Maslov Phase as Geometric Phase in the Time-Dependent Variational Approach with Squeezed Coherent States

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Toward the classical description of quantum fluctuations in quantal systems, the Maslov phase occurring in a semi-classical quantization rule is investigated in a framework of the time-dependent variational principle with squeezed coherent states. In the limit of a semi-classical approximation in this approach, it is definitely shown that the Maslov phase has a geometric nature analogous to Berry's phases or canonical phases including dynamical effects. The Maslov phase appears as a winding number in this framework. It is also indicated that this squeezed coherent state approach is a possible way to go beyond the usual WKB approximation.

§ 1. Introduction

In many-body problems, a great interest is paid to describe quantal systems by means of a few classical variables because of difficulties to treat full degrees of freedom. Another important physical reason to extract a few classical variables from quantal systems is that we are especially interested in some particular characteristic motions in quantal systems, for example, the nuclear collective motions in nucleus as an isolated many-nucleon system and the soliton dynamics in the field theory as a system of infinite degrees of freedom. It is known that, in various quantum many-particle systems, if one takes the limit of large $N$ which represents, for example, the number of "particles", then the quantum theories become equivalent to the classical ones. A similar situation is realized in the quantum chromodynamics (QCD). It is believed that the QCD is reduced to the classical meson theory with infinite species of mesons when the number of color degree of freedom, $N_c$, is infinitely large. Then, it is argued that the baryons are realized as classical solitons. It may be regarded that the Skyrme model, or in general chiral soliton models, are based on the above-mentioned situation. However, since we are interested in the nuclei as finite quantum many-body systems and the realistic situation for the number of color degree of freedom in QCD, it should be considered that the deviations from classical dynamics can never be neglected. Thus, we are confronted with the problem to describe the quantum effects in terms of the classical variables.

Up to the present, there has been much progress in describing the quantal systems in terms of classical variables. In many fermion systems such as nucleus, the time-dependent Hartree-Fock (TDHF) theory has been developed. The TDHF approximation is a powerful technique for describing both of the nuclear collective motion and the nuclear reactions. Also, in quantum field theory, a variational definition of the effective potential was formulated within the Gaussian wave packets. These approaches rely on the time-dependent variational principle (TDVP). There, one restricts oneself to assume an appropriate form for a trial state, according to the
problem under consideration. Then, the trial state is parametrized by certain c-number variables, the time-development of which is determined through the TDVP. Thus, one can obtain the classical counterparts of quantal many-body systems. For example, in the TDHF theory, the trial state is taken to be a single Slater determinant and under the time-dependent variational procedure, Hamilton's equation of motion is derived, which governs time-development of c-number variables parametrizing the Slater determinant. Thus, the classical counterpart of many-fermion system is obtained. These trial states are essentially coherent states. The coherent states are used in order to study the various physical phenomena of dynamical quantum systems. However, in the usual coherent states, the quantum fluctuations are fixed in a certain sense. Since we are interested in describing the classical motion in quantal systems beyond the semi-classical or the WKB scope, there are various ways of improving the above-mentioned framework.

The TDHF theory is equivalent to the random phase approximation (RPA) under the assumption that the quantum fluctuations around the Hartree-Fock ground state are small. With the aim of constructing a microscopic theory beyond the RPA, many authors have proposed new theories such as the self-consistent collective coordinate (SCC) method, the resonating Hartree-Fock and resonating random phase approximation and the extended TDHF theory. In the quantum field theory, generalizations of the Gaussian wave packet approximation are carried out by many authors, which are based on the work by Jackiw and Kerman who have formulated the effective action with the use of the most general Gaussian wave packet.

In the preceding papers, we have proposed a possible description of classical motion in quantal systems with the aim of including the higher order quantal effects in terms of the classical mechanics. Our basic idea has been formulated with the use of the TDVP and the squeezed coherent states paying strong attention to canonicity conditions developed in the TDHF theory. This approach can be regarded as a possible refinement and a systematic treatment of Jackiw-Kerman approach. In one-dimensional boson systems, it has been shown that our approach with the squeezed coherent states works well and, especially, the correct results within the leading term of order \( \hbar \) are reproduced in some potential problems. In the Schrödinger problem, we have proved that the time-evolution of the squeezed coherent state is exactly described if Hamiltonian involves quadratic or simpler potentials. This fact means that it is possible to obtain the quantized spectra of the bound states by evaluating the Fourier transform of the trace of the time-evolution kernel. Also, we have extended this squeezed coherent state approach to the quantum field theory. We have examined the \((1+1)\)-dimensional scalar-field soliton systems and derived a set of coupled equations for solitons and quantum fluctuations as was expected; this may afford a low energy effective theory of QCD to describe baryons and mesons in a single framework. By using the heat kernel method, we have also shown that, in the static one-soliton systems, the usual results of the WKB method are correctly reproduced as a semi-classical limit in our framework. Therefore, the time-dependent variational approach in terms of squeezed coherent states is expected to take account of the higher order quantal effects in \( \hbar \) than the WKB approximation.
In the usual coherent state path integral method, one may often divide original variables into two parts, that is, a classical part and quantum fluctuations around it. The relation between our squeezed coherent state approach and the standard coherent state path integral technique is not yet understood clearly at present. However, since the "classical" part and the "quantum fluctuations" are not divided at the first stage but determined self-consistently through the variational equations in our framework, the higher order quantal effects in $\hbar$ are naturally incorporated. Thus, the squeezed coherent state approach is a possible way to go beyond the approximation of WKB order. However, the direct connection between the squeezed coherent state approach and the WKB method is not clear.

