1cm} (3·18)

and

$$P^{\alpha} B^{(s)}_{\alpha1...\alpha n} = 0.$$  \hspace{1cm} (3·19)

§ 4. Semiclassical particles with half-integer spin

The Lagrangian density in the Riemann-Cartan spacetime is obtained from that in special relativity by the following procedure referred to as the minimal prescription: Replace each $\partial_n$ by the covariant derivative $D_n = e_{k\mu} \nabla \mu$, which acts, for example, on $\phi^{(n)}_{\alpha1...\alpha n}$ like

$$D_n \phi^{(n)}_{\alpha1...\alpha n} = e_{k\mu} \left( \partial_n \phi^{(n)}_{\alpha1...\alpha n} + \frac{i}{2} A_{\mu\nu} \sigma^{\mu\nu} \phi^{(n)}_{\alpha1...\alpha n} - A^{\mu}_{\alpha1...\alpha m} \phi^{(n)}_{\alpha m...\alpha n} \right),$$  \hspace{1cm} (4·1)

and then multiply by $e = \det (e_{k\mu})$. Here $\sigma^{\mu\nu}$ is the Lorentz generator for the spinor index of $\phi$. The $D_n$ used above is the full covariant derivative which operates on both spinor and tensor indices. This means that the $S^{mn\nu}$ of (2·3) reduces to the intrinsic spin density in the special relativistic limit.

The Lagrangian density is then given by

$$L_M = e \left[ \frac{1}{2} \overline{\phi}^{(n)}_{\alpha1...\alpha n} (i\hbar \gamma^k D_k - m) \phi^{(n)}_{\alpha1...\alpha n} + \text{c.c.} \right] + (\text{terms involving the auxiliary fields}),$$  \hspace{1cm} (4·2)

or alternatively by

$$L_M = e \left[ \frac{i\hbar}{2} (\overline{\Psi} \gamma^k D_k \Psi - (D_k \overline{\Psi}) \gamma^k \Psi) - m \overline{\Psi} \Psi \right]$$  \hspace{1cm} (4·3)

in terms of the field $\Psi$. Variation of (4·3) with respect to $\Psi$ gives the Euler-Lagrange equation as follows:

$$\left[ i\hbar Y^k (D_k + \frac{1}{2} v_k) - m Y \right] \Psi = 0,$$  \hspace{1cm} (4·4)

where $v_k = T'_{ik}$ is the vector part of the torsion tensor.
Let us look for a WKB solution of the form

$$\Psi(x) = A(x) \exp \left( \frac{iS(x)}{\hbar} \right),$$  \hspace{1cm} (4.5)

assuming that $|S(x)| \gg \hbar$. In the lowest order of $\hbar$, Eq. (4.4) gives

$$(Y^k \partial_k S(x) + mY)A(x) = 0$$  \hspace{1cm} (4.6)

with $\partial_k S = e_k^\nu \partial_k S$. This equation for $A(x)$ is of the same form as the coupled linear equation (3.7) for the plane wave solution in the Minkowski space-time. Therefore, the results of Singh and Hagen are valid also for (4.6): All the components of $A(x)$ corresponding to the auxiliary fields are vanishing, and the $A^{(n)}_{t_1 \ldots t_n}(x)$ satisfies

$$(\gamma^k \partial_k S + m)A^{(n)}_{t_1 \ldots t_n}(x) = 0$$  \hspace{1cm} (4.7)

and

$$(\partial^1 S)A^{(n)}_{t_1 \ldots t_n}(x) = 0.$$  \hspace{1cm} (4.8)

Thus, the $A^{(n)}_{t_1 \ldots t_n}(x)$ describes a particle with spin $s = n + 1/2$ and mass $m$.

By virtue of (4.7), the function $S$ satisfies the Hamilton-Jacobi equation

$$g^{\mu\nu}(x) \partial_\mu S \partial_\nu S + m^2 = 0.$$  \hspace{1cm} (4.9)

As is well known in the Hamilton-Jacobi theory, the complete solution of (4.9) has three independent parameters $a_i (i = 1, 2, 3)$ besides an additive constant. We shall denote the solution as $S(x, a)$. In conformity with this, we shall write the corresponding solutions as $A^{(n)}_{t_1 \ldots t_n}(x, a)$ and $\phi^{(n)}_{t_1 \ldots t_n}(x, a)$,

$$\phi^{(n)}_{t_1 \ldots t_n}(x, a) = A^{(n)}_{t_1 \ldots t_n}(x, a) \exp \frac{i S(x, a)}{\hbar}.$$  \hspace{1cm} (4.10)

