NOTES AND DISCUSSIONS | NOVEMBER 01 2016

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Michael Nauenberg

Am. J. Phys. 84, 879–882 (2016)
https://doi.org/10.1119/1.4962981

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Einstein's equivalence principle in quantum mechanics revisited

Michael Nauenberg[a)]
Department of Physics, University of California, Santa Cruz, California 95064

(Received 21 February 2016; accepted 30 August 2016)

The gravitational equivalence principle in quantum mechanics is of considerable importance, but it is generally not included in physics textbooks. In this note, we present a precise quantum formulation of this principle and comment on its verification in a neutron diffraction experiment. The solution of the time dependent Schrödinger equation for this problem also gives the wave function for the motion of a charged particle in a homogeneous electric field, which is also usually ignored in textbooks on quantum mechanics.
[http://dx.doi.org/10.1119/1.4962981]

I. INTRODUCTION

Einstein's equivalence principle for a uniform gravitational field states that the motion of an object in an inertial reference frame is indistinguishable from the motion of the object in the absence of this field but with respect to a suitable uniformly accelerated reference system.[1–3] Despite its importance, a discussion of this equivalence principle in quantum mechanics is generally not included in physics textbooks. Already in a 1977 article in this journal, Eliezer and Leach complained: "To what extent the equivalence principle in classical mechanics has an analog in quantum mechanics is a problem to which no attention seems to have been given in the literature."[4] In fact, three years earlier, prompted by an experimental proposal of Overhauser and Colella[5] to test the equivalence principle in an experiment on quantum interference with neutrons, I obtained the analytic solution of the time dependent Schrödinger equation for this problem. My solution appeared in footnote 6 of a paper by Collela *et al.*,[6,7] where their proposed neutron diffraction experiment was first carried out. Although this solution has now been rederived several times (see Refs. 8–14), it still remains largely ignored in the physics teaching literature. To fill this gap, a derivation from first principles is presented here. For information about more recent work in neutron interferometry, see Ref. 15. Experiments with atomic beams have also been carried out to test the equivalance principle and general relativity,[16,17] but this work will not be discussed here.

II. EQUIVALENCE PRINCIPLE IN CLASSICAL AND IN QUANTUM MECHANICS

For the classical treatment of the equivalence principle with the gravitational field directed along the $-z$ axis, we have Newton's equation of motion

$$m_i \frac{d^2 z}{dt^2} = -m_g g, \tag{1}$$

where m_i is the inertial mass, m_g is the gravitational mass, and g is the (local) acceleration due to gravity. The equality of these two masses, $m_i = m_g$, which has been verified[18] to 2 parts in 10^{13}, implies that the mass terms drop out of this equation. Changing to a reference system with coordinate z' undergoing a constant acceleration a along the negative z-axis

$$z' = z + \frac{at^2}{2}, \quad t' = t, \tag{2}$$

results in

$$\frac{d^2 z'}{dt^2} = a - \frac{m_g g}{m_i}. \tag{3}$$

Hence, when $a = m_g g/m_i$, the particle motion with respect to this accelerated reference system—in an elevator falling freely after its suspension is broken—is free motion at a constant velocity determined by its initial value.

For the quantum treatment of this problem, the Schrödinger equation for a particle under the action of a constant gravitational field along the $-z$ axis is[19]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_i}\frac{\partial^2 \psi}{\partial z^2} + V(z)\psi, \tag{4}$$

where the potential $V(z) = m_g g z$. Even with the equality of inertial and gravitational mass ($m_i = m_g$), this equation still depends on the value of this mass; it does not disappear as in the classical equation of motion given in Eq. (1). But it will be shown that this mass dependence does not imply a violation of Einstein's formulation of the equivalence principle. Changing to space and time coordinates z', t' in the accelerated coordinate system given in Eq. (2), the Schrödinger equation for ψ in these variables becomes

$$i\hbar\left(\frac{\partial \psi}{\partial t'} + at'\frac{\partial \psi}{\partial z'}\right) = -\frac{\hbar^2}{2m_i}\frac{\partial^2 \psi}{\partial z'^2} + m_g g\left(z' - \frac{at'^2}{2}\right)\psi. \tag{5}$$

Let

$$\psi(z', t') = \phi(z', t')e^{iS(z', t')}, \tag{6}$$

where ϕ satisfies the Schrödinger equation for a free particle in the accelerated frame

$$i\hbar\frac{\partial\phi}{\partial t'} = -\frac{\hbar^2}{2m_i}\frac{\partial^2\phi}{\partial z'^2}, \tag{7}$$

and S is assumed to be a real function of the coordinates z' and t'. The requirement that ϕ satisfies the Schrödinger equation for a free particle in an accelerated frame of reference for which $a = g$ (when the inertial and gravitational masses are equal) is the mathematical formulation in quantum mechanics of Einstein's equivalence principle in classical mechanics. The assumption that S is a real function follows from the probability interpretation in quantum mechanics that requires $|\psi|^2 = |\phi|^2$, but a proof is given below.

