Nucleon-Trinucleon Scattering in Terms of the Schmidt Expansion. I

—— Faddeev’s Residue Prescription ——

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In order to define the elastic/rearrangement and the breakup amplitudes for nucleon-trinucleon (NT) scattering in conformity with Faddeev’s residue prescription, a pole term decomposition of the amplitudes for 3+1 and 2+2 subsystems is introduced. New Osborn-type equations for these four-nucleon amplitudes are derived. The obtained Osborn equations are expressed in practical form using the Schmidt expansion for each subamplitude. In order to clarify the difference between Faddeev’s formalism and the previously proposed one based on the various pole expansions, a detailed comparison of them is made.

§ 1. Introduction

Although Faddeev\(^1\) established rigorously a three-body scattering theory based on the spectral representation of the mathematical sphere, his theory has not been familiar to many physicists and has been somewhat neglected. Osborn\(^2\) gave a simple physical explanation of Faddeev’s approach and provided a derivation of integral equations for each physical amplitude. Karlson and Zeiger\(^3\) also obtained their own equation and generalized them to the four-body case.\(^4\) Meanwhile Alt, Grassberger and Sandhas (AGS)\(^5\) succeeded in finding appropriate scattering equations for \(N\)-body system and an application of the various pole expansions(PE)\(^6\) to these equations has been presented.

Since then numerical calculation for four-body problem has been practised gradually.\(^7,8\) Nevertheless, in the scattering problem, it was found that there exists a discrepancy in the final solutions\(^9\) of our and PE methods, in spite of starting from the same Yakubovskii equation. Then the elucidation of the difference between them comes up as an important problem. It seems that the important point to consider the difference is an introduction of the pole term decomposition, which corresponds to the spectral representation of the scattering amplitude. The spectral representation is a sum of the pole term and the continuum term. These terms are composed of the projection operators to the eigenstates for the point and continuous spectra respectively. Faddeev proposed the residue prescription for the definition of the three-body amplitudes, in which the pole term contributes to the elastic/rearrangement amplitude and the continuum term does only to the breakup one.

If we assume rank one separable potential, the two-body \(T\)-matrix contains the pole term alone, and the pole term decomposition is required only for the 3+1 and 2+2 subamplitudes. Then in the case of rank one potential, the derivation of Osborn’s equation from the Yakubovskii one becomes more straightforward. Eliminating the unphysical part from the total amplitude, we can extract the NT
scattering amplitudes in which a trinucleon bound state is in the initial configuration. From these NT scattering amplitudes, we must further extract the elastic/rearrangement amplitudes. The remaining parts of the NT amplitudes contribute only to the breakup process. The 'breakup amplitude' is defined as a sum of these amplitudes: the elastic/rearrangement and the remaining parts of the NT amplitudes. New Osborn-type equations for the elastic and rearrangement amplitudes and the parts contributing only to the breakup process are derived in general operator form.

Narodetskiǐ first obtained the practical four-body equations by introducing the Hilbert-Schmidt (HS) expansion, which gives the separable forms of the subamplitudes. He could define correctly the elastic/rearrangement amplitude of the four-nucleon system in terms of the HS expansion. Unfortunately, we cannot use the HS expansion for the scattering problem, because the HS expansion can be applied only to a self-adjoint operator which can be realized only in the bound state problem. The integral kernels of the Faddeev equations for three-body scattering become the non-self-adjoint operator, as the total energy of the scattering state is in the continuous spectrum region. Instead we must introduce the Schmidt expansion theorem, which is the HS theorem generalized to the non-self-adjoint operator.

If we introduce the Schmidt expansion, each subamplitude is expressed in a Fredholm form, which has a Fredholm determinant (FD) as its denominator. Then we proposed the pole term decomposition of each subamplitude, in which only the pole term contains the FD in its denominator. It is important that the residue of this pole term at the pole energy constitutes a bound state projection; this allows us to define the amplitudes of real physical processes in the four-nucleon system.

In the PE approach, to define the elastic/rearrangement amplitudes the author did not employ the pole term decomposition for each subamplitude. The unknown functions are defined as a matrix element of the AGS operator taken between the arbitrarily chosen bases. Then without the derivation of Osborn's equation, the Yakubovskii equation is directly used for numerical calculation; this leads to the main difference between our and the PE approach.

