tion to the maximum surface pressure, but depend upon the geometry of the bodies in contact and their mechanical properties.

2 With two cylinders rolling over each other, the symmetrical location of surfaces with respect to the center of contact (used as a model in this problem) is only one instantaneous relative position of the surfaces, and this position does not necessarily produce the critical stress conditions.

3 The critical point in the contact zone does not necessarily occur at the surface.

4 In actual machines both elements in contact are elastic. That fact affects not only the pressure distribution on the surfaces but also the relation between the stress components in the zone of contact.

5 If the defect is produced by "brinelling" the surface by one of the rolling elements, the surface of depression will have the radius approaching the radius of the rolling element itself. Such form of depression does not correspond to the model used in this analysis. If defects in the form of longitudinal surface depressions are accidentally generated on the surface of one rolling element, these depressions hardly will be oriented as in the model used for this investigation.

Again, let me congratulate the author on this original work and repeat that my concern is only with the possible extension of the quantitative results beyond the assumptions used in the contact problem which is the subject of the investigation.

Authors' Closure

The author wishes to thank Professors Radzimovsky, Liu, and Dr. Poritsky for their interest and comments on the paper.

The solution of the paper provides the contact pressure distribution but not the subsurface stress distribution mentioned by the first two discussers. However, the subsurface stress distribution can be obtained by a numerical integration using the calculated contact pressure as input. As a matter of fact, the subsurface octahedral stress due to the presence of a narrow furrow in a large contact has been computed numerically and presented in a separate paper (reference [12] of paper). Dr. Radzimovsky is certainly correct in stating that the critical stress component in the contact is not necessarily proportional to the maximum contact stress.

The author wishes to clarify that the manufacturing surface defects modeled in this paper are grinding furrows rather than scratches, since the latter exhibit turned-up edges (as pointed out by Dr. Liu) while the former do not. (This has been found from Talysurf traces run across the defects.[7]) Dr. Radzimovsky's comment that defects formed by "brinelling" of the surface by one of the rolling elements are not modeled by this analysis is proper.

Although it was stipulated in the Introduction of the paper that the cylinder with the depression is rigid, the solution obtained in the paper can be generalized by introducing a reduced elastic modulus E' so that both cylinders may be considered elastic as stated at the end of the paper. In the paper, it has also been stated that the problem of stress concentration around a furrow shaped defect on a bearing raceway is essentially three dimensional. The present two-dimensional solutions considering an idealized depression located in the center of contact provides a first approximation. There are many problems in this area not yet solved such as the nonsymmetrically located depression in the two-dimensional contact, the inclusion of friction in the contact, a depression located in a Hertzian point contact as well as the dynamic case of the present problem. The contact problem with shear loading included seems to be approachable by available complex variable theory.

Because of the limited time available for the preparation of this Closure, the author regrets not being able to obtain, as Dr. Liu requests, additional pressure distribution curves for a multitude of values of $\pi PR/E'c^2$ other than the single value 1.8 given in the paper. Such pressure profiles can be evaluated from equation (18) of the paper using the values of a/c, b/c (from Figs. 2 and 3) as inputs.

The validity of the statement in the Discussion of the paper that the stress-concentration factor increases with decreasing load P and increasing R can be seen as follows: For a given r/R and $\bar{P}$ one can locate a point in Fig. 5 of the paper. The stress-concentration factor defined in the paper as the ratio of $p_{\max}$ around a depression and the Hertzian maximum pressure without the depression is equal to the vertical distance between the said point and the dotted straight line denoted "Hertzian case." For increasing values of P, the point moves toward the right along the curve of constant r/R, and one sees that the vertical distance between the displaced point and the dotted line decreases. This means a decrease in the stress-concentration factor. For increasing radius R, the point moves to the right and, simultaneously, to the curve of smaller r/R values. One sees that the displaced point always has a larger vertical distance between it and the dotted line, or a greater stress-concentration factor.