The main purpose of this paper is to show that, when the TDVP and the squeezed coherent states are employed in a simple one-dimensional boson system, the Maslov phase appears as an effect of the quantal fluctuations in the semi-classical approximation. Recently, Littlejohn has originally pointed out\(^{29}\) that the Maslov phase, occurring in the semi-classical quantization procedure, is a kind of Berry's phase by using the generalized Gaussian wave packet dynamics.\(^{27),28}\) He first assumes a general Gaussian wave packet in the Schrödinger equation, and expands the Hamiltonian up to the second order around the center of the wave packet. Finally, he determines the time-evolution of parameters characterizing the wave packet. In our time-dependent variational approach with the squeezed coherent states, higher order corrections in $\hbar$ are already incorporated in the framework. Thus, we need to carry out a possible refinement and extension, in order to realize the geometric nature of the Maslov phase. It will be shown that this situation is analogous to that we encountered when the geometric phases including dynamical effects have been derived by means of the TDVP.\(^{29}\) Thus, it is expected that the relationship and difference between the conceptions of our squeezed coherent state approach and the usual WKB method become clear.

The second purpose of this paper is to demonstrate the usefulness of our approach when we evaluate the ground state energy of quantum systems. Owing to the inclusion of higher order effects in $\hbar$, the zero-point energies are well reproduced by solving a set of coupled equations of motion numerically. Therefore, our approach is a possible way to go beyond the WKB approximation.

This paper is organized as follows. In § 2, we recapitulate the time-dependent variational approach with the squeezed coherent states developed in Refs. 19) and 20). Furthermore, we numerically show the validity of selecting initial conditions for the classical image of quantal fluctuations, i.e., squeezing degree of freedom. In § 3, we discuss the connection between the semi-classical limit in our squeezed coherent state approach and the usual WKB approximation. It is shown that the Maslov phase has a definite geometrical nature similar to the geometric phase and the Planck constant $\hbar$ plays a role of a kind of adiabatic parameter analogous to the situation of Berry's phase including dynamical corrections. In § 4, it is shown that the time-dependent variational approach with the squeezed coherent state is a possible way to go beyond the WKB approximation. The last section is devoted to a summary.
§ 2. Recapitulations of TDVP with squeezed coherent states

In this section, we give the framework of the time-dependent variational approach in terms of squeezed coherent states in order to make this paper self-contained. The detailed discussions have been given in Refs. 19) and 20).

We start with the squeezed coherent state as
\[ |\Phi(a, \beta)\rangle = \exp\{a\hat{a}^\dagger - a^* \hat{a}\} |\Psi(\beta)\rangle, \tag{2.1a} \]
\[ |\Psi(\beta)\rangle = \exp\left\{ \frac{1}{2} (B\hat{a}^2 + B^* \hat{a}^2) \right\} |0\rangle. \tag{2.1b} \]

The following relations are satisfied:
\[ [\hat{a}, \hat{a}^\dagger] = 1 \quad \text{and otherwise} = 0, \tag{2.2a} \]
\[ \hat{a} |0\rangle = 0, \tag{2.2b} \]
where \([ , ]\) denotes the commutator and \(|0\rangle\) is a vacuum state with respect to a boson operator \(\hat{a}\). Here, \(a\) and \(B\) are the time-dependent c-number variables, and \(|\Psi(\beta)\rangle\) is called the squeezed vacuum. For later convenience, it is useful to introduce new variables instead of \(B\) and \(B^*\),
\[ \beta = \frac{1}{\sqrt{2}} \frac{B}{|B|} \sinh|B|, \quad \hat{B} = \frac{\beta}{|B|} \sinh^{-1}(\sqrt{2}|\beta|). \tag{2.3} \]

It is possible to recast the usual squeezed coherent state in Eqs. (2.1a) and (2.1b) to another equivalent form. First, we define the coordinate and momentum operators
\[ \hat{Q} = \sqrt{\frac{\hbar}{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{P} = (i) \sqrt{\frac{\hbar}{2}} (\hat{a} - \hat{a}^\dagger). \tag{2.4} \]

We can show the equivalence between the original squeezed coherent state in Eqs. (2.1a) and (2.1b) and the following Gaussian-type state:\(^{20}\)
\[ |\Phi(\alpha, \beta)\rangle = (2G)^{-1/4} \exp\left\{ i \frac{\hbar}{\sqrt{2}} (\hat{P} - \hat{Q}) \right\} \exp\left\{ \frac{1}{2\hbar} |\hat{Q}|^2 \right\} |0\rangle \]
\[ = e^{-i\phi} |\Phi(a, \beta)\rangle, \tag{2.5} \]
where we define new variables as follows:
\[ q = \sqrt{\frac{\hbar}{2}} (\alpha + \alpha^*), \quad p = (-i) \sqrt{\frac{\hbar}{2}} (\alpha - \alpha^*), \tag{2.6} \]
\[ Q = 1 - \frac{1}{2G} + i2\Pi, \tag{2.7a} \]
\[ G = \frac{1}{2} \left| \cosh|B| + \frac{B}{|B|} \sinh|B| \right|^2 \]
\[ = \sqrt{\frac{1}{2} + |\beta|^2 + \beta}^2, \tag{2.7b} \]
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\[ \Pi = \frac{i}{2} \frac{B^* - B}{|B|} \frac{\sinh |B| \cosh |B|}{|\cosh |B| + \frac{B}{|B|} \sinh |B|}^2 \]

\[ = \frac{i}{2} (\beta^* - \beta) \sqrt{\frac{1}{2} + |\beta|^2} G^{-1} \]  (2.7c)

and

\[ e^{-i\varphi} = \frac{1}{\sqrt{G}} \left( \sqrt{\frac{1}{2} + |\beta|^2} + \beta \right) . \]  (2.8)

Here, we have the time-dependent variables \( q, p, G \) and \( \Pi \) instead of \( a, a^*, \beta \) and \( \beta^* \). We will start from the equivalent form of the squeezed coherent state in Eq. (2.5) in the later discussion. The variables \( (q, p) \) correspond to classical variables and \( (G, \Pi) \) are regarded as the classical image of quantal fluctuations. We treat these variables as dynamical ones.