Now let us form a wave-packet solution by superposing $\phi^{(n)}(x, a)$ over a small region in the $a$-space:

$$\phi^{(n)}_{t_1 \ldots t_n}(x) = \int d^3 a W(a - \bar{a}) \phi^{(n)}_{t_1 \ldots t_n}(x, a)$$

$$\approx A^{(n)}_{t_1 \ldots t_n}(x, \bar{a}) \int d^3 a W(a - \bar{a}) \exp \frac{i S(x, a)}{\hbar}$$

$$\equiv A^{(n)}_{t_1 \ldots t_n}(x, \bar{a}) \mathcal{Q}(x, \bar{a}),$$  \hspace{1cm} (4.11)

where $W(a - \bar{a})$ is a weight function peaked at $a_i = \bar{a}_i$ ($i = 1, 2, 3$) so sharply that $A^{(n)}_{t_1 \ldots t_n}(x, a)$ can be approximated by $A^{(n)}_{t_1 \ldots t_n}(x, \bar{a})$, and the last equation defines $\mathcal{Q}(x, \bar{a})$. Because $|S| \gg \hbar$, the integrand oscillates rapidly in the integrals of (4.11), and therefore compensation takes place almost everywhere in the spacetime: The only exception is a thin tube around the world line satisfying

$$\frac{\partial S(x, \bar{a})}{\partial a_t} = 0.$$  \hspace{1cm} (4.12)

Thus $\phi^{(n)}_{t_1 \ldots t_n}(x)$ of (4.11) indeed represents a wave packet.

We represent the world line defined by (4.12) as $X^\mu(x)$, parametrizing it by the
proper time \( \tau \). Then it satisfies the differential equation

\[
U^\mu(\tau) \equiv \frac{dX^\mu(\tau)}{d\tau} = \frac{1}{m} g^{\mu \nu}(X(\tau)) \partial_\nu S(X(\tau), \vec{a}) .
\]  

(4·13)

It easily follows from (4·9) and (4·13) that the world line \( X^\mu(\tau) \) satisfies the geodesic equation, \( \nabla U^\mu / \nabla \tau = 0 \).

Henceforth we assume that \( e^\nu_k \) and \( A^{(n)}_{l_1 \cdots l_n} \) are slowly varying over the size of the wave packet: Then \( e^\nu_k(x) \) and \( A^{(n)}_{l_1 \cdots l_n}(x, \vec{a}) \) can be approximated by \( e^\nu_k(X(\tau)) \) and \( A^{(n)}_{l_1 \cdots l_n}(X(\tau), \vec{a}) \).

If we use \( U_k = e^\nu_k(X(\tau)) U_k(\tau) \), (4·7) and (4·8) can be rewritten as

\[
(\gamma^k U_k + 1) A^{(n)}_{l_1 \cdots l_n}(X(\tau), \vec{a}) = 0
\]

(4·14) and

\[
U^{l_i} A^{(n)}_{l_1 \cdots l_n}(X(\tau), \vec{a}) = 0.
\]

(4·15)

Use of (4·2) in (2·3) gives the spin tensor density*)

\[
S^{ijk} = e \left[ -2i\hbar \left( n + \frac{1}{2} \right) \bar{\phi}^{(n)}_{l_1 \cdots l_n} \gamma^i \phi^{(n)}_{l_1 \cdots l_n} + i\hbar \bar{\phi}^{(n)}_{l_1 \cdots l_n} \gamma^i \phi^{(n)}_{l_1 \cdots l_n} + i\hbar \bar{\phi}^{(n)}_{l_1 \cdots l_n} \gamma^i \phi^{(n)}_{l_1 \cdots l_n} \right]
\]

(4·16)

where we have employed the relation

\[
\frac{1}{2} \bar{\phi}^{(n)}_{l_1 \cdots l_n} \left( \sigma^{ij}, \gamma^k \right) \phi^{(n)}_{l_1 \cdots l_n}
\]

\[
= \frac{i}{2} \left( \bar{\phi}^{(n)}_{l_1 \cdots l_n} \gamma^i \phi^{(n)}_{l_1 \cdots l_n} + \bar{\phi}^{(n)}_{l_1 \cdots l_n} \gamma^j \phi^{(n)}_{l_1 \cdots l_n} + \bar{\phi}^{(n)}_{l_1 \cdots l_n} \gamma^k \phi^{(n)}_{l_1 \cdots l_n} \right)
\]

(4·17)

which will be proved in Appendix B. The above relation cannot be applied to the Dirac field since \( n=0 \), and so we study this exceptional case separately in Appendix C.