Regarding the slope of the pressure profile at the vicinity of the inner and outer contact edge ($x = b$ and $x = a$), the author found the pressure profile rises almost vertically at the edges as in the case of Hertzian contact of two cylinders. By examination of the complex function Φ (equal to one-half the function $p(x)$ in equation (7) with x replaced by z where $z = x + iy$) for this mixed boundary-value problem, one can find that near the vicinity of the contact edges, $p(x)$ is proportional to $\sqrt{(a^2 - x^2)(x^2 - b^2)}$. In other words, the pressure curves meet the x-axes vertically at the edges of the contacts.

The author appreciates Dr. Poritsky's presentation of an alternative general solution to the mixed boundary-value problem of a half plane with two contact zones.

The basic formulas used in the present paper were taken directly from the results in Muskhelishvili's celebrated work. These formulas can be derived by other means using complex variable theory. It is pertinent to cite the original work by Carter referenced in Dr. Poritsky's discussion and Westergaard[8] which are applications of complex variable theory prior to the publication of Muskhelishvili's books. Dr. Poritsky's contribution in this Discussion is certainly a valuable one. The case of two flat-based punches acting on a half plane in Dr. Poritsky's discussion corresponds to the special case $f'(x) = 0$ of the first formula given in the Appendix of the paper.

[7] Schoeler, E. S., Discussion in *ASLE Transactions*, Vol. 12, No. 2, 1969, pp. 115–116.

[8] Westergaard, H. M., "Bearing Pressures and Cracks," Journal of Applied Mechanics, Trans. ASME, Vol. 59, 1937, pp. A49–A53.

High-Frequency Response of an Elastic Spherical Shell[1]

R. R. GOODMAN.[2] The Sommerfeld-Watson transformation is familiar to those who work in nuclear physics and acoustics. It has given us, in these fields, deeper insight into the nature of resonance and the dynamic responses of a structure. The technique which is introduced by Feit and Junger can offer the structural dynamicist the same refreshing opportunity. It should be noted, however, that more complex structures, such as thick cylinders or spheres, give very complicated integrals

[1] By David Feit and M. C. Junger, published in the December, 1969, issue of the Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, pp. 859–864.

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which differ, essentially, in the form of Z in equation (13). The analytic form of Z then includes an infinite set of zeros and a large number, possibly infinite, of branch lines which must be treated with care if one desires the complete solution. However, each zero and branch line of Z has physical significance and can be identified with a running or standing wave. That, of course, is the power of the method. This same physical picture does not come from computer methods which require hundreds of modes for convergence.

The method is applied, in the paper by Feit and Junger, correctly and carefully, and demonstrates the type of physical insight of the method. The wiggles in the curve in Fig. 4 are disconcerting, however, and the nature of these deviations from a smooth curve should be described. The only serious error, one which does not affect any of the calculations, is a missing factor of 2 in front of $(1 + \mu)$ in equation (22*a*).

J. E. GREENSPON.[3] This very fine paper is one of a series of significant studies conducted by the group at Cambridge Acoustical Associates showing the application of the Watson-Sommerfeld transformation to high-frequency structural and acoustic problems. In the writer's opinion the reader should be made aware of the powerful tool which is employed by the authors in this paper. The type of transformation used in the analysis can save orders of magnitude of time in computing high-frequency acoustic and vibrational response and, at the same time, give more physical insight into the problem than can be obtained by standard series approaches. The main difficulty in the application of this type of solution is the "low cutoff" point; i.e., the frequency or wave number below which it is impractical to use the transformation. A study of this low cutoff would certainly be enlightening. It is not inconceivable that, with slight corrections, this method could be extended down into the medium frequency region.

F. R. NORWOOD.[4] The application of the Watson transformation in the authors' references [1–3] has been to the special case of infinite series which contain Bessel functions of order n or $n + \frac{1}{2}$, and argument kr; for example, $K_{n+1/2}\,(kr)$. The slow high-frequency convergence of these series is due primarily to the behavior of the Bessel functions for large values of the argument. Watson derived fast converging eigenfunction expansions valid in some geometrical regions.

