Rayleigh wave remains unchanged. The half-order singularity associated with the circular Rayleigh wave also remains unchanged. The only difference is that at $y = 0$ and $t = t_R$, the point on the circular Rayleigh wave where the point load is traveling, the singularity is of order three halves (the factor n in (1) contributes a first-order singularity for $c = c_R$ and $y = 0$). This corresponds to a coupling of the singularity under the point load, which exists for all values of c, with the singularity in the circular Rayleigh wave. Meanwhile, the response of the remainder of the surface of the half space is well behaved. In no way does the entire half space respond in a singular manner for a point load traveling steadily at the Rayleigh wave speed.

The expansions in (1) and (2) can also be derived, for Poisson's ratio equal to a quarter, from Payton's results [5] for a point load moving at a constant velocity. Similar couplings of impulsive surface loads traveling at constant velocities with Rayleigh waves have been observed by Ang [6] and Payton [7] for moving line loads, and by the discusser [8] for an expanding ring load.

The only case in which the response of the entire half space can be expected to be singular for a load traveling steadily at the Rayleigh wave speed is for the related steady-state problem, where no initial conditions are employed and the solution is expressed in terms of coordinates moving with the load. Cole and Huth [9] noted this in their steady-state solution for a moving line load. Gol'dshtein [10] observed the same thing for a moving step load. Of interest is the fact that these authors reached their conclusions by different techniques. Cole and Huth formulated a steady-state problem, while Gol'dshtein obtained his steady-state solution as a limit case of a transient solution.

In conclusion, a load traveling steadily at the Rayleigh wave speed does not cause a singular response of the entire half space, if a transient problem is considered as in [1, 5–8]. Rather, such a singular response may only occur for steady-state problems, as in [9, 10]. Therefore, it is not necessary to consider a load with variable velocity to avoid such a singular response. The discusser, however, does not wish to discredit the author's work, since variable velocity loads are of great interest for understanding the transonic effect of a load whose speed varies through one or more of the characteristic wave speeds of the medium.

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6 Ang, D. D., "Transient Motion of a Line Load on the Surface of an Elastic Half Space," *Quarterly of Applied Mathematics*, Vol. 18, No. 3, 1960, pp. 251–256.

7 Payton, R. G., "Transient Motion of an Elastic Half Space Due to a Moving Surface Line Load," *International Journal of Engineering Science*, Vol. 5, 1967, pp. 49–79.

8 Gakenheimer, D. C., "Response of an Elastic Half Space to Expanding Surface Loads," to be published in Journal of Applied Mechanics.

9 Cole, J., and Huth, J., "Stresses Produced in a Half Plane by Moving Loads," Journal of Applied Mechanics, Vol. 25, Trans. ASME, Vol. 80, Dec. 1958, pp. 433–436.

10 Gol'dshtein, R. V., "Rayleigh Waves and Resonance Phenomena In Elastic Bodies," *Journal of Applied Mathematics and Mechanics* (*Prikladnaya Matematika i Mekhanika*), Vol. 29, No. 3, 1965, pp. 516–525.

Author's Closure

The author thanks Mr. Gakenheimer for his kind comments and agrees with the substance of his discussion. It should be emphasized, however, that the main motivation for the investigation was the study of a load moving at variable velocity. The introductory remark with which the discusser disagrees should have been qualified. The discusser's clarification of the issue is greatly appreciated.

On the Contact Problem of Cylinders Containing a Shallow Longitudinal Surface Depression[1]

C. K. LIU.[2] The author is to be complimented highly for having carried out investigations of the more fundamental portion of the problem of contact between two elastic cylinders; i.e., the task of ascertaining the contact area or extent in relationship with the applied load at an "infinite" distance away. The quantitative results are presented, for the most part, in the form of curves which are of value to a designer of bearings and similar machine members. The writer is certain that the author can and probably has generated many more such curves of significant utility.

In particular, the writer would like to know if the author has prepared a Fig. 4 in which the dimensionless pressure is plotted versus x/c for various load factors $\pi PR/E'c^2$. These pressure profile (s) will aid immeasurably in the determination of the stresses beneath the contact area. It would be interesting to know if the present mode of loading will yield a maximum subsurface shear, also.

The writer hopes that the author in his reply to this discussion will kindly clarify the following points:

1 Knowing that the present paper is dealing with normal loading only, the writer would like to know if the author also has a solution for the combined case of normal and shear loading as in the case of rolling contact. To put it differently, would the presence of shear alter significantly the extent of contact?