In § 2, the Osborn equation of the general operator for the NT scattering is presented. Following the method of the general operator, the Osborn equation for NT scattering amplitudes in terms of the Schmidt expansion is derived in § 3. Based on the formalism described in the previous sections, several differences between our and PE approach will be discussed in § 4.

§ 2. Osborn equation for general operators of NT scattering

In this section, we shall derive the Osborn equation for the general operators of the NT scattering. If the two-body potential is taken to be rank one separable form \( V = \lambda |g\rangle \langle g| \), the corresponding \( T \)-matrix is written as

\[
T(z) = |g\rangle \tau(z) \langle g| \quad \text{with} \quad \tau(z) = -[\lambda^{-1} + \langle g|G_0(z)|g\rangle]^{-1}.
\]

The \( 3+1/2+2 \) subamplitude is expressed by \( w(z)/v(z) \).

Let \( N \), \( T \) and \( D \) be a nucleon, a trinucleon and a deuteron. If we shall express
by \( A \) and \( B \) the four-nucleon amplitudes, which describe the reactions \( NT \to NT \) and \( NT \to DD \), respectively, then the amplitudes \( A \) and \( B \) satisfy the following Yakubovskii equation:

\[
A(p', q'; p, q; z) = \left( \frac{3}{\sqrt{3}} \right)^3 w(p', Q; s') \tau(z_1) w(q_z, p; s) + \left( \frac{3}{\sqrt{8}} \right)^3 \int w(p', Q_1; s') \tau(z_0) A(Q_z, q''; p, q; z) dq''
\]

\[
+ 2 \left( \frac{\sqrt{3}}{2} \right)^3 \eta^p \int w(p', R_1; s') \tau(z_0) B(R_z, q''; p, q; z) dq'' ,
\]

\[
B(p', q'; p, q; z) = \left( \frac{3}{\sqrt{2}} \right)^3 \eta^p v(p', S_1; s') \tau(z_0) w(S_z, p; s) + \left( \frac{3}{\sqrt{2}} \right)^3 \eta^p \int v(p', S_1; s') \tau(z_0) A(S_z, q''; p, q; z) dq'' ,
\]

where

\[
Q_1 = q + 3q', \quad Q_2 = q + 3q, \quad S_1 = q + 3q', \quad S_2 = q' + 3q,
\]

\[
R_1 = S_2, \quad R_2 = S_1 ,
\]

\[
Q_1 = q + 3q', \quad Q_2 = q + 3q, \quad S_1 = q + 3q', \quad S_2 = q' + 3q,
\]

\[
z_1 = s' - Q_1^2, \quad z_2 = s' - Q_1^2, \quad z_3 = s' - S_1^2, \quad z_4 = s' - S_1^2.
\]

\[
s' = z - q^2, \quad s = z - q^2, \quad s^2 = q^2 / 2m.
\]

We then decompose \( A \) and \( B \) by explicitly factoring out the primary singularities. To do this, we need to expand \( w \) and \( v \) around their bound-state poles. These pole decompositions are, in the case of only one bound state for each system:

\[
w(p', p; s) = \frac{\Phi_T(p)}{s + E_T} + \bar{w}(p', p; s),
\]

\[
v(p', p; s) = \frac{\Phi_D(p)}{s + E_D} + \bar{v}(p', p; s).
\]

Here \( \Phi_T(p) \) is a 'vertex function' related to the trinucleon bound state wave function \( \Psi_T(k, p) \) with the energy \( -E_T \), by

\[
g(k) \tau(-E_T - \vec{p}^2) \Phi_T(p) = (\vec{k}^2 + \vec{p}^2 + E_T) \Psi_T(k, p),
\]

and \( \Phi_D(p) \) is a vertex function related to a state of two bound pairs. The first terms of the right-hand side of Eq. (3) are the pole terms and the second ones are the continuum terms. With these pole term decompositions (3), we can decompose \( A \) and \( B \) as follows:
\[
A(p', q'; p, q; z) = \mathcal{F}(p', q'; p, q; z) + \frac{\Phi_r(p')}{s' + E_r} \mathcal{U}(q'; pq; z) \\
+ \mathcal{U}(p', q'; p, q; z) - \frac{\Phi_r(p')}{s' + E_r} + \mathcal{A}(q', q; z) - \frac{\Phi_r(p')}{s' + E_r},
\]

\[
B(p', q'; p, q; z) = \mathcal{F}(p', q'; p, q; z) + \frac{\Phi_r(p')}{s' + E_D} \mathcal{C}(q'; pq; z) \\
+ \mathcal{C}(p', q'; p, q; z) - \frac{\Phi_r(p')}{s' + E_D} - \mathcal{B}(q', q; z) + \frac{\Phi_r(p')}{s' + E_D}.
\]