Substituting the wave-packet solution of (4·11) in (4·16) and then putting the result in (2·2), we obtain the spin-current \( N^{\mu \nu \lambda} \).

\[
N^{\mu \nu \lambda} = e^\nu_k e^\lambda_j e^\mu_l \left( -U^0 \int S^{ijk} d^3 x \right)
\]

\[
= \left[ -2i\hbar \left( n + \frac{1}{2} \right) \bar{A}^{(n)}_{l_1 \cdots l_n} \left( A^{(n)}_{l_1 \cdots l_n} \right) \right] + i\hbar \bar{A}^{(n)}_{l_1 \cdots l_n} \left( A^{(n)}_{l_1 \cdots l_n} \right) \right) \left( -U^0 \int |\Omega|^2 d^3 x \right),
\]

(4·18)

*) Here the symbol \( \left[ \right] \) denotes antisymmetrization. For example, \( A^{(gh)}_{ij} = (1/2)(A^{(gh)}_{ij} - A^{(gh)}_{ji}) \).
where $\gamma^I$ denotes $e_k^I \gamma^k$, and $e_k^\nu$ and $A^{(n)}_{l_1\ldots l_n}$ are to mean their values on the world line $X^\nu(t)$. We also note that all the auxiliary fields vanish. The intrinsic spin tensor $S^\mu\nu$ of (2·4) is now given by

$$S^\mu\nu = N^\mu\nu U^\nu,$$

$$= 2i\hbar \left( n + \frac{1}{2} \right) \bar{A}^{(n)}_{l_1\ldots l_n} A^{(n)}_{l_1\ldots l_n} \left( - U^0 \int |\Omega|^2 \epsilon d^3 x \right).$$  \hspace{1cm} (4·19)

Here we have used (4·14) and (4·15): A particularly useful relation is

$$\bar{A}^{(n)}_{k_1\ldots k_n} \gamma^k A^{(n)}_{l_1\ldots l_n} = \bar{A}^{(n)}_{k_1\ldots k_n} U^k A^{(n)}_{l_1\ldots l_n},$$  \hspace{1cm} (4·20)

which follows from (4·14). It should also be noted that $S^\mu\nu U^\nu = 0$ holds owing to (4·15).

Using (4·19) in (4·18), we finally get the following expression for $N^\mu\nu$ in terms of $S^\mu\nu$ and $U^\nu$:

$$N^\mu\nu = - S^\mu\nu U^\nu - \frac{1}{s} U^\mu S^\nu U^\lambda,$$  \hspace{1cm} (4·21)

where $s = n + (1/2)$ has been employed.

The energy-momentum tensor density of (2·3) is given by

$$T^\mu\nu = e^{\gamma^I} \left[ \frac{i\hbar}{2} \left( D^\nu \bar{\psi}^{(n)}_{l_1\ldots l_n} \gamma^I \psi^{(n)}_{l_1\ldots l_n} - \bar{\psi}^{(n)}_{l_1\ldots l_n} \gamma^I D^\nu \psi^{(n)}_{l_1\ldots l_n} \right) 
+ \text{(terms involving auxiliary fields)} \right] + g^\mu\nu L^M,$$  \hspace{1cm} (4·22)

which allows us to calculate the rotational spin $L^\mu\nu$ of (2·4) for the wave packet solution. Since the $L^\mu\nu$ is a tensor as is mentioned in § 2, it is sufficient to evaluate its value in the instantaneous rest system:

$$L^\mu\nu = \int (\delta x^\mu T^{\nu 0} - \delta x^\nu T^{\mu 0}) d^3 x$$

$$= m \bar{A}^{(n)}_{l_1\ldots l_n} A^{(n)}_{l_1\ldots l_n} \int (\delta x^\mu U^\nu - \delta x^\nu U^\mu) |\Omega|^2 |\epsilon| d^3 x,$$  \hspace{1cm} (4·23)

which implies that the space-space components of $L^\mu\nu$ vanish because $U^\nu = (1, 0, 0, 0)$ in this system. Therefore, $L^\mu\nu$ should be vanishing owing to the constraint $L^\mu\nu U^\nu = 0$.

