Dynamical chiral-symmetry-breaking in massless QED with \( N \) fermion species is studied through the numerical solution of the coupled Schwinger-Dyson (SD) equation. We have taken into account the fermion loop effect (at the 1-loop level) in the SD equation for the photon propagator through the vacuum polarization function \( \Pi(k^2) \), with and without the standard approximation: \( \Pi((p-q)^2) \approx \Pi(\max(p^2, q^2)) \). We have found that the scaling law is unchanged by this approximation and that, irrespective of the fermion flavor \( N \), the dynamical fermion mass and chiral order parameter obey the same mean-field type scaling, while the quenched planar QED without the vacuum polarization (\( N = 0 \) limit) obeys the Miransky scaling with the essential singularity.

§ 1. Introduction

In a previous work\(^1\) we studied analytically the Schwinger-Dyson (SD) equation in massless QED with \( N \) fermion species. In this paper we confirm the analytical results by solving the SD equation numerically. The SD equation is the simultaneous integral equation among the fermion propagator \( S(p) \), the photon propagator \( D_{\mu\nu}(k) \) and the vertex function \( \Gamma_\mu(p, q; k) \). The SD equation for the fermion propagator \( S(p) = [\mu A(p^2) - B(p^2)]^{-1} \) is decomposed into a pair of coupled integral equations for \( A \) and \( B \), each of which is a non-linear integral equation containing multiple-integrals.

In the previous analytical treatment,\(^1\) we avoided several difficulties appearing in solving the SD equation for the fermion propagator as follows. (1) Multiple-integral: The presence of the nontrivial vacuum polarization leads to the integral equation containing the double integral. This can be avoided by replacing the kernel by the separated (degenerate) form \( K(p, q) = K(p^2)\theta(p^2 - q^2) + K(q^2)\theta(q^2 - p^2) \) as a consequence of the LAK\(^1\) approximation à la Landau-Abrikosov-Khalatnikov for the vacuum polarization function:

\[
\Pi((p-q)^2) = \Pi(\max(p^2, q^2)) = \Pi(p^2)\theta(p^2 - q^2) + \Pi(q^2)\theta(q^2 - p^2). \tag{1.1}
\]

Then we can carry out the angular integral exactly in the same manner as in the quenched planar case and the SD equation reduces to the integral equation containing the single integral only. (2) Simultaneousness: This has been evaded by taking the Landau gauge. In the Landau gauge \( A(p^2) = 1 \) follows under the LAK approximation, and the SD equation for the fermion propagator reduces to the single integral equation for \( B(p^2) \), as shown in the quenched planar approximation.\(^3\) (3) Non-linearity: In order to study the scaling behavior in the neighborhood of the critical point, we do not have to deal with the non-linear equation and it is sufficient to solve the linearized equation, as guaranteed by the bifurcation theory.\(^3\)
Here we comment on the vertex function $\Gamma$. Some procedure for the truncation of the SD equation is necessary to terminate the infinite hierarchy of coupled SD equation. Here the truncation of the SD equation must be done in the gauge-parameter independent way. The local gauge invariance requires that the vertex function should be chosen so that the Ward-Takahashi (WT) identity is satisfied. In view of this, it is most convenient to take the Landau gauge, which extremely simplifies the actual calculations. In fact, in the Landau gauge $\epsilon = 0$, $A(p^2) \equiv 1$ is a good approximation and, in particular, this follows exactly as an identity in the quenched planar approximation $^4$ and in the LAK approximation.$^1$ Thus from this observation we take the bare vertex approximation in the Landau gauge:

$$\Gamma_\mu(p, q; k) = \gamma_\mu,$$  \hspace{1cm} (1.2)

which is also consistent with the Ward identity: $\Gamma_\mu(p, p; 0) = (\partial / \partial p^\mu)[S(p)]^{-1}$. In our calculation the validity of the bare vertex approximation in the Landau gauge is checked by examining whether or not the solution of the simultaneous equation for $A$ and $B$ supports $A(p^2) \equiv 1$. Improvement of the vertex in other gauges has been studied in the previous works$^5,6$ so that the resulting critical behaviors become gauge-independent even in the presence of the vacuum polarization$^5$ as well as in the quenched planar case.$^6$ Further details on the vertex correction will be discussed elsewhere.$^7$ In what follows, our numerical calculations are limited to the Landau gauge case, although the necessary expressions are given in the arbitrary gauge.

