Stress Concentration Around an Ellipsoidal Cavity[1]

L. H. Donnell.[2] The research which is presented in this excellent paper is part of a long-range program which is being undertaken by the Illinois Institute of Technology under the sponsorship of the David Taylor Model Basin. The aim of this program is the study of the detailed mechanism of brittle-crack propagation, although it is expected that steps in this broader program like the present paper will also have an interest far beyond the program itself. It may add to the interest of this paper to outline this broader program and the physical premises upon which it is proposed to base it.

The program is not concerned with the "starting" of cracks, which usually results from fatigue effects in combination with local stress concentrations due to irregularities in design or fabrication. Such starting of cracks is nearly unavoidable in large structures, such as welded ship structures, but the results will not be catastrophic unless the cracks "propagate" in a brittle manner.

Besides elastic deformations, two types of material behavior are involved in crack propagation, plastic flow under shear and cohesive failure under tension. Crack propagation can only be explained on the basis of cohesive failure, since plastic flow is a healing influence, which tends to reduce the stress concentrations causing propagation, while cohesive failure tends to maintain or increase the stress concentration as the crack progresses from its starting point. However, it is unlikely that crack propagation can occur without important plastic flow taking place prior to the cohesive failure, even though this plastic flow may not be very evident. This means that elastic theory is insufficient for studying the local conditions of stress and strain at the end of the crack as it progresses.

Let the principal stresses at a point be σ_1, σ_2, σ_3, in order of their algebraic values, σ_1 being the largest. Then the occurrence of cohesive failure presumably depends on the value of σ_1, cohesive failure occurring when σ_1 reaches some value of the cohesive strength, which we can call σ_c. Similarly the occurrence of plastic flow can be taken to depend upon the value

$$\sigma_4 = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}}$$

plastic flow occurring when the quantity σ_4 reaches σ_y, the yield point of the material.

Suppose, now, that a body of a certain shape is subjected to a certain type of loading. There will be some critical point (or points) at which the danger of cohesive failure is greatest. As the loading is increased, the ratios to each other of the principal stresses σ_1, σ_2, σ_3 at this point, and hence the ratio σ_1/σ_4 also, will remain constant. The question of whether destructive cohesive failure or a healing plastic flow will occur at this point then depends upon whether the "stress distribution property" σ_1/σ_4 is greater than, or less than, the "characteristics of the material property" σ_c/σ_y. If σ_1/σ_4 is greater than σ_c/σ_y, cohesive failure will occur. If σ_1/σ_4 is less than σ_c/σ_y, plastic flow will occur first.

The value of the "material property" σ_c/σ_y, increases with the "ductility" of the material, and so might be considered to represent a quantitative measure of ductility. It would be a far more significant and quantitatively applicable value than the common

measures of this property. The value of σ_1/σ_4 in simple tension is unity. Hence what we ordinarily speak of as "ductile" materials (that is, materials behaving in a ductile manner under a simple stress condition) have values of σ_c/σ_y greater than unity, while "brittle" materials (behaving in a brittle manner under a simple stress condition) would have values of σ_c/σ_y less than unity. Whether a given material will behave in a ductile or a brittle manner under more complex stress conditions then depends on the exact value of σ_c/σ_y, which should be determined as a function of temperature for each kind of material. Such determinations are outside the scope of the present program, but will presumably be covered eventually by other investigations.

The stress-distribution property σ_1/σ_4 measures the same general condition as what has come to be called "triaxiality," but the significance of its value is perhaps simpler to interpret. The value of σ_1/σ_4 for simple tension is 1, and for equal triaxial tension it is infinity. If its value is known for various structural shapes and loading conditions, and the corresponding information about the properties of various materials is available, designers would be able to select materials or otherwise control the conditions intelligently, to insure ductile behavior.

The purpose of the present investigation might be described as an attempt to determine this stress-distribution property σ_1/σ_4 for the case of a plate containing a crack under simple tension, and to study the effect on this of variables such as crack length and plate thickness. Two cases must be investigated, namely, (1) that of a short crack, such as a fatigue crack perpendicular to the tension and (2) a propagating crack.

The critical point where the maximum value of σ_1/σ_4 will occur for either of these cases is known to be a point in the plane of the crack, but slightly beyond the end of the crack, and halfway between the faces of the plate. At this point the stress will be finite even for a crack of zero width. On the other hand, just at the end of the crack, the stress is theoretically infinite for a crack of zero width. The value of σ_1/σ_4 at the latter point will be low, so that there is no danger of cohesive failure occurring there, but plastic flow will occur there, and this may have an important effect on the conditions at the critical point.