2 Are there any solutions of a similar nature in existence for dynamic contact?

3 Does the author have a more detailed plot of Fig. 4 in which the slopes of the curves on meeting the x-axis are more clearly shown? It is known from equation (22) and is verified by the author's dotted curve $r/R = 1.0$ in Fig. 4 that $\left.\frac{d\rho}{d\chi}\right|_{\chi=\alpha} = \infty$. Do all the other curves in the same figure also meet the x-axis vertically at the edge of contact region?

4 By direct integration the stresses in the contacting member(s) can be computed if the pressure profile $p(x)$ is known. Much labor could be saved if $p(x)$ is given as a function of x with p as parameter. Has the author already done that?

5 The author mentions in the Introduction "...defects generated in manufacturing or handling." Unless these defects are all smooth at the edges, some of them are actually scratches. With the presence of burrs or scratches with turned-up edges, the mechanism or rolling contact will have to include shear loading as well. Does the author plan to investigate this as a sequel?

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[1] By Y. P. Chiu, published in the December, 1969, issue of the Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, pp. 852–858.

[2] Professor, Department of Mechanical Systems Engineering, College of Engineering, University of Alabama, University, Ala.

6 Under item 5 of the author's Discussion, the author states "...that the stress-concentration factor increases monotonically with increasing elastic modulus E' and *decreasing* load P, and corner radius r. It also increases moderately with increasing R." A few words of explanation would be appreciated regarding the increase of the stress-concentration factor with decreasing load P and increasing R.

The writer acknowledges with pleasure having had the opportunity to review the significant findings contained in this paper.

HILLEL PORITSKY.[3] The following discussion relates to studies on multiple contacts carried out about a year ago, and it is described briefly in the paper "Microslip and Creep in Contacts," presented at the NASA Symposium, Interdisciplinary Approach to the Lubrication of Concentrated Contacts, held at RPI, Troy, N. Y., July 15–17, and will appear in the *Proceedings* of this symposium.

For contacts with purely normal, pressure forces, we utilize the integral

$$f(Z) = F + iF' = \int p(s) \ln [(Z - s)/i] ds, \qquad (1)$$

essentially due to F. W. Carter,[4] (see also, Poritsky[5]) where

$i = \sqrt{-1}$
$Z = x + iy$
$p(x)$ = applied pressure over one or several intervals of the boundary $y = 0$
$f(Z)$ = analytic function of Z in $y > 0$, defined by equation (1)
F, F' = real and imaginary parts of $f(Z)$

The functions F, F' are conjugate harmonic functions in $y > 0$. Under the assumption that there are no shear forces applied over the contacts, the normal displacement n and the pressure p are related to F, $\partial F'/\partial x$ over the boundary as follows:

$$n = \frac{1 - \nu}{\pi G} F(x, 0), \qquad (2)$$

$$p = -\frac{1}{\pi} \frac{\partial F'(x, 0)}{\partial x}. \qquad (3)$$

In contact with a rigid base in $y < 0$, pressing against an elastic solid in $y > 0$, the normal displacement n is prescribed over L, the contact portion of the boundary, while over L', the free portion, p vanishes and F' is constant. Hence F is prescribed over L, F' over L'.

As an example, consider the function defined by the integral

$$f(Z) = F + iF' = -Pi \int_{(b+c)/2}^{Z} \frac{dZ}{\sqrt{(Z - b)(c - Z)}}, \qquad (4)$$

where $0 < b < c$, and the integrand is positive for Z real and $b < Z < c$, and has the determination consistent with Z staying in $y > 0$. It is readily shown that F is constant for x between b and c, and that $F' = \pi P/2$ for $x < b$, and $F' = -\pi P/2$ for $x > c$. Hence (4) corresponds to a rigid, horizontal die, pressing against the upper solid $y > 0$ over $b < x < c$, with a force P per unit z.