The purpose of the present paper is to confirm the analytical results on the scaling law obtained analytically in the previous paper, by numerically solving the SD equation without the above approximations adopted in the analytical study. Our results justify the LAK approximation which is taken also for the running coupling constant in asymptotically free gauge theories.$^8$

In this paper we solve numerically the SD equation for the fermion propagator with the 1-loop vacuum polarization function in the photon propagator. The actual calculations are carried out in the Landau gauge under the bare vertex approximation. The validity of the bare vertex approximation in the Landau gauge is also checked by solving a pair of integral equations for $A$ and $B$. In § 2, the SD equation of (QED)$_D$ in $D$-euclidean dimension is given in the arbitrary covariant gauge. In § 3, we give the numerical result under the approximation (1·1) for the vacuum polarization function. In § 4, we present the result of numerical calculations carried out without such an approximation. The final section is devoted to conclusions and discussion.

§ 2. SD equation

In euclidean (QED)$_D$, the SD equation for the full fermion propagator $S(p)$ in momentum space is given by

$$[S(p)]^{-1} = [S^{00}(p)]^{-1} - \Sigma(p)$$  \hspace{1cm} (2·1)

with the fermion self-energy part
\[ \Sigma(p) := e^2 \int \frac{d^D q}{(2\pi)^D} \gamma_{\nu} S(q) D_{\mu\nu}(q-p) \Gamma_{\nu}(q, p; q-p). \]  
(2.2)

Here \( S^{(0)}(p) = (p + m_0)^{-1} \) is the bare fermion propagator with bare fermion mass \( m_0 \), \( \Gamma_{\nu}(q, p; q-p) \) the vertex function, and \( D_{\mu\nu}(k) \) the full photon propagator with the covariant gauge-fixing parameter \( \alpha \):

\[ D_{\mu\nu}(k) = \frac{d(k^2)}{k^2} \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] + \frac{\alpha}{k^2} \frac{k_\mu k_\nu}{k^2}, \]  
(2.3)

where \( \alpha = 0 \) corresponds to the Landau gauge. Here the vacuum polarization effect is included through the transverse function \( d(k^2) \):

\[ d(k^2) := \frac{1}{1 + \Pi(k^2)}, \]  
(2.4)

from the vacuum polarization function:

\[ \Pi(k^2) = \frac{\Pi_\mu^\nu(k)}{(1 - D)k^2}. \]  
(2.5)

The vacuum polarization function is obtained from the self-energy part of the photon

\[ \Pi_{\mu\nu}(k) = N e^2 \int \frac{d^D p}{(2\pi)^D} \text{Tr}[\gamma_\mu S(p) \Gamma_{\nu}(p, p-k; k) S(p-k)], \]  
(2.6)

since the self-energy part of the photon is determined through the SD equation for the photon propagator

\[ D^\mu_\nu(k) = D^{(0)}_{\mu\nu}(k) + \Pi_{\mu\nu}(k), \]  
(2.7)

where \( D^{(0)}_{\mu\nu}(k) \) is the free photon propagator defined from (2.3) by setting \( d(k^2) \equiv 1 \). Thus we have a set of simultaneous integral equations.

For the fermion propagator, we solve the SD equation as a self-consistent equation of the form:

\[ S(p) = [\not{p} A(p^2) - B(p^2)]^{-1}. \]  
(2.8)