§ 5. Semiclassical particles with integer spin

We can repeat nearly the same arguments as those for the particles with half-integer spin, using the same notation as in § 4. The Lagrangian density of the integer-spin field in the Riemann-Cartan spacetime is obtained by applying the minimal prescription to (3·12), or (3·14): namely,

$$L^\mu = e^{\gamma^I} \left[ - \frac{1}{2} H^I_{l_1\ldots l_k} H^{l_1\ldots l_k} + (s - 1) D^m \phi^I_{l_1\ldots l_s} D_k \phi^{k l_1\ldots l_s} \right].$$
from (3·12), where \( H_{l_{1}...l_{k}} \) is defined by

\[
H_{l_{1}...l_{k}} = D_{k} \phi_{l_{1}...l_{k}} - D_{l_{k}} \phi_{l_{1}...l_{k}}
\]

and \( \phi \) denotes \( \phi^{(s)} \) for simplicity; or alternatively

\[
L_{M} = e \hbar^{2} \left[ -D_{k} \phi^{+} Z^{m} D_{m} \phi - \frac{i m}{\hbar} \left( \phi^{+} Z^{k} D_{k} \phi - D_{k} \phi^{+} Z^{k} \phi \right) - \left( \frac{m}{\hbar} \right)^{2} \phi^{+} Z \phi \right]
\]

from (3·14). The latter gives the Euler-Lagrange equation of the form,

\[
\left[ Z^{m n} (D_{n} + v_{n}) D_{m} + i \left( \frac{m}{\hbar} \right) Z^{k} (D_{k} + \frac{1}{2} v_{k}) - \left( \frac{m}{\hbar} \right)^{2} Z \right] \phi = 0.
\]

Let us look for a WKB solution of the form

\[
\phi(x) = B(x) \exp(i S(x)/\hbar),
\]

assuming that \(|S(x)| \gg \hbar\). In the lowest order of \( \hbar \) (5·4) gives

\[
(Z^{m n} \partial_{n} S \partial_{m} S + m Z^{k} \partial_{k} S + m^{2} Z) B(x) = 0,
\]

which is of the same form as the coupled linear equation (3·17) for the plane wave solution in the Minkowski spacetime. Accordingly, it follows from (5·6) that (i) all the auxiliary fields vanish, that (ii) the \( S(x) \) satisfies the Hamilton-Jacobi equation (4·9), and finally that (iii) \( B^{(s)}_{l_{1}...l_{k}} \) obeys

\[
(\partial^{i} S) B^{(s)}_{l_{1}...l_{k}}(x) = 0.
\]

In the same way as in § 4, we can form a wave packet by superposing the WKB solution: The result is nearly the same as (4·11), the only difference being that the tensor is used here instead of the tensor-spinor therein. The world line \( X^{\mu}(r) \) of the peak of the wave packet is determined by (4·13). In terms of the four-velocity \( U_{k} = e^{\mu}_{k} U_{\nu}, \) (5·7) reads

\[
U^{i} B^{(s)}_{l_{1}...l_{k}}(r) = 0,
\]

where \( e^{\mu}_{k} \) and \( B^{(s)}_{l_{1}...l_{k}} \) are to mean their values on the world line \( X^{\mu}(r) \).

Use of (5·1) in (2·3) gives the spin tensor density as

\[
S^{ijk} = 2 e \hbar^{2} [(s - 1) H^{+ \{ j \{ l_{k} \} \} k} - \{ l_{k} \} l_{i} \} \phi_{\ldots l_{k}} + H^{\{ j \{ l_{k} \} \} k} \phi_{\ldots l_{k}}] - (s - 1)^{2} D_{m} \phi^{\dagger m} \phi \phi_{\ldots l_{k}} + (s - 1) D_{m} \phi^{\dagger m} \phi \phi_{\ldots l_{k}}
\]

\[+ \text{c.c.} + \text{(terms involving the auxiliary fields)}.\]

Substituting the wave packet solution of \( \phi \) in (5·9), and then putting the result in (2·2), we obtain the spin-current \( N^{\mu \nu l} \)

\[
N^{\mu \nu l} = i \hbar m (4 s B^{\dagger \{ \mu \} l_{k} B^{\nu} \phi_{\ldots l_{k}} U^{l_{k}} - 2 U^{\mu} B^{\dagger + l_{k} B^{\nu} \phi_{\ldots l_{k}}})
\]
\[ +2 \mathcal{U}[^{\mu} B^{\nu 
abla} - t_{\lambda} B_{\nu} - t_{\lambda}] \left( - U^\rho \int |\Omega|^2 e^{3x} \right), \] (5·10)

where \( B_{t_{\lambda} - t_{\lambda}} \) denotes \( B^{(s)}_{t_{\lambda} - t_{\lambda}} \) for short. With the help of (5·8) the intrinsic spin tensor \( S^\mu \nu \) of (2·4) is now given by

\[ S^\mu \nu = N^{\mu \nu} U \]

\[ = -4i \hbar m B^{t^{[\mu \nu} - t_{\lambda]} B^\nu - t_{\lambda}] \left( - U^\rho \int |\Omega|^2 e^{3x} \right). \] (5·11)