Note that the variable \( G \) is positive definite and never takes zero. This fact is important in order to understand the interpretation of the usual WKB approximation within our framework in the next section.

In general, the squeezed coherent states have the character of "squeezing" the uncertainty relations. In the coherent states which we write as \( |\Phi_c\rangle \), we can calculate the square of the standard deviation as

\[ (\Delta q)^2 = \langle \Phi_c | (\bar{Q} - q)^2 | \Phi_c \rangle = \frac{1}{2} \hbar , \]  (2.9a)

\[ (\Delta p)^2 = \langle \Phi_c | (\bar{P} - p)^2 | \Phi_c \rangle = \frac{1}{2} \hbar . \]  (2.9b)

Then, we obtain the uncertainty relation in the coherent state

\[ (\Delta q)(\Delta p) = \frac{\hbar}{2} . \]  (2.10)

Thus, we can see that the minimum uncertainty is realized at any time.

In the squeezed coherent states under consideration, the square of the standard deviation is modified from the coherent states due to the incorporation of the squeezing degree of freedom. We can calculate the square of the standard deviation with the use of the squeezed coherent state in Eq. (2.1):

\[ (\Delta q)^2 = \langle \Phi | (\bar{Q} - q)^2 | \Phi \rangle = \hbar \xi , \]  (2.11a)

\[ (\Delta p)^2 = \langle \Phi | (\bar{P} - p)^2 | \Phi \rangle = \hbar \eta , \]  (2.11b)

where \( \xi \) and \( \eta \) are defined by

\[ \xi = G , \]  ( > 0 )  (2.12)

\[ \eta = \frac{1}{4G} + 4G \Pi^2 . \]  ( > 0 )  (2.13)

Thus, we arrive at the uncertainty relation with respect to the squeezed coherent state.
\[ (\Delta q)(\Delta p) = \hbar \sqrt{\xi} \]
\[ = \frac{\hbar}{2} \sqrt{1 + 4 \xi^2} \geq \frac{\hbar}{2} \]  
(2.14)

with
\[ \xi = 2GI. \]  
(2.15)

Since \( G \) and \( II \) are the time-dependent dynamical variables, the minimum uncertainty is not realized through all the time. Therefore, we can only set a requirement that the minimum uncertainty should be satisfied at some specific time, which we take the initial time \( t_0 \).

It is necessary to evaluate various expectation values with respect to the squeezed coherent states. We can easily calculate them since we have kept the second quantized form consistently. For example,
\[ \langle \Phi(t) | \hat{Q} | \Phi(t) \rangle = q(t), \quad \langle \Phi(t) | \hat{P} | \Phi(t) \rangle = p(t), \]  
(2.16)
\[ \langle \Phi(t) | \hat{Q}^2 | \Phi(t) \rangle = q(t)^2 + \hbar \xi, \]  
(2.17a)
\[ \langle \Phi(t) | \hat{P}^2 | \Phi(t) \rangle = p(t)^2 + \hbar \eta, \]  
(2.17b)
\[ \langle \Phi(t) | \frac{1}{2} (\hat{Q}\hat{P} + \hat{P}\hat{Q}) | \Phi(t) \rangle = q(t)p(t) + \hbar \xi, \]  
(2.17c)
\[ \langle \Phi(t) | \partial_z | \Phi(t) \rangle = \frac{i}{\hbar} (q \partial_z p - p \partial_z q) + iG \partial_z II, \]  
(2.18)

where \( \partial_z = \partial / \partial Z \).

In general, we can calculate the expectation values for arbitrary operators in terms of the Wigner transform,
\[ \langle \Phi(t) | \hat{O} | \Phi(t) \rangle = \int_{-\infty}^{\infty} dQ dP \rho_w(Q, P) \rho_{w}(Q, P) \]
\[ = \exp \left\{ \frac{\hbar}{2} \xi \left( \frac{\partial}{\partial q} \right)^2 + \hbar \eta \left( \frac{\partial^2}{\partial q \partial p} \right) + \frac{\hbar}{2} \eta \left( \frac{\partial}{\partial p} \right)^2 \right\} \rho_{w}(q, p), \]
(2.19)

where
\[ \rho = |\Phi(t)\rangle \langle \Phi(t)|, \]  
(2.20)

\[ \rho_{w}(Q, P) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} ds e^{i\phi \sinh} \left( \left| Q - \frac{s}{2} \right| \rho \left| Q + \frac{s}{2} \right| \right) \]
\[ = \frac{1}{2\pi \hbar} \exp \left\{ -\frac{2}{\hbar} \xi (P - p)^2 + \frac{4}{\hbar} \xi (P - p)(Q - q) - \frac{2}{\hbar} \eta (Q - q)^2 \right\}, \]
\[ \rho_{w}(Q, P) = \int_{-\infty}^{\infty} ds e^{i\phi \sinh} \left( \left| Q - \frac{s}{2} \right| \partial \left| Q + \frac{s}{2} \right| \right). \]
(2.21)

Here, \(|Q\rangle\) satisfies the relation \( \hat{Q} |Q\rangle = Q |Q\rangle \).