In what follows, we take the Landau gauge and the bare vertex approximation \( \Gamma_{\nu}(q, p; k) \equiv \gamma_{\nu} \). Then the SD equations (2.1) \( \sim \) (2.3) are decomposed into a pair of integral equations for \( A(p^2) \) and \( B(p^2) \):

\[ A(p^2) = 1 + \frac{\text{tr} \left[ \not{p} \gamma_{\nu} \right]}{\text{tr}(1)p^2}, \]  
(2.9)

\[ B(p^2) = m_0 + \frac{\text{tr} \Sigma(p)}{\text{tr}(1)}, \]  
(2.10)

by using the \( D \)-dimensional Clifford algebra \( \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = -2\delta_{\mu\nu}1, (\mu, \nu = 1, \cdots, D) \). Thus we obtain

\[ A(p^2) = 1 + e^2 \int \frac{d^D q}{(2\pi)^D} \frac{A(q^2)}{q^2 A^2(q^2) + B^2(q^2)}. \]
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Using the formula for the function $F(p^2, q^2, \theta)$ with the angle $\theta$ defined by the inner product

$$(p-q) = p^2 + q^2 - 2 \sqrt{p^2 q^2} \cos \theta,$$

the above integral equations are simply written in terms of $x := p^2$ and $y := q^2$:

$$A(x) = 1 + C_d e^2 \int_{\mu^2}^{A^2} dy \frac{y^{(D-2)/2} A(y)}{y A^2(y) + B^2(y)} Y_0(x, y),$$

$$B(x) = m_0 + C_d e^2 \int_{\mu^2}^{A^2} dy \frac{y^{(D-2)/2} B(y)}{y A^2(y) + B^2(y)} Y_0(x, y),$$

where $A$ (resp. $\mu$) is the ultraviolet (resp. infrared) cutoff and

$$C_d = \frac{1}{2^D \pi^{(D+1)/2} \int} \left( \frac{D-1}{2} \right).$$

In particular, $C_4 = 1/(8\pi^3)$. For this pair of integral equations, the respective integral kernels are given by

$$L_0(x, y) = \frac{1}{2x} \left[ \delta(x + y) I_0(x, y) \delta(x - y) I_0(x, y) \right]$$

$$- [(D-2) I_0(x, y; II) \delta - (D-3)(x+y) I_0(x, y; II) \delta]$$

$$- \left[ (x-y)^2 I_0(x, y; II) \delta \right],$$

$$K_0(x, y) = a I_0(x, y) \delta + (D-1) I_0(x, y; II) \delta,$$

where we have introduced the definite integrals,

$$I_0(x, y; II) = \int_0^\infty d\theta \frac{\sin^{D-2} \theta \cos^m \theta}{(x+y-2\sqrt{xy} \cos \theta)^s} \frac{1}{1 + II(x+y-2\sqrt{xy} \cos \theta)},$$

and as a special case

$$I_0(x, y; II = 0) = \int_0^\infty d\theta \frac{\sin^{D-2} \theta \cos^m \theta}{(y-x-2\sqrt{xy} \cos \theta)^s}.$$
\[
\Pi(k^2) = \frac{N\alpha^2}{12\pi^2} \left[ C - \ln \frac{k^2}{\Lambda_p^2} \right] = \frac{N}{3\beta} \left[ C + \rho + \ln \frac{\Lambda_p^2}{k^2} \right], \tag{2.21}
\]

which is obtained by taking \(S(p) = S^{(0)}(p)\), \(\Gamma_\alpha = \gamma_\alpha\) in the vacuum polarization part (2.6) with the cutoff \(\Lambda_p\) (the "photon" momentum-cutoff). Here \(C\) is a constant depending in general on the regularization scheme,\(^9\) and

\[
\rho := \ln \eta = \ln \left(\frac{\Lambda_p^2}{\Lambda^2}\right). \tag{2.22}
\]

Here we have introduced the inverse coupling constant

\[
\beta := \frac{4\pi^2}{\alpha^2}. \tag{2.23}
\]

Once knowing both functions \(A(p^2)\) and \(B(p^2)\) for all \(p\), the chiral order parameter is obtained by performing the numerical integration according to the formula

\[
\langle \bar{\psi} \psi \rangle = \int \frac{d^D p}{(2\pi)^D} \text{tr}[S(p)] = C_D \text{tr}(1) \Theta_D \int_\mu^2 \frac{d^D q}{p^2} \frac{B(p^2)^{1/2} B(q^2)}{A(p^2) + B(p^2)^{1/2}}, \tag{2.24a}
\]

where

\[
\Theta_D := \int_0^\pi d\theta \sin^{D-2} \theta = \begin{cases} \frac{\pi}{(D-3)!} & \text{for } D = \text{even} \\ \frac{2}{(D-2)!} & \text{for } D = \text{odd} \end{cases}, \tag{2.24b}
\]

which leads to \(\Theta_4 = \pi/2\). In this paper we use the word "the quenched planar approximation" in the sense that \(\Pi(k^2) = 0\) or \(d(k^2) = 1\), which corresponds to the \(N=0\) case in our formulation.