\(\mathcal{A}\) gives elastic scattering amplitude and \(\mathcal{B}\) gives rearrangement amplitude. As the rank of our potential is one, we cannot distinguish between partial and full breakup amplitude but they must share common amplitudes, which are both defined as a breakup amplitude. They are

\[
qu + \frac{\Phi_r A}{s' + E_r} \quad \text{and} \quad qv + \frac{\Phi_r B}{s' + E_D}.
\]

We have a coupled equation for amplitudes \(\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{C}\):

\[
\mathcal{A}(q', q; z) \\
= \left(\frac{3}{\sqrt{8}}\right)^3 \bar{\Phi}_r(Q_1) \tau(z_1) \Phi_r(Q_2) \\
+ \left(\frac{3}{\sqrt{8}}\right)^3 \int \bar{\Phi}_r(Q_1) \tau(z_2) \left\{ \frac{\Phi_r(Q_2^1)}{s'' + E_r} \mathcal{A}(q'', q; z) + \mathcal{U}(Q_2^2, q''; q; z) \right\} dq'' \\
+ 2\left(\sqrt{\frac{3}{2}}\right)^3 \eta^r \int \bar{\Phi}_r(S_1^1) \tau(z_3) \left\{ \frac{\Phi_r(S_2^1)}{s'' + E_D} \mathcal{B}(q'', q; z) + \mathcal{C}(S_2^2, q''; q; z) \right\} dq''.
\]

\[
\mathcal{U}(p', q'; q; z) \\
= \left(\frac{3}{\sqrt{8}}\right)^3 \bar{\mathcal{w}}(p', Q_1; s') \tau(z_1) \Phi_r(Q_2) \\
+ \left(\frac{3}{\sqrt{8}}\right)^3 \int \bar{\mathcal{w}}(p', Q_1; s') \tau(z_2) \left\{ \frac{\Phi_r(Q_2^1)}{s'' + E_r} \mathcal{A}(q'', q; z) + \mathcal{U}(Q_2^2, q''; q; z) \right\} dq'' \\
+ 2\left(\sqrt{\frac{3}{2}}\right)^3 \eta^r \int \bar{\mathcal{w}}(p', S_1^1; s') \tau(z_3) \left\{ \frac{\Phi_r(S_2^1)}{s'' + E_D} \mathcal{B}(q'', q; z) + \mathcal{C}(S_2^2, q''; q; z) \right\} dq''.
\]

\[
\mathcal{B}(q', q; z) \\
= \left(\sqrt{\frac{3}{2}}\right)^3 \eta^r \Phi_r(S_1) \tau(z_2) \Phi_r(S_2) \\
+ \left(\sqrt{\frac{3}{2}}\right)^3 \eta^r \int \Phi_r(S_1^1) \tau(z_3) \left\{ \frac{\Phi_r(S_2^1)}{s'' + E_r} \mathcal{A}(q'', q; z) + \mathcal{U}(S_2^2, q''; q; z) \right\} dq''.
\]
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\[ C_{\nu}(p', q', q; z) = \left( \sqrt{\frac{3}{2}} \right)^3 \eta^p \bar{u}(p', S_1; s') \tau(z_\nu) \Phi_T(S_2) \]
\[ + \left( \sqrt{\frac{3}{2}} \right)^3 \eta^p \int \bar{u}(p', S_1; s') \tau(z_\nu) \left\{ \Phi_T(S_2') \frac{\Phi_T(S_2)}{s'' + E_T} \overline{\lambda}(q'', q; z) + U(S_2', q'', q; z) \right\} dq'', \]

with \( s'' = z - q''^2 \).

This is the Osborn equation generalized to the four-body case.

§ 3. Four-nucleon Osborn equation in terms of the Schmidt expansion

In this section, the practical version of Osborn equation for the NT scattering will be derived along the procedure described in the previous section. We shall start to obtain the separable expression for each subamplitude.