Using Eq. (6) for ψ in Eq. (5), we find for the left-hand side

$$A = i\hbar\left[\frac{\partial\phi}{\partial t'} + i\frac{\partial S}{\partial t'}\phi + at'\left(\frac{\partial\phi}{\partial z'} + i\frac{\partial S}{\partial z'}\phi\right)\right], \tag{8}$$

and for the right-hand side

$$B = -\frac{\hbar^2}{2m_i}\left[\frac{\partial^2\phi}{\partial z'^2} + 2i\frac{\partial S}{\partial z'}\frac{\partial\phi}{\partial z'} - \left(\frac{\partial S}{\partial z'}\right)^2\phi + i\frac{\partial^2 S}{\partial z'^2}\phi\right] + m_g g\left(z' - \frac{at'^2}{2}\right)\phi, \tag{9}$$

with $A = B$. This condition requires that to satisfy Eqs. (4) and (7), the remaining coefficients of ϕ and $\partial\phi/\partial z'$ in the quantity A–B must vanish. The coefficient of $\partial\phi/\partial z'$ is

$$-\frac{\hbar^2}{2m_i}\left(2i\frac{\partial S}{\partial z'}\right) - i\hbar at', \tag{10}$$

which implies that

$$S = -\frac{m_i a t' z'}{\hbar} + f(t'), \tag{11}$$

where $f(t')$ is a function of t' only. The vanishing of the coefficient of ϕ requires that

$$a = \frac{m_g g}{m_i}, \tag{12}$$

which is the same condition in classical mechanics for uniform motion in the prime coordinate system [Eq. (3)] and that

$$\frac{df}{dt'} = \frac{m_i a^2 t'^2}{\hbar}. \tag{13}$$

Integrating this equation with the boundary condition $f(0) = 0$ yields

$$f(t') = \frac{m_i a^2 t'^3}{3\hbar}. \tag{14}$$

Hence, the solution of the Schrödinger equation in the accelerated system with coordinates z', t' is

$$\psi(z', t') = \phi(z', t')\exp\left[-\frac{i m_i a t'}{\hbar}\left(z' - \frac{at'^2}{3}\right)\right], \tag{15}$$

and the corresponding solution in the system with coordinates z, t, stationary in a gravitational potential $V = m_g gz$, is

$$\psi(z, t) = \phi\left(z + \frac{at^2}{2}, t\right)\exp\left[-\frac{i m_i a t}{\hbar}\left(z + \frac{at^2}{6}\right)\right], \tag{16}$$

where the relation between the gravitational constants g and the relative coordinate acceleration a is given by Eq. (12). The equality of inertial and gravitational mass, $m_i = m_g$, then leads to the gravitational equivalence principle in quantum mechanics with $a = g$. Substituting Eq. (16) for ψ into Eq. (4), it can be readily shown that ψ then satisfies the Schrödinger equation for a particle moving in a potential $V = m_i az$.

To complete the proof of this equivalence, consider the form of the Schrödinger equation for a free particle in an inertial frame with respect to an accelerated frame according to the change of coordinates in Eq. (2). It can be shown by a transformation similar to Eq. (6) that in this accelerated frame the Schrödinger equation acquires a potential $V(z) = m_i az$. For the case that $a = g$, an observer in the accelerated frame would then find that $m_i = m_g$, where m_g is the gravitational mass.