§ 3. Separation of the integral kernel (LAK approximation)

In the previous works,\(^{1,5}\) the following LAK approximation \( \text{\& la Landau-Abrikosov-Khalatnikov}^{10,11} \) was taken:

\[
d((p-q)^2) \approx d(\max(p^2, q^2)) = d(p^2)\theta(p^2-q^2) + d(q^2)\theta(q^2-p^2), \tag{3.1}
\]

which avoids the analytically intractable angular integral and allows one to convert the problem of solving the integral equation into the boundary value problem of the differential equation. Carrying out the angular integration, we obtain the integral kernels:

\[
L_D(x, y) = \frac{1}{2x} \left\{ a[\alpha + (x+y)I_D(x, y)_1 - (x-y)^2 I_D(x, y)_2] - d(\max(x, y)) T_D(x, y) \right\}, \tag{3.2a}
\]

\[
T_D(x, y) := (D-2)I_D(x, y)_0 - (D-3)(x+y)I_D(x, y)_1 - (x-y)^2 I_D(x, y)_2 \tag{3.2b}
\]

and

\[
K_D(x, y) = [\alpha + (D-1)d(\max(x, y))] I_D(x, y)_1. \tag{3.3}
\]

Note that in the Landau gauge \(a = 0\), the identity \(L_D(x, y) = 0\) follows, which implies


\[ A(x) = 1, \text{ since the following mathematical identity is proved}^{4} \text{ that for any } D \geq 3, \]
\[ T_D(x, y) = 0. \quad (3.4) \]

Then in the Landau gauge a pair of simultaneous equations for \( A(x) \) and \( B(x) \) decouples each other, under the LAK approximation. Then, using the formula for \( D = 4 \),

\[ I_4(x, y) = \frac{\pi}{x + y + |x - y|} = \frac{\pi}{2} \left[ \frac{1}{x} \theta(x - y) + \frac{1}{y} \theta(y - x) \right], \quad (3.5) \]

the SD equation of QED\(_4\) in the Landau gauge is given as

\[ A(x) = 1, \quad (3.6) \]
\[ B(x) = m_0 + \frac{3e^2}{16\pi^2} \int_{m^2}^{\Lambda^2} dy \left[ \frac{d(y)}{x} \theta(x - y) + \frac{d(y)}{y} \theta(y - x) \right]. \quad (3.7) \]

In the quenched planar QED\(_4\), the SD equation has the simple form

\[ B(x) = m_0 + \frac{3e^2}{16\pi^2} \int_{m^2}^{\Lambda^2} dy \left[ \frac{\theta(x - y)}{x} + \frac{\theta(y - x)}{y} \right]. \quad (3.8) \]

In this case, the dynamical fermion mass \( m_d \) defined by the solution of \( m = B(m^2) \) obeys the Miransky scaling with an essential singularity at the critical point \( e_c = 2\pi / \sqrt{3} \),

\[ f(e^2) = \frac{m_d}{A} \exp \left\{ -\frac{n\pi}{\sqrt{e^2/e_c^2 - 1}} \right\}, \quad (n = 1, 2, \ldots) \quad (3.9) \]

which was first derived by the analytic method\(^2\) and is also confirmed by the numerical calculations of the gap equation \(3.8\).

Our main interest is how the scaling law may change when the vacuum polarization effect is included and how it depends on the fermion flavors \( N \). The numerical results presented in this section are elaboration of the previous calculation\(^5\) where the LAK approximation is adopted, simply because this enables us to obtain the analytic form for the integral kernel after integrating out the angular integral and hence the double integral can be reduced to the single integral with respect to the squared momentum \( p^2 \). In this approximation, the SD equation is greatly simplified, and above all the SD equation in the Landau gauge reduces to the self-consistent equation for the single function \( B(x) \). Furthermore, by this reduction, the SD equation for \( B(p^2) \) can be converted into the boundary value problem of the second order differential equation. Moreover linearization is necessary to solve the differential equation analytically.