The constraint \( S^\mu \nu U_v = 0 \) is automatically satisfied due to (5·8). Using (5·11) in (5·10), we finally see that \( N^{\mu \nu} \) can be represented by \( S^\mu \nu \) and \( U^\mu \) as follows:

\[ N^{\mu \nu} = - S^\mu \nu U \frac{1}{s} U^{[\mu S^\nu \lambda]}, \] (5·12)

In the same way as in § 4, we can show that the rotational spin \( L^\mu \nu \) is negligibly small for the wave packet.

§ 6. Spin precession due to torsion

It has now been shown for semiclassical particles with arbitrary spin \( s(=j/2 \text{ with } j \text{ being a natural number}) \) described by wave packets that the spin-current is represented as

\[ N^{\mu \nu} = - S^\mu \nu U \frac{1}{s} U^{[\mu S^\nu \lambda]}, \] (6·1)

and that the rotational spin is negligibly small, \( L^\mu \nu = 0 \). Thus we have

\[ J^\nu = S^\nu. \] (6·2)

Substituting (6·1) and (6·2) in (2·7), we get the equation of spin precession due to torsion:

\[ \frac{\nabla S^\nu \nu}{\nabla \tau} = \left( U^\mu \nabla U^\nu \nu - U^\nu \nabla U^\mu \nu \right) S^\nu + f_1 \varepsilon_{\mu \nu \rho \sigma} U_\rho a_\sigma S^\nu + f_2 (U_{\mu \nu} t^{[\nu] \rho \sigma} U_\rho U_\sigma) S^\nu, \] (6·3)

where \( f_1 \) and \( f_2 \) are dimensionless, spin-dependent parameters defined by

\[ f_1 = - \frac{1}{2} \left( 1 + \frac{1}{s} \right), \quad f_2 = \frac{4}{3} \left( 1 - \frac{1}{2s} \right), \] (6·4)

and \( a_\sigma \) and \( t^{[\mu \nu]} \) are the axial-vector part and the tensor part of the torsion tensor, respectively. In deriving (6·3) we have used the fact that the contorsion tensor is represented as

\[ K_{\lambda \mu} = - \frac{2}{3} (t_{\lambda \mu} - t_{\mu \lambda}) - \frac{1}{3} (g_{\lambda \mu} v_{\nu} - g_{\mu \nu} v_{\lambda}) + \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} a^\rho \] (6·5)

in terms of the three irreducible components of the torsion tensor, \( v_\mu \), \( a_\mu \) and \( t_{\lambda \mu}. \)
Equation (6·3) agrees with our previous result for spin 1/2 and 1, which was derived by the WKB method alone. The expression (6·4) for \( f_1 \) and \( f_2 \) is linear in \( 1/s \), and does not support a conjecture\(^1\) that \( f_1 \) and \( f_2 \) might be linear in \( s \). Furthermore, since it is impossible to have both \( f_1=0 \) and \( f_2=0 \) at the same time, the intrinsic spin vector is not Fermi-Walker transported along the world line with respect to the ordinary, symmetric connection. Thus, the intrinsic spin necessarily precesses due to torsion irrespective of the spin value. We also note that \( S^\nu S_\nu \) is a constant of motion as is seen from (6·3).

We can rewrite (6·3) into an alternative form in terms of the covariant derivative \( D/D\tau \) with respect to the nonsymmetric connection:

\[
\begin{align*}
\frac{DS^\nu}{D\tau} &= \left( U^\nu \frac{DU^\mu}{D\tau} - U^\nu \frac{DU^\mu}{D\tau} \right) S_\nu \\
&\quad - \frac{1}{2S} e^{\rho\sigma\gamma} U^\rho a^\sigma S_\gamma - \frac{2}{3S} \left( U^{[\mu} U_\nu] - 3 U^{[\mu} U^{\nu]} U_\rho U_\sigma \right) S_\gamma.
\end{align*}
\]  
\[ (6·6) \]

This equation shows that the intrinsic spin vector is not Fermi-Walker transported along the world line relative to the nonsymmetric connection: We notice, however, that the Fermi-Walker transport is realized in the limiting case \( s \to \infty \). It should also be emphasized that the equation of spin precession due to torsion depends on the spin \( s \).