We need to determine the time-development of the variables \( q(t), p(t), G(t) \) and \( II(t) \). We can carry this out with the aid of the time-dependent variational principle.
similar to the TDHF theory
\[ \delta \int_{t_1}^{t_2} dt \langle \Phi(t) | i\hbar \frac{\partial}{\partial t} - \hat{H} | \Phi(t) \rangle = 0. \]  \hspace{1cm} (2.23)

Here, we assume the trial state $|\Phi(t)\rangle$ being the squeezed coherent state. Furthermore, we impose the canonicity conditions\(^{10}\) well known in the TDHF theory, in order to extract canonical variables. Taking the freedom of canonical transformations into account, we can express the canonicity conditions in the following form:

\[ \langle \Phi(t) | i\hbar \partial_y | \Phi(t) \rangle = \delta_y (X, Y), \]  \hspace{1cm} (2.24a)

\[ \langle \Phi(t) | i\hbar \partial_x | \Phi(t) \rangle = \delta_x (X, Y), \]  \hspace{1cm} (2.24b)

where $s(X, Y)$, which represents the freedom of the canonical transformation, is an arbitrary function of $X$ and $Y$. We can take possible solutions of the above canonicity conditions as

\[ (X, Y) = (q, p) \quad \text{and} \quad (\hbar G, \Pi), \]  \hspace{1cm} (2.25)

that is, when explicitly written,

\[ \langle \Phi(t) | i\hbar \partial_q | \Phi(t) \rangle = \frac{1}{2} p, \quad \langle \Phi(t) | i\hbar \partial_p | \Phi(t) \rangle = -\frac{1}{2} q, \]  \hspace{1cm} (2.26)

\[ \langle \Phi(t) | i\hbar \partial_\Pi | \Phi(t) \rangle = 0, \quad \langle \Phi(t) | i\hbar \partial_G | \Phi(t) \rangle = -\hbar G. \]  \hspace{1cm} (2.27)

Therefore, the resultant equations of motion derived from the variational principle in Eq. (2.23) are nothing but the canonical equations of motion

\[ \dot{q} = -\frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \]  \hspace{1cm} (2.28)

\[ \hbar \dot{G} = \frac{\partial H}{\partial \Pi}, \quad \hbar \dot{\Pi} = -\frac{\partial H}{\partial G}, \]  \hspace{1cm} (2.29)

where the dot denotes the time derivative and the classical Hamiltonian function $H$ is defined by

\[ H = \langle \Phi(t) | \hat{H} | \Phi(t) \rangle. \]  \hspace{1cm} (2.30)

Therefore, our main task is reduced to solving the classical equations of motion under appropriate initial conditions in the canonical formalism.

If the Hamiltonian is of a simple Schrödinger type, which consists of separate kinetic and potential terms like

\[ \hat{H} = \frac{1}{2} \hat{p}^2 + V(\hat{q}), \]  \hspace{1cm} (2.31)

then the Hamiltonian function is written as

\[ H = \langle \Phi(t) | \hat{H} | \Phi(t) \rangle = \frac{1}{2} p^2 + V(q) + \hbar \left( \frac{1}{8G} + 2G \Pi^2 \right) \left[ \exp \left\{ \frac{\hbar}{2} G \left( \frac{\partial}{\partial q} \right)^2 \right\} - 1 \right] V(q), \]  \hspace{1cm} (2.32)
and the equations of motion are given by

\[ \dot{p} = \dot{q} , \]
\[ \dot{q} + \exp \left\{ \frac{\hbar}{2} G \left( \frac{\partial}{\partial q} \right)^2 \right\} V^{(1)}(q) = 0 , \] \hspace{1cm} (2·33)
\[ \dot{\Pi} = \frac{\dot{G}}{4G} , \]
\[ \dot{\Pi} + 2\Pi^2 - \frac{1}{8G^2} + \frac{1}{2} \exp \left\{ \frac{1}{2} \hbar G \left( \frac{\partial}{\partial q} \right)^2 \right\} V^{(2)}(q) = 0 , \] \hspace{1cm} (2·34)

where \( V^{(n)}(q) \) denotes the \( n \)-th order derivatives with respect to the variable \( q \). In the limit of \( \hbar \to 0 \), Eq. (2·33) is reduced to the usual classical equation of motion. Therefore, we can regard \( G \) and \( \Pi \) as a classical image of quantal fluctuations.

From Eq. (2·18) or the canonicity conditions in Eqs. (2·26) and (2·27), the action function in Eq. (2·23) is expressed as

\[ S = \int_{t_0}^{t} dt' \left\{ i \hbar \frac{\partial}{\partial t'} - \dot{\Pi} \right\} \Phi(t') \]
\[ = \int_{t_0}^{t} dt' \left\{ \frac{1}{2} (\dot{p} \dot{q} - \dot{p} \dot{q}) - \hbar \dot{\Pi} G - H \right\} \]
\[ = \int_{t_0}^{t} dt' (\dot{p} \dot{q} + \hbar \dot{\Pi} \dot{G} - H) + \text{(surface terms)} . \] \hspace{1cm} (2·35)

In order to solve Hamilton's equations of motion, appropriate initial conditions should be selected. Especially, we need to have initial conditions for the newly-introduced classical motion of quantal fluctuations.