In fact, the Miransky scaling in the quenched planar approximation has been first obtained by this method\(^2\) for the differential equation of the Euler type:

\[ B''(p^2) + 2 \frac{B'(p^2)}{p^2} + \frac{3e^2}{16\pi^2} \frac{B(p^2)}{(p^2)^2} = 0 \quad (3.10) \]

with the boundary conditions:
On the other hand, in the presence of the vacuum polarization, the corresponding differential equation is a linear second order differential equation with three singular points \( z = 0, 1, \infty \), of which 0 and 1 are regular singular points and \( \infty \) is an irregular one:

\[
B''(z) + \left[ \frac{\frac{2}{z} - \frac{1}{z-1}}{z-1} \right] B'(z) + \sigma \left[ \frac{1}{z} - \frac{1}{z^2} \right] B(z) = 0; \quad \sigma := \frac{9}{4N}
\]  

(3.12)

with the boundary conditions:

\[
m_0 = B(z) + \frac{z}{1-z} B'(z) = 0 \quad \text{at} \quad z = z_a ,
\]

(3.13a)

\[
B'(z) = 0 \quad \text{at} \quad z = z_p ,
\]

(3.13b)

where

\[
z = z_p := z_a + \ln \frac{A^2}{\beta}; \quad z_a := \frac{3\beta}{N} + C + \rho .
\]

(3.14)

However the general solution of this differential equation cannot be expressed through the known special function. Even in this case, we can obtain the asymptotic solution for large \( z_a \), which is able to give the critical value \( e_c(N) \) of the bare coupling constant in good accuracy, at least for \( N = 1 \) and 2 (see Ref. 1).

It is shown\(^{13}\) that the scaling law is given by the MF type, irrespective of the fermion flavor \( N \):

\[
m_d \sim A(N_c(e) - N)^{1/2},
\]

(3.15)

\[
\langle \bar{\psi} \psi \rangle \sim A^2 m_d \sim A^3(N_c(e) - N)^{1/2}.
\]

(3.16)

In general the critical behavior of the model is characterized by a set of critical exponents. We can define the critical exponent \( \nu_m \) and \( \nu_{ch} \) for the dynamical fermion mass and the chiral order parameter by

\[
m_d \sim A(e^2 - e_c^2(N))^{\nu_m} ,
\]

(3.17)

\[
\langle \bar{\psi} \psi \rangle \sim A^3(e^2 - e_c^2(N))^{\nu_{ch}}.
\]

(3.18)

In the numerical calculations, the non-linear integral equation (3.7) for \( B \) is solved when \( C = 0, \rho = 0 \) (i.e., \( \Lambda_\rho = 1 \)). By this choice of the parameters generality is not lost in the one-loop level, as shown in the previous work.\(^{13}\) As can be seen from the data, the scaling window becomes very narrower for smaller \( N \). Hence, if the dynamical masses for different \( N \) are plotted in Fig. 1 on the same scale, the critical exponents seemed to have small-dependence on \( N \), as was wrongly reported in the preliminary paper.\(^{5}\) Therefore we must approach sufficiently near the critical point, which considerably consumes the CPU time for the computer calculations. In Figs. 2~4, we plot the chiral order parameter \( \langle \bar{\psi} \psi \rangle \) vs \( \beta \) and \( \langle \bar{\psi} \psi \rangle^2 \) vs \( \beta \) where \( \langle \bar{\psi} \psi \rangle \):
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Our numerical calculations indicate that in the unquenched case the scaling law is given by the MF type in the sense that the critical exponents are given by the classical MF-values\textsuperscript{1}) $1/2$ for all $N=1, 2, 4$, as shown in Figs. 2~4:

$$
\nu_m = \frac{1}{2} = \nu_{ch}.
$$

(3·19)

Thus the dynamical fermion mass and chiral order parameter have the same critical exponent, $\nu_m = \nu_{ch}$, which agrees with the analytical prediction\textsuperscript{1}) that the dynamical fermion mass and the chiral order parameter (normalized to dimensionless by UV cutoff $\Lambda$) obeys the same scaling law:

$$
\langle \bar{\psi} \psi \rangle \sim \Lambda^2 m_d.
$$

(3·20)

Thus the essential-singularity type scaling is obtained only in the quenched planar case, which corresponds to the limit $N=0$ in our formulation.