The eigenvalues in references [1–3] of the paper are the zeros of a transcendental equation containing Bessel functions, e.g., $d[\log K_{n+1/2}\,(kr_0)]/dr_0 + \alpha\, kr_0 = 0$. In general, these zeros can be found either by the use of a computer or in an asymptotic fashion as in references [1–3] of the paper. For the present paper, the eigenvalues s_i satisfy the bicubic equation (9), subject to the change of variables $\lambda + \frac{1}{4} = s^2$. Thus, subject to algebraic manipulations, for given values of Poisson's ratio and beta, the eigenvalues s_i can be found exactly; the explicit form of equation (9) indicates the existence of a real root. This root requires a modification of Watson's transformation which was not indicated by the authors. Also, the approximate root s_1 given by equation (24*a*) violates the conditions under which the Watson transformation is applicable.

Authors' Closure

The authors would like to express their appreciation to the discussers for their interest in the paper, and especially to Drs. Greenspon and Goodman for their kind remarks. Dr. Greenspon is correct in his belief that the method presented can be extended to deal with problems in a lower frequency range. However the convenient physical interpretation of the results as presented in this paper can no longer be justified. The authors agree with Dr. Greenspon that it would be useful to establish a "low cutoff point," but this in itself would require a large effort unless performed empirically.

We are grateful to Dr. Goodman for pointing out the error in equation (22). The oscillations in the curves shown in Fig. 4 are an incipient standing wave pattern which becomes more pronounced as θ approaches 180 deg. This standing wave pattern can be explained as an interference between waves which travel in opposite directions on the sphere. In the region depicted in Fig. 4 the response is primarily dominated by the first term of (30), and so the interference phenomenon is not as pronounced as is the case in the region close to $\theta = 180$ deg where the relative contributions of the two terms of (30) are comparable.

Dr. Norwood's remarks are of an entirely different kind. They seem to be irrelevant when they are correct and wrong when they are relevant. His first paragraph and a half discusses references [1–3], but is not particuarly germane to the subject matter of the present paper. With regard to the application of Watson's transformation to the problem at hand, it is not the roots of (9) that are needed, but the residue contribution for the roots of (23). In the small β and large Ω range assumed roots of (9) not included in (24) give rise to vanishing residue contributions. It is therefore not necessary to compute these roots from the characteristic equation (9) without first making the thin-shell high-frequency approximations. As far as the root s_1 is concerned the modification of Watson's transformation is indicated using an explanatory statement following equation (13) and the deformed contour shown in Fig. 3. Therefore the statement to the effect that Watson's transformation is not applicable is very puzzling.

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Free Vibrations of Reinforced Elastic Shells[1]

D. M. EGLE.[2] The authors have presented an interesting study of the vibration of ring-stiffened cylindrical shells. There are two comments which I think are appropriate. First, it seems that, because the method of analysis employs a series in the normal modes of the unstiffened shell to satisfy Hamilton's principle, this approach would have the same end result, equation (22), as an analysis using Hamilton's principle expressed in terms of the kinetic and potential energies of the structure and the same series of normal modes. This assumes, of course, that the generalized forces $Q^{(s)}$ are derived from the potential and kinetic energies of the stiffeners. Thus it appears to me that although the formulation is different, the results of the formulation, equation (22) is the same as that which would be obtained by a Rayleigh-Ritz analysis (such as that in references [1, 2],[3] of the Discussion) using the same assumed modes and the same shell and ring theories. This is not to detract from the authors' approach but to point out that the same conclusions (i.e., the computed frequencies are upper bounds of the actual frequencies) and problems associated with the Rayleigh-Ritz technique also apply to the authors' technique.

[1] By Hyman Garnet and Alvin Levy, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 835–844.

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[3] Numbers in brackets designate References at end of Discussion.