Since we are interested in classical motions in quantal systems, it may be natural to require that, at least, the initial state \( |\Phi(t_0)\rangle \) satisfies the minimal uncertainty. Furthermore, since we are interested in the least quanta effects, we assume that the quantal fluctuations are minimal at the very beginning. Thus, we set the following two criteria.

1) **Minimal Uncertainty**

From Eq. (2·14),

\[ (\Delta q)(\Delta p)|_{t=t_0} = \frac{\hbar}{2} \sqrt{1 + (4G_0 \Pi_0)^2} \]
\[ = \frac{\hbar}{2} \]

which means, because \( G_0 > 0 \),

\[ \Pi_0 \equiv \Pi(t=t_0) = 0 , \] \hspace{1cm} (2·36)

where \( t_0 \) denotes the initial time.
Fig. 1. The validity for initial conditions for $G(t)$ and $\Pi(t)$ is shown in the case of Morse potential, $V(Q) = W_0(e^{-2\mu q} - 2e^{2\mu q})$. Parameters are taken to be $W_0 = 1.0$ and $\mu = 0.1$ and the initial values for $p(t)$ and $q(t)$ are taken to be $0$. The abbreviations $L. Q. E.$ and $M. U.$ represent that the initial conditions of Least Quantal Effects and Minimal Uncertainty are satisfied, respectively. Exact represents the energy eigenvalue of the exact ground state.

ii) Least Quantal Effects

In the systems under consideration, we require that the absolute contribution to the energy due to quantal fluctuations is as small as possible at the initial time. That is, we demand

$$\left| H(q_0, p_0, G_0, \Pi_0) - H_\alpha(q_0, p_0) \right| : \text{Minimal},$$

(2.37)

where $H_\alpha$ denotes the classical part of the expectation value of $\hat{H}$, which only includes the terms of order of $\hbar^0$. Thus, $G_0$ is connected to the initial conditions $q_0$ and $p_0$ for the variables $q(t)$ and $p(t)$ of the classical part.

$$G_0 = G(t=\tau_0) = G(q_0, p_0).$$

(2.38)

The initial condition $G_0$ is regarded as a function of initial values of $q(t)$ and $p(t)$ in a classical phase space.

In Fig. 1, we show the energy expectation values for the Morse potential, which are obtained by several choices of initial conditions. The Hamiltonian has the form of Eq. (2.31) and the potential form is $V(\hat{Q}) = W_0(e^{-2\mu \hat{q}} - 2e^{2\mu \hat{q}})$. Here, we set the parameters $W_0 = 1$ and $\mu = 0.1$ for simplicity. In this figure, $L. Q. E.$ indicates that the initial condition for $G(t)$ is taken according to our criterion in Eq. (2.37) and $M. U.$ indicates that the initial condition for $\Pi(t)$ is taken according to our criterion in Eq. (2.36), respectively. Since we are interested in the least quantal effect, we set the initial conditions for the classical parts $q(t)$ and $p(t)$ to be $q_0 = 0$ and $p_0 = 0$, respectively. Good reproduction of the exact result shows that it may be plausible to take these criteria for the initial conditions with respect to the classical image of quantal fluctuations.
§ 3. Maslov phase within squeezed coherent state approach

In this section, we give a connection between the usual WKB approximation and our framework of the time-dependent variational approach with the squeezed coherent states.

The time-dependent variational principle in Eq. (2·23) leaves an ambiguity of a time-dependent phase $\lambda(t)$. When the trial state $|\Phi(t)\rangle$ is transformed to $|\tilde{\Phi}(t)\rangle = e^{i\lambda(t)/\hbar}|\Phi(t)\rangle$, the derived variational equations of motion remain invariant. In order to compare the approximate state with the exact one, we fix the phase $\lambda(t)$ with the aid of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\tilde{\Phi}(t)\rangle = \hat{H}|\tilde{\Phi}(t)\rangle,$$

$$|\tilde{\Phi}(t)\rangle = e^{i\lambda(t)/\hbar}|\Phi(t)\rangle. \quad (3·1)$$

Substituting Eq. (3·2) into Eq. (3·1), taking the inner product with $|\Phi(t)\rangle$ and solving the resultant differential equation, we can determine the $\lambda(t)$ as

$$\lambda(t) = \int_0^t dt' \langle \Phi(t')|i\hbar\frac{\partial}{\partial t'} - \hat{H}|\Phi(t')\rangle. \quad (3·3)$$

This phase factor is nothing but the action function in Eq. (2·35). Thus, we can exponentiate the action function on the state in the framework of TDVP.

From the result in Eq. (2·19) for general Hamiltonians, we obtain the expectation value of the energy as

$$H = \langle \Phi(t)|\hat{H}|\Phi(t)\rangle = \exp \left\{ \frac{\hbar}{2} G \left( \frac{\partial^2}{\partial q^2} \right) + \hbar 2 G I \left( \frac{\partial^2}{\partial q \partial p} \right) + \frac{\hbar}{2} \left( \frac{1}{4G} + 4GI^2 \right) \left( \frac{\partial}{\partial p} \right)^2 \right\} H_w(q, p), \quad (3·4)$$

where $H_w(q, p)$ is the Wigner transform of Hamiltonian operator $\hat{H}$

$$H_w(q, p) = \int_{-\infty}^{\infty} ds e^{iqs/\hbar} \left\langle q - \frac{s}{2} | \hat{H} | q + \frac{s}{2} \right\rangle. \quad (3·5)$$