Finally, we briefly comment on the world line equation derived by the Fock-Papapetrou method. (More details will be given in Appendix D.) The intrinsic spin \( S^\nu \) is of the first-order in \( \hbar \), and hence it is negligible in the semiclassical limit \( \hbar \to 0 \): According to (6·1) and (6·2), this means that the spin-current \( N^{\nu\lambda} \) and the total spin \( J_\nu \) are also negligibly small. Thus, the world line equation is reduced to the geodesic one with respect to the Christoffel symbol in the semiclassical limit. This result is consistent with the Hamilton-Jacobi equation (4·9), since the latter equation is derived by the WKB method in the lowest order in \( \hbar \).

\[ § 7. \text{Conclusion} \]

We have studied the motion of wave packets with arbitrary spin in the Riemann-Cartan spacetime by means of the Fock-Papapetrou method and the WKB method. We have first calculated the spin-current \( N^{\nu\lambda} \) of the wave packets using the WKB method:

\[
N^{\nu\lambda} = - S^{\mu\nu} U^\lambda - \frac{1}{s} U^{[\nu} S^{\mu\lambda]}, \quad (7·1)
\]

\(^*1\) The relation linear in \( s \) was conjectured based on our result that \( N^{\nu\lambda} = - S^{\nu\mu} U^\lambda \) for spin-3/2 particles. However, this result is based on the massive version of the Lagrangian employed in the supergravity theory, in which the covariant derivative acts only on the spinor index of the Rarita-Schwinger field. If we use instead the full covariant derivative operating on both the spinor and the vector indices, then the resulting spin-current is shown to agree with (6·1), as was mentioned in a note added to Ref. 12). In view of the fact that the spin tensor density of (2·3) is reduced to the canonical spin tensor in the special relativistic limit only when the full covariant derivative is used in the minimal prescription, we take the latter result physically more reasonable.
where \( s = j/2 \) with \( j \) being a natural number) is the intrinsic spin in units of \( \hbar \), and \( S^{\mu\nu} \) is the intrinsic spin tensor orthogonal to the four-velocity \( U^\mu \). The rotational spin of the wave packets is found negligibly small compared to the intrinsic spin in the semiclassical limit: Namely, we can put

\[
J^\mu = S^\mu .
\]  

(7·2)

Using these results in the equations of spin precession and world line derived in a previous paper\(^{12}\) by the Fock-Papapetrou method, we have shown that the world line is the geodesics of the metric, and that the equation of spin precession is given by (6·3) with two spin-dependent parameters \( f_1 \) and \( f_2 \) of (6·4). It is worth emphasizing that the spin vector is not Fermi-Walker transported along the world line relative to the nonsymmetric connection, and that the spin precession due to torsion depends on the intrinsic spin \( s \).

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Appendix A

--- The Constraint on \( J^{\mu\nu} \) ---

It has been argued in a previous paper\(^{12}\) that one can put three constraints on \( J^{\mu\nu} \) since the equation for its temporal evolution has only three independent components. We shall show below that these constraints can indeed be fulfilled by choosing the world line \( X^\nu(\tau) \) appropriately.

Let us remember for this purpose that we are assuming the pole-dipole approximation for \( T^{\mu\nu} \) and the single pole approximation for \( S^{\mu\nu} \). In this appendix we denote by \( I[f] \) the integral of a function \( f(x) \) over three dimensional space,

\[
I[f] = \int f(x) d^3x .
\]  

(A·1)

Then we are assuming that \( I[(\delta x)^p T] \) (or \( I[(\delta x)^q S] \)) is vanishingly small if \( p \geq 2 \) (or \( q \geq 1 \)).