Next consider a double horizontal die, pressing against $y > 0$ over the *two* intervals

$$b < x < c; \qquad -c < x < -b, \qquad (5)$$

with a force P over each one. It is readily shown that F is even in x for $y = 0$, while F' vanishes between the intervals (5), and that F' is odd in x and hence vanishes along the y-axis. Now carry out the transformation

$$Z_1 = Z^2, \qquad (6)$$

mapping the first quadrant of the Z-plane into the upper half of the Z_1-plane. Then along the real Z_1-axis, F' vanishes for x_1 less than b^2, is equal to $-\pi P$ for x_1 greater than c^2, while for x_1 between b^2 and c^2, F is constant. Thus, in the Z_1-plane, $F + iF'$ corresponds to a die pressing against $y_1 = 0$ over the interval between b^2 and c^2, with a force P. Hence $F + iF'$ in the Z_1-plane can be derived from (4), and expressed as follows in terms of Z:

$$\begin{aligned} f = F + iF' &= -Pi \int_0^{Z_1} \frac{dZ_1}{\sqrt{(Z_1 - b^2)(c^2 - Z_1)}} \\ &= -Pi \int_0^{Z_2} \frac{ZdZ}{\sqrt{(Z^2 - b^2)(c^2 - Z^2)}} \end{aligned} \qquad (7)$$

There results for the pressure

$$p = \frac{2Px}{\sqrt{(x^2 - b^2)(c^2 - x^2)}} \qquad (8)$$

over the intervals (5).

The transformation (6) is similarly effective for the symmetrical contact configuration used by Chiu in converting the double contact into a single contact case.

For a general, nonsymmetric contact problem, with contact over the two intervals

$$x_1 < x < x_2, \qquad x_3 < x < x_4, \qquad x_1 < x_2 < x_3 < x_4, \qquad (9)$$

the elliptic integral transformation

$$W = u + iv = \int^{Z} \frac{dZ}{[(Z - x_1)(Z - x_2)(Z - x_3)(Z - x_4)]^{1/2}} \qquad (10)$$

can be used to convert the upper half plane into a rectangle, with the contacting intervals going into two opposite rectangle sides. Expansions in product harmonics in u, v, in the plane of the rectangle can then be resorted to solve for F and F'.

E. RADZIMOVSKY.[6] The author applied an original and interesting mathematical technique for the solution of the problem, which is the subject of this investigation. There is no doubt that this analytical investigation is a valuable contribution to the literature on the Hertzian problem.

However, for the following reasons, one should be careful in using directly the results of this investigation for predicting the effect of surface defects on the stress field in the zone of contact and upon the development of a fatigue failure in the real machine members subjected to repeated stresses by rolling under the load.

1 The fatigue failure phenomena such as pitting or "peeling" of the working surfaces of rolling machine elements depend upon all stress components in the area of contact and the type of stress cycle (the range ratio) of these components rather than upon the static value of the maximum stress (pressure) on the surface. These stress components are not necessarily in the same propor-

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[4] Carter, F. W., *Proceedings of the Royal Society*, Series A, Vol. 112, 1926, p. 151.

[5] Poritsky, H., "Microslip and Creep in Contacts," *Proceedings of NASA Symposium on Interdisciplinary Approach to the Lubrication of Concentrated Contacts*, July 15–17, 1969 at RPI, Troy, N. Y.

[6] Professor of Mechanical Engineering, University of Illinois, Urbana, Ill. Mem. ASME.

tion to the maximum surface pressure, but depend upon the geometry of the bodies in contact and their mechanical properties.

2 With two cylinders rolling over each other, the symmetrical location of surfaces with respect to the center of contact (used as a model in this problem) is only one instantaneous relative position of the surfaces, and this position does not necessarily produce the critical stress conditions.

3 The critical point in the contact zone does not necessarily occur at the surface.

4 In actual machines both elements in contact are elastic. That fact affects not only the pressure distribution on the surfaces but also the relation between the stress components in the zone of contact.

5 If the defect is produced by "brinelling" the surface by one of the rolling elements, the surface of depression will have the radius approaching the radius of the rolling element itself. Such form of depression does not correspond to the model used in this analysis. If defects in the form of longitudinal surface depressions are accidentally generated on the surface of one rolling element, these depressions hardly will be oriented as in the model used for this investigation.

Again, let me congratulate the author on this original work and repeat that my concern is only with the possible extension of the quantitative results beyond the assumptions used in the contact problem which is the subject of the investigation.

Authors' Closure

The author wishes to thank Professors Radzimovsky, Liu, and Dr. Poritsky for their interest and comments on the paper.

The solution of the paper provides the contact pressure distribution but not the subsurface stress distribution mentioned by the first two discussers. However, the subsurface stress distribution can be obtained by a numerical integration using the calculated contact pressure as input. As a matter of fact, the subsurface octahedral stress due to the presence of a narrow furrow in a large contact has been computed numerically and presented in a separate paper (reference [12] of paper). Dr. Radzimovsky is certainly correct in stating that the critical stress component in the contact is not necessarily proportional to the maximum contact stress.