A. Schmidt expansion for each subamplitude

As assumed in the previous section, taking the two-body potential to be of rank one separable form, the \( l \)-th partial wave three-nucleon off-shell amplitude satisfies the symmetrized Faddeev equation:\(^9\)

\[ u^l(p', p; z) = \frac{1}{4\pi} K^l(p', p; z) + \int_0^\infty K^l(p', p''; z) u^l(p'', p'; z) p''^2 dp''. \] (4)

From the Schmidt theorem, the kernel \( K^l(p', p; z) \) can be expanded as follows:

\[ K^l(p', p; z) = \sum_{i=1}^\infty \eta_i(z) \phi_i(p'; z) \overline{\psi}_i(p; z). \]

If we insert this \( K^l_i \) with the terms of \( i > n \) truncated, into Eq. (4), the solution can be expressed in a Fredholm form:\(^9\)

\[ u^{(n)}_{l}(p', p; z) = \frac{D^{(n)}(p', p; z, 1)}{D^{(n)}(z, 1)}, \] (5)

where

\[ D^{(n)}(z, 1) = \begin{vmatrix} 1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1 - a_{nn} \end{vmatrix} \]

is Fredholm's determinant with the matrix elements

\[ a_{ii}(z) = \eta_i(z) \int_0^\infty \overline{\psi}_i(p; z) \phi_i(p; z) p^2 dp \]

and
B. Pole term decomposition of the subamplitude in Fredholm’s form

For the following Fredholm integral equation of the second kind:

$$\chi(x) = f(x) + \lambda \int_{\Omega} N(x, y) \chi(y) dy,$$

we have the resolvent kernel $R$, being the ratio of the second Fredholm series to the first one,

$$R(x, y; \lambda) = D\left(\frac{x}{y}; \lambda\right)/D(\lambda).$$

We now state the following two theorems\(^{(6)}\) about the pole term decomposition of the resolvent kernel $R(x, y; \lambda)$ in the Fredholm form.

**Theorem I.** Under the condition $D(\lambda) \neq 0$, $D(\frac{s}{t}; \lambda) \neq 0$ for $s, t \in \Omega$, we have the following relation:

$$\frac{D\left(\frac{x}{y}; \lambda\right)}{D(\lambda)} = \frac{D\left(\frac{x}{t}; \lambda\right)D\left(\frac{s}{y}; \lambda\right)}{D(\lambda)D\left(\frac{s}{t}; \lambda\right)} + \frac{D\left(\frac{x}{y}; \lambda\right)}{D(\lambda)}.$$

**Theorem II.** When $\lambda_0$ is an eigenvalue of rank one and then the resolvent $R(x, y; \lambda)$ has a pole of order one at $\lambda_0$, we designate the residue of $R$ at this pole $c(x, y)$. Then it has the form

$$c(x, y) = \frac{D\left(\frac{x}{t}; \lambda_0\right)D\left(\frac{s}{y}; \lambda_0\right)}{D(\lambda_0)D\left(\frac{s}{t}; \lambda_0\right)}, \quad \left[D\left(\frac{s}{t}; \lambda_0\right) \neq 0\right],$$

satisfying the relation

$$\int_{\Omega} \frac{D\left(\frac{x}{t}; \lambda_0\right)D\left(\frac{s}{y}; \lambda_0\right)}{D(\lambda_0)D\left(\frac{s}{t}; \lambda_0\right)} dx = -1 \quad \text{with} \quad D^{(1)}(\lambda) = \frac{d}{d\lambda} D(\lambda).$$

Equation (6) indicates that the resolvent $R$ is decomposed into two ratios, the pole term and the continuum term: The pole term has the Fredholm determinant (FD) in its denominator, while on the contrary the continuum term does not have the FD. Here, we should note that the residue $-c(x, y)$ gives a projection operator to the
bound state, which enables us to define each physical amplitude correctly.

For the case $n=1$, the last term of Eq. (6) is identically zero, and so the resolvent has only the pole term. Then the reason why in the case of rank one the separable expression of the subamplitude is not a realistic approximation, is such that the breakup process must share the same amplitude with the elastic/rearrangement processes. The same argument holds for the two-body $T$-matrix to define the three-body amplitudes. If the rank $n(n\geq2)$ separable potential is assumed, the corresponding two-body $T$-matrix can be expressed as the sum of the pole term and the continuum one. This allows us not only to define each three-body amplitude separately, but also to distinguish the partial and full breakup amplitudes of the four-body system.