Now take another world line \( X'^\nu(\tau') \) in the world tube slightly shifted from \( X^\nu(\tau) \):

\[
X'^\nu = X^\nu + \Delta X^\nu , \quad U'^\nu = U^\nu + \Delta U^\nu ,
\]  

(A·2)

where \( \Delta X^0 = 0 \), and all the components of \( \Delta X^\nu \) and \( \Delta U^\nu \) are at most of the same order of magnitude as the size of the world tube. Then \( U_{\mu}J^{\mu\nu} \) is changed to

\[
U_{\mu}'J'^{\mu\nu} = U_{\mu}J^{\mu\nu} + [U_{\mu}P^\mu] \Delta X^\nu - (U_{\mu} \Delta X^\nu)P^\nu ,\]  

(A·3)

where \( P^\mu = I[T^{\mu0}] \), and we have used that \( \Delta U^\nu I[\delta x^\nu T^{\sigma\nu}] \), \( \Delta U^\nu I[S^{\mu\nu\sigma}] \) and \( \Delta U^\nu \Delta X^\nu I[T^{\mu0}] \) are negligibly small by assumption. Thus, we have

\[
U_{\mu}'J'^{\mu\nu} = 0
\]  

(A·4)
if we require $\Delta X^\nu$ to satisfy
\[ \Delta X^\nu = -\frac{U_\mu}{U_\lambda P^\lambda} P^\nu - \frac{U_\nu}{U_\lambda P^\lambda} J^\mu \phi^\nu. \] (A.5)

The solution of (A.5) satisfying the condition $\Delta X^0 = 0$ is
\[ \Delta X^\nu = \frac{1}{U_\lambda P^\lambda} U_\mu J^\mu \phi^\nu. \] (A.6)

We notice that the above argument can also be used to get the constraint $P_\nu J^\mu = 0$ instead of $U_\mu J^\mu = 0$. Similarly, the constraint $U_\nu L^\mu = 0$ (or $P_\nu L^\mu = 0$) can be postulated instead of that on $J^\nu$.

**Appendix B**

--- Proof of Eq. (4·17) ---

We use the following properties of the $\gamma$-matrices:
\[ \{ \gamma^i, \gamma^j \} = -2\eta^{ij}, \quad \gamma^0 = \gamma^i, \quad \gamma^a = -\gamma^a (a = 1, 2, 3), \]
\[ \gamma^0 \gamma^i \gamma^0 = \gamma^i, \quad \sigma^{ij} = \frac{i}{4} [\gamma^i, \gamma^j], \quad \gamma^5 = i\gamma^1 \gamma^2 \gamma^3. \] (B·1)

For simplicity, we shall consider the spin-3/2 Rarita-Schwinger field $\phi_m$. First of all, we note that the following equations can be verified:
\[ \frac{1}{2} \bar{\psi}^m \gamma^5 \phi^m = \frac{1}{2} e^{ijkl} \bar{\psi}^m \eta_{mn} \gamma^5 \phi^n \]
\[ = -\frac{i}{2} e^{ijkl} \bar{\psi}^m \{ \gamma_m, \sigma_{nl} \} \gamma^5 \phi^n \]
\[ = -\frac{i}{2} e^{ijkl} \epsilon_{mnkl} \bar{\psi}^m \gamma^5 \phi^n. \] (B·2)

The first and second equations hold owing to $\{ \sigma^{ij}, \gamma^k \} = e^{ijkl} \gamma^l$ and $[\gamma_m, \sigma_{nl}] = -i(\eta_{mn} \gamma_l - \eta_{ml} \gamma_n)$, respectively. We have also used the spinor-trace condition $\gamma_m \phi^m = 0$ in the second equation, and $(\gamma^5)^2 = 1$ in the last equation. Then using the well-known formula for the product of two Levi-Civita tensors in the last term of (B·2), we have
\[ \frac{1}{2} \bar{\psi}^m \{ \sigma^{ij}, \gamma^k \} \phi^m = \frac{i}{2} \left( \bar{\psi}^j \gamma^i \phi^k + \bar{\psi}^i \gamma^k \phi^j + \bar{\psi}^k \gamma^j \phi^i \right) \]
\[ - \frac{i}{2} \left( \bar{\psi}^j \gamma^i \phi^k + \bar{\psi}^k \gamma^i \phi^j + \bar{\psi}^i \gamma^j \phi^k \right). \] (B·3)

This proves (4·17) for the Rarita-Schwinger field. We can proceed quite in the same manner to show (4·17) for tensor-spinor fields with two or more tensor indices.