There is probably little chance of analyzing the effect of this plastic flow completely, but such a condition of combined elastic and plastic flow can be represented approximately as the superposition of two elastic conditions: Condition A, in which the region of plastic flow is neglected, but only enough load is applied to bring the average stresses over it up to the yield point of the material; and condition B, in which the region of plastic flow is regarded as a hole, and the stresses are found for the remaining loads.

Fig. 1 of this discussion shows (1) first, a rectangular crack *abcd*, which represents an initial fatigue crack; (2) regions S where plastic flow is likely to occur and which could be considered as holes for the B condition; and finally, an elliptic-prismatic hole, which would give about the same effect as the crack plus the holes S. To be able to analyze this case, we need a three-dimensional elastic theory for the case of a plate with an elliptic-prismatic hole, having any ratio of major to minor axes. This would cover both the initial rectangular crack used in condition A and the elliptic hole used in condition B.

After cohesive failure has occurred at such a critical point and spread back to the original crack, the shape of the end of the crack will evidently be something like that shown in Fig. 2. Fig. 2 shows (1) the end of a crack *abcde* which has "propagated" for some distance; (2) a region S where plastic flow is likely to occur; and (3) an ellipsoidal hole which would have about the same effect as the crack plus the plastic regions. To analyze this case we evidently need a three-dimensional theory for the case of a plate with an ellipsoidal hole having any ratio of its axes.

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[1] "Stress Concentration Around an Ellipsoidal Cavity in an Infinite Body Under Arbitrary Plane Stress Perpendicular to the Axis of Revolution of Cavity," by M. A. Sadowsky and E. Sternberg, published in the September, 1947, issue of the JOURNAL OF APPLIED MECHANICS, Trans. ASME, vol. 69, p. A-191.

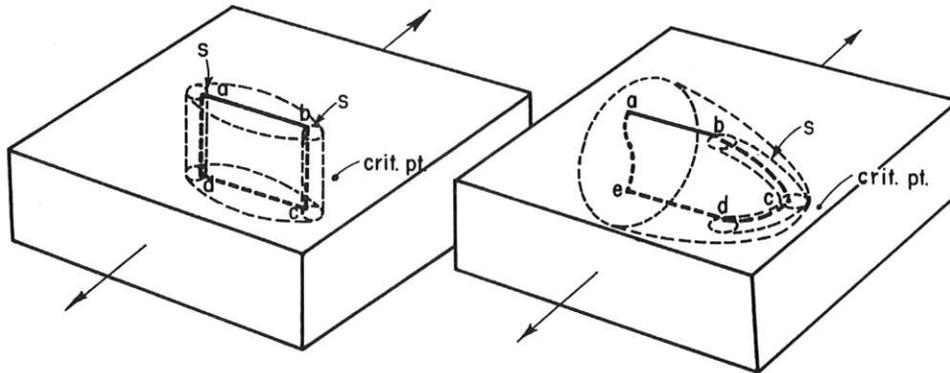[2] Research Professor of Mechanics, Illinois Institute of Technology, Chicago, Ill. Mem. ASME.

FIG. 1

FIG. 2

The theory of an ellipsoid of revolution in an infinite solid, which is the subject of the present paper, is an important step in the development of such a theory. It is not expected that exact solutions of the general cases just outlined will be obtainable, but any exact solutions of special cases which can be obtained are of great value in setting up and interpreting more general approximate solutions.

R. E. PETERSON.[3] It is interesting to note from the authors' Fig. 7 that, for a very slender ellipsoidal cavity perpendicular to the stress direction, a maximum value of stress-concentration factor of approximately 2.83 is obtained. This value seems to be surprisingly low, so that it is natural to look for a physical explanation.

If one considers a rough analogy with flow lines, the result seems reasonable. Consider first a disk of ellipsoidal contour of radius a and thickness b with the axis along the direction of stress. A flow line at the axis would need to go out a distance a and around a small radius, resulting in a major flow disturbance,

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corresponding to a high stress-concentration factor. In the case of the present ellipsoid, however, the flow line at the axis does not go out a distance a, but takes the short route around b. Near the end of the ellipsoid the flow disturbance would be much less than for the case of a disk, and a corresponding relatively low stress-concentration factor may be expected for even a rather sharp-ended ellipsoid.

It is realized that this discussion does not present anything basically new and that the authors appreciate these relationships without resort to such a crude analogy, but it is thought that for the benefit of readers interested in application of the results, it may be worth while to stimulate a discussion along these lines.

AUTHORS' CLOSURE

The authors are greatly indebted to Prof. Lloyd H. Donnell for indicating his views as to the significance of the solution in connection with the important problem of brittle crack propagation. Mr. Peterson's attempt to account for the low stress concentration at the poles of a very slender ellipsoidal cavity, by means of a flow analogy, ought to contribute to the physical understanding of the results obtained.