For the later convenience, we define the derivative operator $\mathcal{D}$

$$\mathcal{D} = \frac{1}{2} G \left( \frac{\partial}{\partial q} \right)^2 + 2G I \left( \frac{\partial^2}{\partial q \partial p} \right) + \frac{1}{2} \left( \frac{1}{4G} + 4GI^2 \right) \left( \frac{\partial}{\partial p} \right)^2.$$

Therefore, the classical Hamiltonian in Eq. (3·4) is written as $H = e^{\mathcal{D}} H_w(q, p)$ and the derivative operator $\mathcal{D}$ depends on $G$ and $I$ only. Thus, the equations of motion in Eqs. (2·28) and (2·29) are

$$\dot{q} = \frac{\partial H}{\partial p} = e^{\mathcal{D}} \frac{\partial H_w}{\partial p},$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -e^{\mathcal{D}} \frac{\partial H_w}{\partial q}, \quad (3·6)$$
\[ \dot{G} = \frac{1}{\hbar} \frac{\partial H}{\partial \Pi} \]
\[ = e^{i\phi} \left\{ 2G\left( \frac{\partial^2}{\partial q \partial p} \right) + 4G\Pi\left( \frac{\partial}{\partial p} \right)^2 \right\} H_w, \]
\[ \dot{\Pi} = -\frac{1}{\hbar} \frac{\partial H}{\partial G} \]
\[ = -e^{i\phi} \left\{ \frac{1}{2} \left( \frac{\partial}{\partial q} \right)^2 + 2\Pi\left( \frac{\partial^2}{\partial q \partial p} \right) + \frac{1}{2} \left( -\frac{1}{4G^2} + 4\Pi^2 \right) \left( \frac{\partial}{\partial p} \right)^2 \right\} H_w. \] (3.7)

In order to solve these equations of motion, we have to fix the initial conditions as mentioned in the previous section. As for the initial conditions for the classical parts \( q(t) \) and \( p(t) \), we may select them in a similar way to usual TDHF theory. However, for the newly-introduced variables, the initial values of the classical image of quantal fluctuations \( G(t) \) and \( \Pi(t) \) should be selected according to the discussion in Eqs. (2·36) \( \sim \) (2·38):

\[ q(t=t_0) = q_0, \quad p(t=t_0) = p_0, \] (3.8)
\[ G(t=t_0) = G_0 = G(q_0, p_0), \quad \Pi(t=t_0) = \Pi_0 = 0, \] (3.9)

that is,

\[ q(t) = q(q_0, p_0, t), \quad p(t) = p(q_0, p_0, t), \] (3.10)
\[ G(t) = G(q_0, p_0, t), \quad \Pi(t) = \Pi(q_0, p_0, t). \] (3.11)

From now on, we consider the limit, \( \hbar \to 0 \), in Eqs. (3·6) and (3·7). This treatment corresponds to the approximation up to the order of \( \hbar \), since the variables \( G \) and \( \Pi \) always appear accompanied with \( \hbar \). Up to the order of \( \hbar \), the variational equations of motion in Eqs. (3·6) and (3·7) are reduced to

\[ \dot{q} = \frac{\partial H_w}{\partial \dot{p}}, \]
\[ \dot{p} = -\frac{\partial H_w}{\partial q}, \] (3·12)
\[ \dot{G} = 2G \frac{\partial^2 H_w}{\partial q \partial \dot{p}} + 4G\Pi \frac{\partial^2 H_w}{\partial \dot{p}^2}, \]
\[ \dot{\Pi} = -\frac{1}{2} \frac{\partial^2 H_w}{\partial q^2} - 2\Pi \frac{\partial^2 H_w}{\partial q \partial \dot{p}} - \frac{1}{2} \left( -\frac{1}{4G^2} + 4\Pi^2 \right) \frac{\partial^2 H_w}{\partial \dot{p}^2}. \] (3·13)

Furthermore, we can evaluate how \( \dot{q} \) and \( \dot{p} \) depend on the initial conditions \( q_0 \) and \( p_0 \) as follows:

\[ \frac{\partial \dot{q}}{\partial q_0} = \frac{\partial^2 H_w}{\partial q \partial \dot{p}} \frac{\partial q}{\partial q_0} + \frac{\partial^2 H_w}{\partial \dot{p}^2} \frac{\partial p}{\partial q_0}, \] (3·14a)
\[ \frac{\partial \dot{q}}{\partial p_0} = \frac{\partial^2 H_w}{\partial q \partial \dot{p}} \frac{\partial q}{\partial p_0} + \frac{\partial^2 H_w}{\partial \dot{p}^2} \frac{\partial p}{\partial p_0}, \] (3·14b)
For the later convenience, the following matrices are introduced:

\[
M = \begin{pmatrix}
\frac{\partial q}{\partial q_0} & \frac{\partial q}{\partial p_0} \\
\frac{\partial p}{\partial q_0} & \frac{\partial p}{\partial p_0}
\end{pmatrix} = \begin{pmatrix}
A_{qq_0} & B_{qp_0} \\
C_{pq_0} & D_{pp_0}
\end{pmatrix},
\]

(3.15)

\[
H^{(2)} = \begin{pmatrix}
\frac{\partial^2 H_w}{\partial q^2} & \frac{\partial^2 H_w}{\partial q \partial q} \\
\frac{\partial^2 H_w}{\partial q \partial p} & \frac{\partial^2 H_w}{\partial p^2}
\end{pmatrix} = \begin{pmatrix}
H^w_{qq} & H^w_{qp} \\
H^w_{qp} & H^w_{pp}
\end{pmatrix},
\]

(3.16)

\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = -J^{-1}.
\]

(3.17)

Here, the matrix \( M \) is often called monodromy matrix. The matrix \( J \) appears due to the symplectic structure of a phase space since the variables are taken as canonical ones.