In 1974, Overhauser and Collela proposed an interference experiment with neutrons scattering from a lattice to check experimentally whether neutrons satisfy the predictions of the equivalence principle in quantum mechanics.[5] Their idea was to build a neutron interferometer that splits a neutron beam into two beams traveling horizontally through the same distance d but at a different height z, and to observe the interference fringes when these beams are reunited. Setting $t = d/v$, where $v = 2\pi\hbar/m_i\lambda$ is the neutron's horizontal velocity and λ is the neutron wavelength, then according to Eq. (16) the phase shift is given by

$$\Delta = \frac{m_i^2 az\lambda d}{2\pi\hbar^2}. \tag{17}$$

Setting $m_i = m_g$, so that by Eq. (12) we have $a = g$ (in accordance also with the classical formulation equivalence principle), and the phase shift given by Eq. (17) becomes equal to the expression derived by Overhauser and Colella.[5] It should be pointed out, however, that their derivation of this phase shift was semiclassical, based on the momentum difference δp of two neutron beams differing by an amount z in height, which is valid only to first order in δp, while the quantum mechanical derivation is exact. By energy conservation, to first order this difference in momentum between the two neutron beams is

$$\delta p = \frac{m_i^2 gz}{p}, \tag{18}$$

and according to quantum mechanics the momentum is

$$p = \frac{2\pi\hbar}{\lambda}. \tag{19}$$

Hence, the phase shift over a horizontal distance d is

$$\Delta_c = \frac{d\,\delta p}{\hbar}, \tag{20}$$

and substituting Eqs. (18) and (19) we find that $\Delta_c = \Delta$ from Eq. (17).

As a simple example of the application of Eq. (15), consider the case that $\phi(z,t)$ is a plane wave with momentum p_0[20]

$$\phi(z,t) = \exp\left[\frac{i}{\hbar}\left(p_0 z - \frac{p_0^2 t}{2m_i}\right)\right], \tag{21}$$

which is a solution of Eq. (7). Substituting Eq. (16) for ψ [with Eq. (21) for ϕ] into the Schrödinger equation, Eq. (4), it can be verified that this equation is satisfied and that ψ is an eigenstate of the quantum mechanical momentum operator $-i\hbar\partial/\partial z$

$$-i\hbar\frac{\partial\psi(z,t)}{\partial z} = p(t)\psi(z,t), \tag{22}$$

and the energy operator $i\hbar\partial/\partial t$

$$i\hbar\frac{\partial\psi(z,t)}{\partial t} = \left(\frac{p(t)^2}{2m_i} + m_i g z\right)\psi(z,t), \tag{23}$$

with the momentum eigenvalue $p(t) = (p_0 - m_i a t)$ varying linearly in time. This result is in accordance[23] with the increase in velocity $v = at$ of a particle in classical mechanics under the action of a constant gravitational force $m_i g$.

A more general solution of Eq. (7) consisting of a linear superposition of these states can then also localize the particle in conformance with the uncertainty principle $\Delta z \Delta p \geq \hbar/2$.

Setting the initial state to be

$$\phi(z,0) = (2\pi\sigma^2)^{-1/4}\exp\left(\frac{ip_0 z}{\hbar} - \frac{z^2}{4\sigma^2}\right), \tag{24}$$

where $\sigma = \Delta z$, we obtain the time dependent solution for a free particle [see Eq. (7)]

$$\phi(z,t) = \left[2\pi\left(\sigma^2 + t^2\hbar^2/4m^2\sigma^2\right)\right]^{-1/4} \times \exp\left[\frac{i}{\hbar}\left(p_0 z - \frac{p_0^2 t}{2m}\right) - \frac{(z - p_0 t/m)^2}{4\sigma^2 + 2i\hbar t/m}\right]. \tag{25}$$

The absolute square of such a wave-packet then spreads in the same manner as the corresponding classical probability distribution, and substituting $z \to z + at^2/2$ [see Eq. (16)] this wave-packet is then centered at $z = v_0 t - at^2/2$ corresponding to the motion of a classical particle.[21,22] Multiplying this wave-packet along the z-axis with a corresponding wave packet for free motion along the transverse x or y axis localizes the trajectory along the well known Galilean parabola found by solving the equations of classical mechanics.

In conclusion, we remark that the neutron interference experiment discussed here verifies only the validity of the Schrödinger equation for neutrons moving in the gravitational potential $V(z) = mgz$. An actual test of the quantum mechanical equivalence principle would require that this experiment be repeated in a region where there is no gravitational field, e.g., in outer space inside a rocket ship that is moving at a constant acceleration or inside a freely falling container dropped from some height above the surface of Earth. More practically, such a test has been carried out by Bonse and Wroblewsi,[27] by setting the entire neutron interferometer into harmonic oscillations, and measuring the phase shift at the end points of the oscillation as a function of the apparatus maximal acceleration a. The results are in good agreement with the theoretical relation given by Eq. (17), closing the loophole in the experiment of Collela *et al.*[5]

ACKNOWLEDGMENTS

The author would like to thank Michael Berry, David Griffiths, Robert Johnson, Wayne Saslow, and Samuel Werner for helpful comments, and John Faulkner for a careful reading of this manuscript and helpful comments.