The pole term decompositions of the $3+1$ and $2+2$ subamplitudes are written as follows:

$$w_1(p', z) = -\frac{1}{4\pi} \sum_{k=1}^{n+1} w_k(p', z) d_k(z) \chi_k(p; z),$$

$$v(p', z) = -\frac{1}{4\pi} \sum_{k=1}^{n+1} v_k(p', z) d_k(z) \theta_k(p; z),$$

where

$$w_1(p; z) = p^{-1} \tau^{-(1/2)}(t) D^{(n)}(p; s; z; 1),$$

$$\chi_1(p; z) = p^{-1} \tau^{-(1/2)}(t) D^{(n)}(s; p; z; 1),$$

$$(d_1^{-1})(z) = D^{(n)}(z; 1) D^{(n)}(s; s; z; 1),$$

$$(d_k^{-1})(z) = D^{(n)}(s, s; z; 1), \quad (k \geq 2)$$

$$w_2(p; z) = p^{-1} \tau^{-1/2}(t) \{ \varphi_1(p; z) \varphi_2(s; z) - \varphi_2(p; z) \varphi_1(s; z) \},$$

and so on. In the case $z = E_T$, both $w_1(p; E_T)$ and $\chi_1(p; E_T)$ give the $n$-th approximation of the trinucleon bound state wave function. In Eq. (7), the term with $k=1$ corresponds to the pole term and the terms with $k \geq 2$ correspond to the continuum term. $n_c$ is the combination in which we take two numbers of the set $1, 2, \cdots, n$.

C. Osborn equation for the NT amplitudes

The partial wave decompositions of the functions $A$ and $B$ have the forms:

$$A(p', q'; p, q; z) = (4\pi)^2 \sum_{\ell, \ell', \ell''} A_{\ell \ell' \ell''}(p', q'; p, q; z) Y^{*}_{\ell'} Y_{\ell''}(\Omega_{p'q'}) Y_{\ell''}(\Omega_{pq}),$$

$$B(p', q'; p, q; z) = (4\pi)^{3/2} \sum_{\ell \ell''} B_{\ell \ell''}(p', q'; p, q; z) Y_{\ell}(\Omega_q) Y_{\ell''}(\Omega_{pq}),$$

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The angular momentum decomposition analysis reduces Eq. (2) to the following system of two-dimensional integral equations for $A^h_{l,v,u}$ and $B^h_{l}$:

$$
A^h_{l,v,u}(p', q'; p, q; z) = \left(\frac{3}{\sqrt{8}}\right)^3 \int w''(p', Q_1; s') \tau(z_3) w'(Q_2, p; s) \mathcal{W}_{l,v,u}(x) dx
$$

$$
+ \left(\frac{3}{\sqrt{8}}\right)^3 \int \int w''(p', Q; s') \tau(z_3) \sum_{p'} \mathcal{A}^l_{v', v''}(x')
$$

$$
\times A^h_{l,v', u}(Q_2, q''; p, q; z) dx' q''^2 dq''
$$

$$
+ \left(\frac{3}{2}\right)^3 \int w''(p', R_1; s') \tau(z_3) \mathcal{B}^l_{v'}(x')
$$

$$
\times B^h_{l}(R_2, q''; p, q; z) dx' q''^2 dq''
$$

$$
B^h_{l}(p', q'; p, q; z) = \left(\frac{3}{2}\right)^3 \int v(p', S_1; s') \tau(z_3) w'(S_2, p; s) \mathcal{B}^l_{v'}(x) dx
$$

$$
+ \left(\frac{3}{2}\right)^3 \int \int v(p', S; s') \tau(z_3) \sum_{p''} \mathcal{D}^l_{v'}(x')
$$

$$
\times A^h_{l,v', u}(S_2, q''; p, q; z) dx' q''^2 dq''
$$

with $x = q \cdot q'$, $x' = q' \cdot q''$. (8)

The functions for the angular momentum part, $\mathcal{W}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are shown in Ref. II.