We simply rewrite Eqs. (3.14a)~(3.14d) by the use of Eqs. (3.15)~(3.17)

\[
\dot{M} = JH^{(2)}M.
\]

(3.18)

According to the discussions of Littlejohn, we can proceed with the semi-classical treatment of our framework of TDVP with the squeezed coherent states. Note that the time-evolution from \((q_0, p_0)\) to \((q(t), p(t))\) is represented as the canonical transformation, the generating function of which is the action itself. Therefore, the Poisson bracket for \((q(t), p(t))\) is unity:

\[
\{q(t), p(t)\} = \frac{\partial q}{\partial q_0} \frac{\partial p}{\partial p_0} - \frac{\partial q}{\partial p_0} \frac{\partial p}{\partial q_0} = 1,
\]

(3.19)

that is,

\[
A_{qq_0}D_{pp_0} - B_{qp_0}C_{pq_0} = 1.
\]

(3.20)

We can derive this relation directly differentiating \( \det M \) with respect to time \( t \) by the use of Eq. (3.14) and initial conditions. Thus, the inverse matrix of \( M \) is explicitly written by

\[
M^{-1} = \begin{pmatrix}
D_{pp_0} & -B_{qp_0} \\
-C_{pq_0} & A_{qq_0}
\end{pmatrix}
\]
Therefore, the second derivatives of $H_w$ are expressed in terms of $(q, \dot{q})$ and their time-derivatives $(\dot{q}, \ddot{q})$. (For the typographical convenience, the subscripts in $A, B, C$ and $D$ are often omitted in the following notations.)

$$H^{(2)} = J^{-1} \dot{MM}^{-1}$$

\[ \begin{pmatrix}
\frac{\partial^2 H_w}{\partial q^2} & \frac{\partial^2 H_w}{\partial q \partial \dot{q}} \\
\frac{\partial^2 H_w}{\partial q \partial \dot{p}} & \frac{\partial^2 H_w}{\partial \dot{p}^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial \dot{H}}{\partial q} & \frac{\partial \dot{H}}{\partial \dot{q}} \\
\frac{\partial \dot{H}}{\partial \dot{p}} & \frac{\partial \dot{H}}{\partial p}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{ p(t), \dot{p}(t) \} \\
\{ q(t), \dot{q}(t) \}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{ \dot{p}(t), q(t) \} \\
\{ \dot{q}(t), \dot{p}(t) \}
\end{pmatrix}
\]

The equations of motion with respect to the quantal fluctuation part $G(t)$ and $\Pi(t)$ in Eq. (3·13) are rewritten as

\[
\begin{align}
\dot{G} &= 2G(\dot{A}\dot{D} - \dot{B}\dot{C}) + 4G\Pi(AB - \dot{A}\dot{B}), \\
\dot{\Pi} &= -\frac{1}{2}(CD - CD) - 2\Pi(\dot{A}\dot{D} - \dot{B}\dot{C}) - \frac{1}{2} \left( -\frac{1}{4G^2} + 4\Pi^2 \right)(AB - \dot{A}\dot{B}).
\end{align}
\]

We can seek the solutions of the above equations of motion under the initial conditions in Eqs. (2·36) and (2·37) or (2·38). Here we only show the resultant solutions. We can directly check the validity by substituting $G$ and $\Pi$ in Eqs. (3·24a) and (3·24b) with the following solutions:

\[
\begin{align}
G &= \frac{1}{2} \left[ 2G_0 A_{\alpha\alpha}^2 + \frac{B^2_{\alpha\beta}}{2G_0} \right], \\
\Pi &= \frac{1}{4G} \left[ 2G_0 A_{\beta\alpha} \beta_{\alpha\beta} + \frac{B_{\alpha\beta} D_{\beta\alpha}}{2G_0} \right].
\end{align}
\]

Here, $A, B, C$ and $D$ are defined in Eq. (3·15). Since $A(t=t_0) = D(t=t_0) = 1$ and $B(t=t_0) = C(t=t_0) = 0$, the initial conditions are satisfied. Thus, if we know the classical orbit $(q(t), \dot{q}(t))$ depending on the initial values $(q_0, \dot{q}_0)$, then we can calculate the classical image of quantal fluctuations $G(t)$ and $\Pi(t)$.

Now, since we take the semi-classical order into consideration, that is, up to the order of $\hbar$, it is necessary to calculate the expectation value of the Hamiltonian up to the order of $\hbar$. From Eq. (3·4), we obtain the approximate energy expectation value
\[ H = H_c(q, p) + \hbar H_{st}(q, p, G, \Pi) \]  

(3·26)

with

\[ H_c = H_w(q, p), \]  

(3·27a)

\[ H_{st} = \frac{1}{2} G \frac{\partial^2 H_w}{\partial q^2} + 2GI\frac{\partial^2 H_w}{\partial q \partial p} + \frac{1}{4} \left( \frac{1}{4G} + 4GI^2 \right) \frac{\partial^2 H_w}{\partial p^2}. \]  

(3·27b)

By using the solutions in Eqs. (3·25a) and (3·25b) and the second derivatives in Eq. (3·22) for \( H_{st} \) in Eq. (3·27b), we can derive the following result:

\[ H_{st} = \frac{1}{2} \left[ 2G_0 \dot{A}C + \frac{\dot{B}D}{2G_0} \right] - \frac{1}{4} \frac{\partial}{\partial t} \left[ 2G_0 AC + \frac{BD}{2G_0} \right]. \]  