The author wishes to clarify that the manufacturing surface defects modeled in this paper are grinding furrows rather than scratches, since the latter exhibit turned-up edges (as pointed out by Dr. Liu) while the former do not. (This has been found from Talysurf traces run across the defects.[7]) Dr. Radzimovsky's comment that defects formed by "brinelling" of the surface by one of the rolling elements are not modeled by this analysis is proper.

Although it was stipulated in the Introduction of the paper that the cylinder with the depression is rigid, the solution obtained in the paper can be generalized by introducing a reduced elastic modulus E' so that both cylinders may be considered elastic as stated at the end of the paper. In the paper, it has also been stated that the problem of stress concentration around a furrow shaped defect on a bearing raceway is essentially three dimensional. The present two-dimensional solutions considering an idealized depression located in the center of contact provides a first approximation. There are many problems in this area not yet solved such as the nonsymmetrically located depression in the two-dimensional contact, the inclusion of friction in the contact, a depression located in a Hertzian point contact as well as the dynamic case of the present problem. The contact problem with shear loading included seems to be approachable by available complex variable theory.

Because of the limited time available for the preparation of this Closure, the author regrets not being able to obtain, as Dr. Liu requests, additional pressure distribution curves for a multitude of values of $\pi PR/E'c^2$ other than the single value 1.8 given in the paper. Such pressure profiles can be evaluated from equation (18) of the paper using the values of a/c, b/c (from Figs. 2 and 3) as inputs.

[7] Schoeler, E. S., Discussion in *ASLE Transactions*, Vol. 12, No. 2, 1969, pp. 115–116.

The validity of the statement in the Discussion of the paper that the stress-concentration factor increases with decreasing load P and increasing R can be seen as follows: For a given r/R and $\bar{P}$ one can locate a point in Fig. 5 of the paper. The stress-concentration factor defined in the paper as the ratio of $p_{\max}$ around a depression and the Hertzian maximum pressure without the depression is equal to the vertical distance between the said point and the dotted straight line denoted "Hertzian case." For increasing values of P, the point moves toward the right along the curve of constant r/R, and one sees that the vertical distance between the displaced point and the dotted line decreases. This means a decrease in the stress-concentration factor. For increasing radius R, the point moves to the right and, simultaneously, to the curve of smaller r/R values. One sees that the displaced point always has a larger vertical distance between it and the dotted line, or a greater stress-concentration factor.

Regarding the slope of the pressure profile at the vicinity of the inner and outer contact edge ($x = b$ and $x = a$), the author found the pressure profile rises almost vertically at the edges as in the case of Hertzian contact of two cylinders. By examination of the complex function Φ (equal to one-half the function $p(x)$ in equation (7) with x replaced by z where $z = x + iy$) for this mixed boundary-value problem, one can find that near the vicinity of the contact edges, $p(x)$ is proportional to $\sqrt{(a^2 - x^2)(x^2 - b^2)}$. In other words, the pressure curves meet the x-axes vertically at the edges of the contacts.

The author appreciates Dr. Poritsky's presentation of an alternative general solution to the mixed boundary-value problem of a half plane with two contact zones.

The basic formulas used in the present paper were taken directly from the results in Muskhelishvili's celebrated work. These formulas can be derived by other means using complex variable theory. It is pertinent to cite the original work by Carter referenced in Dr. Poritsky's discussion and Westergaard[8] which are applications of complex variable theory prior to the publication of Muskhelishvili's books. Dr. Poritsky's contribution in this Discussion is certainly a valuable one. The case of two flat-based punches acting on a half plane in Dr. Poritsky's discussion corresponds to the special case $f'(x) = 0$ of the first formula given in the Appendix of the paper.

[8] Westergaard, H. M., "Bearing Pressures and Cracks," Journal of Applied Mechanics, Trans. ASME, Vol. 59, 1937, pp. A49–A53.

High-Frequency Response of an Elastic Spherical Shell[1]

R. R. GOODMAN.[2] The Sommerfeld-Watson transformation is familiar to those who work in nuclear physics and acoustics. It has given us, in these fields, deeper insight into the nature of resonance and the dynamic responses of a structure. The technique which is introduced by Feit and Junger can offer the structural dynamicist the same refreshing opportunity. It should be noted, however, that more complex structures, such as thick cylinders or spheres, give very complicated integrals

[1] By David Feit and M. C. Junger, published in the December, 1969, issue of the Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, pp. 859–864.

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