\textsuperscript{1}) Quite recently, all the critical exponents have been shown to take the classical MF-values in the unquenched case,\textsuperscript{17}) as well as the quenched case.\textsuperscript{18})

Fig. 1. Scaling function $f := m_d/\Lambda$ in the LAK approximation for fermion flavors $N=0, 1, 2$ and 4 at $\Lambda^2=10^{10}$.

Fig. 2. LAK approximation $(N=1)$ for $\Lambda^2=10^{10}$: $\langle \bar{\psi} \psi \rangle := \langle \bar{\psi} \psi \rangle /\Lambda^2$ vs $\beta$ and $\langle \bar{\psi} \psi \rangle^2$ vs $\beta$.

Fig. 3. LAK approximation $(N=2)$.

Fig. 4. LAK approximation $(N=4)$. 

$\langle \bar{\psi} \psi \rangle /\Lambda^3$. Our numerical calculations indicate that in the unquenched case the scaling law is given by the MF type in the sense that the critical exponents are given by the classical MF-values\textsuperscript{1}) $1/2$ for all $N=1, 2, 4$, as shown in Figs. 2~4:

$$
\nu_m = \frac{1}{2} = \nu_{ch}.
$$

(3·19)

Thus the dynamical fermion mass and chiral order parameter have the same critical exponent, $\nu_m = \nu_{ch}$, which agrees with the analytical prediction\textsuperscript{1}) that the dynamical fermion mass and the chiral order parameter (normalized to dimensionless by UV cutoff $\Lambda$) obeys the same scaling law:

$$
\langle \bar{\psi} \psi \rangle \sim \Lambda^2 m_d.
$$

(3·20)

Thus the essential-singularity type scaling is obtained only in the quenched planar case, which corresponds to the limit $N=0$ in our formulation.

\textsuperscript{1}) Quite recently, all the critical exponents have been shown to take the classical MF-values in the unquenched case,\textsuperscript{17}) as well as the quenched case.\textsuperscript{18})
§ 4. Numerical solutions of the double integral equation

In this section we present the results of numerical calculations of the SD integral equations (2.14)~(2.20) in the Landau gauge, without using the LAK approximation: \( II((p-q)^2) \approx \Pi(\max(p^2, q^2)) \). We have carried out directly the double integral with respect to the squared momentum \( q^2 \) and the angle \( \theta \) coming from the volume element \( d^4q \) of the SD equation, which should be compared with the results in the previous section under the LAK approximation for the vacuum polarization function.

Without the LAK approximation, however, we cannot calculate the kernels in advance for all \( x \) and \( y \), in contrast with the quenched planar case, and hence we must calculate them, every time the coupling constant \( \beta \) and the fermion flavor \( N \) are changed. This makes the numerical calculations a little harder. Furthermore, in this case, \( A(p^2) \) is no longer equal to one (the wave-function renormalization), even in the Landau gauge and hence we must solve a pair of integral equations for \( A(p^2) \) and \( B(p^2) \). Even in this case, however, it is shown that \( A(p^2) = 1 \) holds approximately in good accuracy.

In the first calculation, for comparison, we have put \( A(p^2) = 1 \) a priori and solved (2.15) with the integral kernel (2.18) under the bare vertex approximation in the Landau gauge. The results are shown in Figs. 5 and 6 for \( N=1 \) and 2, respectively.

In evaluating Eqs. (2.15), we chose a finite number of sample points according to the Simpson method with unequal interval and sum up the values of integrand at these points with the appropriate weights. In order to estimate the finite-sample effect, we varied the number of samples both for angular and for radial integrations. Results are tabulated in Table I. Based on these observations, we determined the number of sample points: \( N_p = 400 \) for the radial part \( p^2 \) and \( N_\theta = 80 \) for the angular part \( \theta \). In our calculations, we estimate the absolute error for the dynamical fermion mass as \( \delta \beta \sim O(1) \) for \( A = 10^6 \).