If we insert the pole term decomposition (7) into the amplitudes $A$ and $B$, we have the following decompositions for them, which give each physical amplitude as the residues at the singularities of $A$ and $B$:

$$
A^h_{l,v,u}(p', q'; p, q; z) = \sum_n w^{(p', s')}_n \delta^{(s')} \delta^{(p, s)} A^{l(v, u)}_{n, l}(q', q; z) d^{(p, s)} n \chi^{(l)}(p; s)
$$

$$
B^h_{l}(p', q'; p, q; z) = \sum_n v^{(p', s')}_n \delta^{(s')} \delta^{(p, s)} B^{l(v, u)}_{n, l}(q', q; z) d^{(p, s)} n \chi^{(l)}(p; s)
$$

It is in the case $n = 1$ that $A^{l(v, u)}_{n, l}$ and $B^{l(v, u)}_{n, l}$ give the physical NT scattering amplitudes. $A$ and $B$ with $n \geq 2$ correspond to the unphysical amplitudes, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$, defined in § 2. If we put $n' = n = 1$ in the $A^{l(v, u)}_{n, l}$ and $B^{l(v, u)}_{n, l}$, they give the elastic and rearrangement amplitudes, respectively. From the explicit form

$$
A^{1}_{l,v,u}(q', q; z) = \int \chi^1(p; z - \bar{q}^2) F_{l,v,u}(p, q, q'; q, z) w_i(p; z - \bar{q}^2) dp^2 dp'\,
$$

with $F = \tau\langle g|G_0 U G_0|p\rangle \tau$, we can see that $A^{1}_{l,v,u}$ is a matrix element of the four-nucleon AGS operator $U$ taken between the trinucleon bound state wave function. $A^{1}_{l,v,u}$ and $B^{1}_{l,v,u}$ ($n' \geq 2$) correspond to the amplitudes $\mathcal{U}$ and $\mathcal{V}$, which contribute only to the breakup process as defined in § 2. $A^{1}_{l,v,u}$ and $B^{1}_{l,v,u}$ satisfy the following equation:
where

\[
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\]

\[A_{\ell_{123}^n}(q', q; z) = \left( \frac{3}{\sqrt{8}} \right)^3 \left( \frac{-1}{4\pi} \right)^2 \sum_{m_1} \int_0^1 X_{\ell_{123}^n}(q', q, x; z) \mathcal{W}_{\ell_{123}^n}(x) dx \]

\[+ \left( \frac{3}{\sqrt{8}} \right)^3 \left( \frac{-1}{4\pi} \right)^2 \sum_{m_1} \sum_{m_2} \int_0^1 X_{\ell_{123}^n}(q', q', x'; z) d_{m_1 m_2}^n(s') \]

\[\times \mathcal{A}_{\ell_{123}^n}(x') A_{\ell_{123}^n}(q', q; z) dx' q'^2 dq' \]

\[+ 2\sqrt{3} \left( \frac{3}{2} \right)^3 \left( \frac{-1}{4\pi} \right)^2 \sum_{m_1} \sum_{m_2} \int_0^1 Y_{\ell_{123}^n}(q', q', x'; z) d_{m_1 m_2}^n(s') \]

\[\times \mathcal{B}_{\ell_{123}^n}(x') B_{\ell_{123}^n}(q', q; z) dx' q'^2 dq' , \]

\[B_{\ell_{123}^n}(q', q; z) = \left( \frac{3}{2} \right)^3 \left( \frac{-1}{4\pi} \right)^2 \int_0^1 Z_{\ell_{123}^n}(q', q, x; z) \mathcal{W}_{\ell_{123}^n}(x) dx \]

\[+ \left( \frac{3}{2} \right)^3 \left( \frac{-1}{4\pi} \right)^2 \sum_{m_1} \sum_{m_2} \int_0^1 Z_{\ell_{123}^n}(q', q', x'; z) d_{m_1 m_2}^n(s') \]

\[\times \mathcal{D}_{\ell_{123}^n}(x') A_{\ell_{123}^n}(q', q; z) dx' q'^2 dq' , \]

(10)

where

\[X_{\ell_{123}^n}(q', q, x; z) = \chi_{\ell_{123}^n}(Q_1; s') t(z_1) w_n'(Q_2; s) , \]

\[Y_{\ell_{123}^n}(q', q, x; z) = \chi_{\ell_{123}^n}(R_1; s') t(z_2) w_n(R_2; s) , \]

\[Z_{\ell_{123}^n}(q', q, x; z) = \theta_n(S_1; s') t(z_3) w_n(S_2; s) . \]

Finally, we can get the Osborn equation for the NT scattering amplitudes in terms of the Schmidt expansion once we extract the coupled equation for the amplitudes \(A_{\ell_{123}^n}, B_{\ell_{123}^n}\) and \(A_{\ell_{123}^n}, B_{\ell_{123}^n}(n' \geq 2)\) from Eq. (10).