APPENDIX: STATIONARY SOLUTION OF THE SCHRÖDINGER EQUATION IN A HOMOGENEOUS FIELD

The wavefunction given in Eq. (16) is also the general solution for a free particle with mass m and charge e, moving in a uniform electric field of magnitude $\mathbf{E}$ in the $-z$ direction, with $a = e\mathbf{E}/m$. The stationary treatment for this problem can be found in some quantum mechanics textbooks (e.g., Refs. 24–26), where it is shown that the solution of the time independent Schrödinger equation in a homogeneous field in coordinate space is an Airy function,[28] which in momentum space p is

$$A(p,E) = C\exp\left[\frac{i}{\hbar F}\left(Ep - \frac{p^3}{6m}\right)\right], \tag{A1}$$

where C is a constant and F is the constant force directed in the $+z$ direction. Then its Fourier transform,

$$\psi(z,t) = C\int_{-\infty}^{+\infty}\frac{dp}{2\pi\hbar}\int_{-\infty}^{+\infty}\frac{dE}{2\pi\hbar}A(p,E)\exp\left[\frac{i}{\hbar}(pz - Et)\right], \tag{A2}$$

is the time dependent solution of the Schrödinger equation, Eq. (4), with the potential $V(z) = -Fz$. Integrating first over the variable E, which gives a delta function $2\pi\delta((p/F - t)\hbar)$, one obtains

$$\psi(z,t) = \frac{CF}{2\pi\hbar}\exp\left[\frac{iFt}{\hbar}\left(z - \frac{Ft^2}{6m}\right)\right], \tag{A3}$$

which for $F = -ma$ corresponds to our solution [Eq. (16)] in the case that ϕ is a constant.[29]

In his popular introduction to quantum mechanics textbook,[26] David Griffiths writes that "*Classically*, a linear potential means a constant force, and hence a constant acceleration—the simplest nontrivial motion possible, and the *starting* point for elementary mechanics. It is ironic that the same potential in *quantum* mechanics gives rise to unfamiliar transcendental functions, and plays only a peripheral role in the theory." But as we have shown here, although the transcendental Airy function appearing in the *stationary* solution of the Schrödinger equation in coordinate space does not have any classical analog, the *time dependent* quantum solution corresponds closely to the classical one.

[a]Electronic mail: michael@physics.ucsc.edu

[1]A. Einstein, "On the relativity principle and the conclusions drawn from it," Jahr. Radioaktivitität Elektron. **4**, 411–462 (1907). Reprinted in volume 2: the Swiss Years writings, 1900–1909, of the Collected Papers of

Albert EinsIein, edited by J. Stachel, D. Cassidy, Jürgen Renn, and Robert Schulmann (Princeton U. P.), pp. 252–315. Available online in an English translation at: <http://einsteinpapers.press.princeton.edu/vol2-trans/266?ajax>. In this review article on his special relativity principle, Einstein first introduce the equivalence principle, stating that "… we shall therefore assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system."

[2]J. Norton, "What was Einstein's principle of equivalence?," in *Einstein and the History of General Relativity*, edited by D. Howard and J. Stachel (Birkhaáuser, Boston, 1986), pp. 5–47.

[3]The equivalence principle appears to have been stated centuries ago, well before Einstein, in Sheherazade's tale of Sindbad the Sailor: "[the bird]… rose and rose until I thought that I was about to touch the vault of heaven, then suddenly it dropped, so swiftly that I could not feel my own weight…" (Sinbad speaking, in *Book of the Thousand and One Nights*, translated by J. C. Mardrus and Powys Mathers Routlledge, paperback 1986, volume 2, p. 190). I am indebted to Michael Berry for this interesting information, but this wording does not appear in some other translations of this book.

[4]C. J. Eliezer and P. G. Leach, "The equivalence principle and quantum mechanics," Am. J. Phys. **45**, 1218–1220 (1977).

[5]A. W. Overhauser and R. Collela, "Experimental test of gravitational induced quantum interference," Phys. Rev. Lett. **33**, 1237–1239 (1974).

[6]R. Colella, A. W. Overhauser, and S. A. Werner, "Observation of gravitationally induced quantum interference," Phys. Rev. Lett. **34**, 1472–1474 (1975).

[7]In Ref. 6 my solution was quoted somewhat incorrectly stating that ϕ, instead of ψ, is the solution of the Schrödinger equation in the Newtonian potential $V = mgz$. Actually, ϕ is the solution for a free particle, as shown in Eq. (7).