Furthermore we have solved a pair of double-integral equations for \( A \) and \( B \). In Fig. 7, the solutions \( A(p^2), B(p^2), M(p^2) = B(p^2)/A(p^2) \) are plotted for \( N=1 \). The data support that \( A(p^2) = 1 \) is a good approximation in the Landau gauge, even

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![Fig. 5. Numerical results without LAK approximation (N=1) at A^2=10^6.](image)

![Fig. 6. Numerical results without LAK approximation (N=2) at A^2=10^6.](image)
without the LAK approximation. Our numerical calculations show that the scaling law in the unquenched QED is given by the MF type, irrespective of the fermion flavor \( N \) except \( N=0 \) (the quenched case).

Without the LAK approximation, however, we encounter a new problem. The positivity of the vacuum polarization function \( \Pi(k^2) \) is not always guaranteed for all \( k^2 \in [0, \Lambda^2] \). This is because the range of \( k^2 = (p-q)^2 = p^2 + q^2 - 2\sqrt{p^2 q^2 \cos \theta} \) is \( [0, 4\Lambda^2] \) for \( p^2, q^2 \in [0, \Lambda^2] \). Thus for \( k^2 \in [0, \Lambda^2] \)

\[
\min_{0 \leq p, q \leq \Lambda; 0 \leq \eta \leq \Lambda} \Pi(k^2) = \frac{N}{3\beta} \left(C + \ln \frac{2}{4}\right).
\]  

(4.1)

Then the requirement of positivity of the dielectric function of the vacuum,

\[
\varepsilon(k^2) = 1 + \Pi(k^2),
\]  

(4.2)

for all \( k^2 \in [0, \Lambda^2] \) imposes the following constraint on the allowed region of bare parameters in the phase diagram \( (\beta, N) \):

\[
\frac{N}{3\beta} < \frac{1}{\ln(4/\eta) - C}.
\]  

(4.3)

Therefore, when we take \( C=0 \) and \( \eta=1 \), i.e., \( \Lambda^2 = 1 \), the allowed region is restricted to

\[
\frac{N}{3\beta} < 0.7213\ldots,
\]  

(4.4)

which is satisfied only for \( N=1 \) and 2 according to the numerical calculation.\(^{14,15}\)
This violation of the positivity of the dielectric function of the vacuum implies the negative integral-kernel $K(p^2, q^2)$ in the gap equation, for some choices of $p$ and $q$. This may change considerably the nature of the solution. Actually, as observed in our numerical calculation, the convergence of the iteration is problematical for $N \geq 3$, when $C=0$ and $\eta=1$.

Note that different choices of $C$ or $\eta$ change the critical coupling constant. However, the physically meaningful concept is not the position of the critical point but the type of the scaling law. The critical value of the bare coupling constant has no direct physical meaning in the continuum limit, while the renormalized coupling constant obtained in the limit has the direct relevance to the physics. So the problem is whether or not the scaling law may change for different choices of $C$ or $p$. At the 1-loop level, the scaling is not affected by changing $C$ or $p$, since the non-trivial solutions with $C\neq 0$ and $p\neq 0$ are obtained by shifting the coupling constant $\beta$ in the solution with $C=0$, $p=0$ as follows:

$$\beta \rightarrow \beta - \frac{N}{3} (C + p) = \beta'. \quad (4.5)$$

Therefore, if $C + p > 0$, there is a critical $N_c$ obtained as the intersection point of the critical line in the case $C=0$, $p=0$ with the straight line $\beta = (N/3)(C + p)$ so that for $N > N_c$ the chiral symmetry is not broken spontaneously for all gauge coupling. On the other hand, if $C + p < 0$, there does not exist such a restriction.

Thus it is confirmed that the unquenched QED obeys the scaling law of the MF type, irrespective of the fermion flavor and of the renormalization scheme in the sense explained above.