§ 4. Comparison with the pole expansion approach

In this section the comparison between our method and various pole expansion (PE) methods will be discussed. Haberzettl and several groups proposed the PE methods as a simplification of the HS expansion and studied the application to the four-nucleon problem. Fonseca worked on the numerical analysis of these theories. As investigated in the previous section our method is obtained by generalizing the Faddeev-Osborn three-body theory to the four-body case. This brings out the basic differences between our method and the PE methods in the definition of each physical amplitude and in the practical techniques to get solutions of the equations.

In the PE method the 3+1 subamplitude \(w(z)\) is written as an operator of rank \(N\)

\[w^m(z) = \sum_{n=1}^{N} |w_n(B)\rangle D_{mn}(z, B) \langle w_n(B)| , \]

(11)

where \(w_n(B)\) is the Strumian function calculated at the arbitrary energy \(B\) below the continuous spectrum. \(D_{mn}(z, B)\) is the element of an \(N \times N\) matrix,

\[D_{nm}^{-1}(z, B) = \Delta_{nm}(z, B) - \langle w_n(B)| t(z)| w_m(B) \rangle , \]
where \( [A^{-1}(z, B)]_{nm} = \langle \psi_n(B) | r(z) K(z) r(z) | \psi_m(B) \rangle \).

Comparing this with our bases \( \varphi_n(p; z) \) or \( \phi_n(p; z) \) in the expression (5), we can see that the bases \( | \psi_n(B) \rangle \) of (11) do not depend on the energy \( z \) and the \( z \) dependence is forced into the propagators \( D_{mn}(z, B) \); these simplifications of the PE method make the matter more difficult in the scattering problem.

In our method we have already introduced the pole term decomposition (6) to get the spectral representation for our \( 3+1 \) subamplitude. As the numerator of the pole term becomes a bound state projection on the energy shell, our NT elastic amplitude can be defined as \( A^{11}(q', q; z) \) given by Eq. (9):

\[
A^{11}(q, q'; z) = \langle \chi_t(z - \bar{q}^2) | F(q, q'; z) | \psi_t(z - \bar{q}^2) \rangle.
\]

On the other hand, without obtaining the pole term decomposition of \( w^N(z) \), Fonseca expanded the four-body total amplitude \( T \) with the expansion (11),

\[
T(q', q; z) = \sum_{m, n, l, k=1}^N | \psi_m(B) \rangle D_{mn}(z, B) T_{nl}(z, B) D_{lk}(z, B) | \psi_k(B) \rangle.
\]

In this expansion, he defined the elastic amplitude as \( \sum_{l, i=1}^N T_{ni}(z, B) \), where

\[
T_{ni}(z, B) = \langle \psi_n(B) | F(z) | \psi_l(B) \rangle.
\]

Since Fonseca's elastic amplitudes are obtained by summation of total \( n \) and \( l \) \( (n, l = 1, 2, ..., N) \), it becomes hard to distinguish the breakup amplitude from the elastic one. Then, it is considered that in the PE approach the Yakubovskii equation is used for the practical calculation, without deriving Osborn's equation: Osborn's equation is a simultaneous equation for the elastic/rearrangement and the breakup amplitudes.

One more difference of our formalism is, as seen from the expressions of \( A^{11}(z) \) and \( T_{ni}(z, B) \), the elastic amplitudes of the PE methods have \( B \) dependence. This fact is mentioned repeatedly not only in the introduction but also in the text of Fonseca's paper. In his paper he showed further, that he had gotten different results of \( DD \rightarrow DD \) phase shifts for different values of \( B \). Consequently, in our approach the improvement of the points described above brought about considerable improvement in our numerical results.

References