[8]D. M. Greenberg and A. W. Overhauser, "Coherence effects in neutron diffraction and gravity experiments," Rev. Mod. Phys. **51**, 43–78 (1979). Our solution, Eq. (16), was re-derived in an Appendix in this paper as Eq. (A23), but the factors 1/3 and 1/6 that appear in the exponents in Eqs. (15) and (16) are given in reverse order in Eq. (A23).

[9]J. L. Staudeman, S. A. Werner, R. Collela, and A. W. Overhauser, "Gravity and inertia in quantum mechanics," Phys. Rev. A **21**, 1419–1438 (1980).

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[11]C. Lämmerzahl, "On the equivalence principle in quantum theory," Gen. Relativ. Gravitation **28**, 1043–1070 (1996).

[12]R. W. Robinett, "Quantum mechanical time-development operator for the uniformly accelerated particle," Am. J. Phys. **64**, 803–808 (1996).

[13]L. Viola and R. Onofrio, "Testing the equivalence principle through freely falling quantum objects," Phys. Rev. D **55**, 455–462 (1997). In Appendix B of this paper the result given in our Eq. (16) is presented without a derivation.

[14]A. Camacho and A. Camacho-Guardia, "Quantal definition of the weak equivalence principle," AIP Conf. Proc. **1122**, 209–212 (2009).

[15]H. Rauch and S. A. Werner, *Neutron Interferometry, Lessons in Experimental Quantum Mechanics* (Clarendom Press, Oxford, 2000), pp. 211–255.

[16]M. Kasevich and S. Chu, "Atomic interferometry using stimulated Raman transitions," Phys. Rev. Lett. **67**, 181–184 (1991).

[17]S. Dimopoulos, P. W. Graham, J. M. Hogan, and M. A. Kasevich, "General relativistic effects in atomic interferometry," Phys. Rev. D **78**, 042003-1–042003-29 (2008).

[18]T. A. Wagner, S. Schlamminger, J. H. Gundlach, and E. G. Adelberger, "Torsion-balance tests of the weak equivalence principle," Classical Quantum Gravity **29**, 184002-1–184002-15 (2012).

[19]For simplicity we ignore the dependence of the Schrödinger equation on the transverse (x and y) axes, which is separable and corresponds to the motion of a free particle in the absence of a potential.

[20]In the neutron interference experiment, the initial neutron momentum is taken along the transverse (x, y) direction, and therefore $p0 = 0$. Hence, ϕ does not contribute to the phase shift, Eq. (17), when the neutron beam is split into two beams by the interferometer.

[21]M. Nauenberg, "Wave packets: Past and present," in *The Physics and Chemistry of Wave Packets*, edited by J. A. Yeaszell and T. Uzer (Wiley, New York, 2000), Chap. 1, pp. 1–30.

[22]E. Okon and C. Callender, "Does quantum mechanics clash with the equivalence principle-and does it matter?," Eur. J. Philos. Sci. **1**, 133–145 (2011). These authors conclude that according to quantum mechanics "the time to descent of very light particles is expected to deviate greatly from the Newtonian value…" p. 136, which is shown here to be incorrect.

[23]In their textbook, *Solid State Physics* (Saunders College, Philadelphia, 1976), N. W. Ashcroft and N. D. Mermin argue, on the basis of a semiclassical model of electron dynamics in a Bravais lattice, that this property applies also to the motion of electrons in a homogeneous electric field. The students are assign the problem "to prove this theorem" from the time dependent Schrodinger equation (problem 6, p. 241), but a footnote 46 admits that this "theorem" is not exact.

[24]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non relativistic theory* (Addison-Wesley, MA, 1958), pp. 70–72.

[25]J. Schwinger, *Quantum Mechanics, Symbolism of Atomic Measurements*, edited by Berthokd-Georg Englert (Springer, Heidelberg, 2001), pp. 237–242.

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[28]M. Berry and N. L. Balazs, "Nonspreading wave packets," Am. J. Phys. **47**, 264–267 (1979); M. V. Berry, "Wavelength-independent fringe spacing in rainbows from falling neutrons," J. Phys. A **15**, L385–L388 (1982).

[29]In one dimension the probability interpretation in quantum mechanics requires that $\int dz|\psi(z,t)|^2 = 1$. Hence, ψ must have the dimension inverse square root of length, and correspondingly its Fourier transform $A(p, E)$ must be multiplied by a constant C with the dimensions of square root of length times time. But we are considering here a plane wave, which cannot be normalized.