§ 5. Conclusion and discussion

We have obtained the numerical solutions of the SD equation for the fermion propagator in massless QED with $N$ fermion flavors, taking into account the 1-loop vacuum polarization in the photon propagator. In this case, the SD equation becomes the simultaneous, non-linear integral equation with double integration. Enumerating the results,

1) There is a critical point $e_c(N)$ separating the perturbative phase from the non-perturbative strong coupling phase where the chiral symmetry is spontaneously broken. Once the vacuum polarization function is given, the critical value of the coupling constant can be calculated. It is shown that the critical point shifts monotonically into the stronger coupling region as the fermion flavor increases. Our numerical calculations show that the unquenched QED obeys the MF scaling, irrespective of the fermion flavor $N$.

2) We have checked the validity of the LAK approximation for the vacuum polarization function, which is usually taken in the analysis of the running coupling constant. The scaling law is unchanged; the MF one, although the critical coupling constant slightly shifts in the LAK approximation.

3) There is an ambiguity due to the renormalization scheme, i.e., corresponding to
the choice of the ratio of the fermion cutoff to the photon cutoff, \( \eta = \Lambda_\phi^2/\Lambda^2 \), and a constant \( C \) in the vacuum polarization function \( \Pi(k^2) \). The type of scaling is unchanged for different choices of \( \eta \) and \( C \) as long as the convergent solution exists, although the explicit value of the critical coupling may change depending on \( \eta \) and \( C \). The existence of the 2nd order chiral phase transition is limited to the relative small region of the fermion flavor, as claimed in the Monte Carlo simulation. In fact, changing \( \rho = \ln \eta \) and \( C \) from \( \rho = 0 \) and \( C = 0 \) is simply equivalent to shifting the coupling constant: \( \beta \to \beta - \left( N/3 \right) (C + \rho) \) at the 1-loop level. The renormalization condition \( C + \rho > 0 \) implies the existence of the critical value \( N_c \) above which the chiral-symmetry is not broken even in the strong coupling region.

The next challenging step is to obtain the vacuum polarization function as a self-consistent solution of the simultaneous integral equation. By this, the above ambiguity due to the vacuum polarization function will be fixed.

It is important to remark that QED\(_4\) with the 1-loop vacuum polarization function has the Landau ghost, so that the continuum theory obtained in the limit \( \Lambda \to \infty \) is expected to become a non-interacting ("trivial") theory. On the other hand, we should mention that the mean-field (MF) scaling is believed to be a signal of the triviality of the continuum theory. The triviality of the continuum theory and the exactness of the MF prediction are deeply connected, as proved rigorously for a class of scalar field theories including the \( \lambda(\phi^4) \) theory in \( D \geq 4 \) dimensions. In the framework of the SD equation, this connection is demonstrated as follows. In the massive vector meson model with the mass \( \mu \), the interquark potential is given by the Yukawa type, \( V(|x - y|) = e^2 \exp[-\mu |x - y|]/|x - y| \), and the scaling law in the Landau-like gauge is given by

\[
 f(e^2) \propto \sqrt{\exp \left[ \frac{-2(n\pi + \phi)}{\sqrt{e^2 - e_c^2}} \right] - \frac{r}{1 + r}} ; \quad r : = \frac{\mu^2}{\Lambda^2} ,
\]

which reduces to the Miransky scaling in the limit \( r \to 0 \), as expected. The zero-charge model is defined by setting \( \mu \) proportional to the ultraviolet (UV) cutoff \( \Lambda \), i.e., fixing \( r \neq 0 \). As a result the interquark potential of the zero-charge model converges to zero in the infinite cutoff limit \( \Lambda \to \infty \). The critical point \( e_c(r) \), which is \( r \)-dependent, is given by the condition \( f(e_c(r)) = 0 \), where \( e_c(0) = e_c \). Then, near the critical point, the scaling law is given by the MF type

\[
 f(e^2) \sim \sqrt{e^2 - e_c^2(r)} .
\]

In \( D \geq 5 \) dimensions, QED\(_4\) is non-renormalizable and converges to a trivial theory, which exhibits the MF behavior even in the quenched planar approximation. Accordingly, QED\(_4\) with the vacuum polarization function (2·21) obeys the MF type scaling, irrespective of the fermion species \( N \). Therefore our numerical and previous analytical results are consistent with the above claim.

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References