**Appendix C**

--- The Spin-Current $N_{\mu \nu \lambda}^\alpha$ for Dirac Particles ---

The spin tensor density of the Dirac field takes the form:
Putting (C·1) in (2·2), we have

\[ N_{\mu\lambda} = \frac{1}{2} \epsilon_{\mu\lambda\sigma} N_\sigma, \]  

where \( N_\sigma \) is defined by

\[ N_\sigma = -\hbar U^0 \int (\bar{\psi} \gamma_\sigma \psi) e^3 x \]

\[ \approx \hbar A \gamma_\sigma A \left( -U^0 \int |\psi|^2 e^3 x \right). \]  

(C·3)

Multiplying (C·2) by \( U_\lambda \), we get

\[ S^{\mu\nu} = N^{\mu\lambda} U_\lambda = \frac{1}{2} \epsilon^{\mu\lambda\sigma} U_\lambda N_\sigma. \]

(C·4)

Since \( N_\mu U^\mu = 0 \) holds in the lowest-order approximation of the WKB method, multiplying (C·4) by \( \epsilon_{\mu\nu\lambda\tau} \) gives

\[ N_\tau = \epsilon_{\mu\nu\lambda\tau} S^{\mu\lambda} U^\nu. \]

(C·5)

Putting (C·5) in (C·2), we obtain

\[ N^{\mu\lambda} = -S^{\mu\nu} U_\lambda - 2U_\lambda S^{\nu\lambda}. \]

(C·6)

This is just Eq. (4·21) for \( s=1/2 \).

---

Appendix D

The Equation of World Line

The covariant equation of world line has been obtained by means of the Fock-Papapetrou method as follows: \(^{12}\)

\[ \frac{\partial \bar{P}_\mu}{\partial x} + \frac{1}{2} R^{\rho\sigma\mu\nu} f_{\rho\sigma} U_\nu - \frac{1}{2} (\partial^{\mu} K^{\rho\sigma}) N_{\rho\sigma} = 0, \]

(D·1)

where the four-vector \( \bar{P}_\mu \) means

\[ \bar{P}_\mu = \bar{m} U_\mu - U_\nu \left( \frac{\partial J^{\mu}}{\partial x} + K_\rho \sigma_\mu N^{\rho\sigma} - K_{\rho} \sigma_\nu N^{\rho\sigma} + \frac{1}{2} K_{\rho} \sigma_\nu N^{\rho\sigma} - \frac{1}{2} K_{\rho} \sigma_\nu N^{\rho\sigma} \right). \]

(D·2)

and the scalar \( \bar{m} \) is defined by

\[ \bar{m} = -P^\mu U_\mu + U_\nu \{U^{\mu} \partial^{\nu} U^{\sigma} + U^{\nu} \partial^{\rho} \} N^{\rho\sigma} U^{\mu} - \frac{1}{2} K_{\rho\sigma} U^\nu N^{\rho\sigma} U^{\mu} \]

\[ = -\bar{P}^\mu U_\mu \]

(D·3)

with \( P^\mu \) being given by

\[ P^\mu = \int T_{\mu\nu} d^3 x. \]

(D·4)
Here $R^\rho{}_{\sigma\mu\nu}$ denotes the Riemann-Christoffel curvature tensor formed of the metric.

If we use (6·1) and (6·2), and if we retain only those terms which are lowest-order in $\hbar$, (D·1) is reduced to

$$\frac{\partial \tilde{m} U^\mu}{\partial \tau} = 0 . \tag{D·5}$$

It follows from (D·5) that $\tilde{m}$ is a constant of motion:

$$\frac{\partial \tilde{m}}{\partial \tau} = \frac{d \tilde{m}}{d \tau} = 0 . \tag{D·6}$$

Therefore (D·5) becomes the geodesic equation:

$$\frac{\partial U^\mu}{\partial \tau} = 0 . \tag{D·7}$$

This result is consistent with that in the lowest-order approximation of the WKB method.

If we neglect in (D·3) all the terms of order of $\hbar$, the $\tilde{m}$ is given by $\tilde{m} = -P^\mu U_\mu$.

It can be shown in the same approximation that the energy-momentum tensor for the wave packet is expressed by

$$T^{\mu\nu} = m U^\mu n^\nu , \tag{D·8}$$

where $n^\nu$ is the particle-number current density defined by

$$n^\nu = -\frac{i}{\hbar} \left( \frac{\partial L}{\partial q_{\nu}} q - \bar{q} \frac{\partial L}{\partial \bar{q}_{\nu}} \right) \tag{D·9}$$

and satisfies the conservation law $\partial_\mu n^\mu = 0$. Here $q$ denotes $\Psi$ and $\Phi$ for fields with half-integer and integer spin, respectively. Consequently, when the total particle number is normalized to one, $\tilde{m}$ coincides with the mass $m$.

References

9) W. Pauli, Helv. Phys. Acta 5 (1932), 179; Die Allgemeine Prinzipien der Wellenmechanik in
10) For a Dirac particle in the magnetic field, see
11) For spin precession of a Dirac particle due to torsion, see
    (1980), 883.