(3·28)

In the requantization procedure, the action function plays an important role. We evaluate the semi-classical contribution with the order of \( \hbar \) in \( \langle \Phi(t)|i\hat{h}\partial\partial t|\Phi(t)\rangle \), that is, \( \hbar II\dot{G} \) in Eq. (2·35).

\[ II\dot{G} = \frac{1}{4G} \left( 2G_0 AC + \frac{BD}{2G_0} \right) \left( 2G_0 \dot{A}C + \frac{\dot{B}D}{2G_0} \right) \]

\[ = \frac{\dot{A}B - A\dot{B}}{4G} + \frac{1}{2} \left( 2G_0 \dot{A}C + \frac{\dot{B}D}{2G_0} \right). \]  

(3·29)

Here, we have used the relation in Eq. (3·20). Therefore, the exponentiated action function which is here divided into two parts is

\[ \int dt \langle \Phi(t)|i\hat{h}\frac{\partial}{\partial t}|\Phi(t)\rangle = \int dt \left[ \dot{p} \dot{q} + \hbar II\dot{G} \right] \quad (+ \text{surface terms}) \]

\[ \simeq \int dt \left\{ \dot{p} \dot{q} + \hbar \left[ \frac{\dot{A}B - A\dot{B}}{4G} + \frac{1}{2} \left( 2G_0 \dot{A}C + \frac{\dot{B}D}{2G_0} \right) \right] \right\}, \]  

(3·30a)

\[ \int dt \langle \Phi(t)|\hat{H}|\Phi(t)\rangle = \int dt [H_c + \hbar H_{st}] \]

\[ \simeq \int dt \dot{H}_w \]

\[ + \hbar \int dt \left\{ \frac{1}{2} \left( 2G_0 \dot{A}C + \frac{\dot{B}D}{2G_0} \right) - \frac{1}{4} \frac{\partial}{\partial t} \left( 2G_0 AC + \frac{BD}{2G_0} \right) \right\}. \]  

(3·30b)

Since we are now interested in the bound state problems, we assume that the classical orbit \( (q(t), p(t)) \) is a periodic one and the period is \( T_c \). Obviously from Eqs. (3·25a) and (3·25b), \( G(t) \) and \( II\dot{G} \) are also periodic. However, its period is not always the same as that of the classical orbit.

In the usual WKB considerations, the energy is kept in the classical form which does not include \( \hbar \). Therefore, in our framework of the TDVP with the squeezed coherent states, \( \hbar \int dt H_{st} \) in Eq. (3·30b) should be combined with the requantized phase factor \( \langle \Phi(t)|i\hat{h}\partial\partial t|\Phi(t)\rangle \), in order to compare our treatment with the usual WKB one.
Maslov Phase as Geometric Phase in the Time-Dependent Variational Approach

properly. That is, the action function, which appears when the phase of the squeezed coherent state is fixed in Eqs. (3·2) and (3·3), is rewritten as

\[ S = \int_0^{t_a} dt \langle \Phi(t) | i\hbar \frac{\partial}{\partial t} - \hat{H} | \Phi(t) \rangle = \int_0^{t_a} dt \left\{ [p\dot{q} + \hbar \hat{A}\hat{B} - \frac{\hat{A}\hat{B}}{4G}] - H_{\text{ex}} \right\} (+ \text{total time-derivative term}) \quad (3\cdot31) \]

According to the requantization procedure in the TDHF theory,

\[ \int_0^{t_a} dt \left\{ p\dot{q} + \hbar \frac{\hat{A}\hat{B} - \frac{\hat{A}\hat{B}}{4G}}{2(2G_0A^2 + \frac{B^2}{2G_0})} \right\} = 2\pi \hbar n \quad n: \text{integer} \quad (3\cdot32) \]

We rewrite the above relation as

\[ \oint_C p dq = 2\pi \hbar \left( n - \frac{\Gamma}{2\pi} \right) \quad (3\cdot33) \]

where

\[ \Gamma = \int_0^{t_a} dt \frac{\hat{A}\hat{B} - \frac{\hat{A}\hat{B}}{4G}}{2(2G_0A^2 + \frac{B^2}{2G_0})} \quad (3\cdot34) \]

Here, \( \mathcal{C} \) denotes the integral contour along the classical orbit. We can understand that the phase \( \Gamma \) has the geometric aspect as follows. In the previous section, we have indicated that the denominator \( G \) in Eq. (3·34) never has the value zero. Here, we rewrite \( G \) as

\[ G = \frac{1}{2} \left( 2G_0A^2 + \frac{B^2}{2G_0} \right) = \frac{1}{2} |z|^2 \quad (3\cdot35) \]

where the complex variable \( z \) is defined as

\[ z = \sqrt{2G_0} A + i \frac{B}{\sqrt{2G_0}} \quad (3\cdot36) \]

From the previous consideration, \( z \) will never pass through the point of origin \( z = 0 \) during the periodic time-evolution. Therefore, Eq. (3·34) is recast to

\[ \Gamma = \frac{1}{2} \int_0^{t_a} dt \text{Im} \left\{ - \frac{\partial}{\partial t} \ln \left( \sqrt{2G_0} A + i \frac{B}{\sqrt{2G_0}} \right) \right\} = -\pi \nu \quad \nu: \text{integer} \quad (3\cdot37) \]
Here, $G$ or $z$ undergoes the time-evolution accompanied by the classical motion $q(t)$ through the variables $A$ and $B$. The integer $\nu$ is nothing but the winding number around the origin $z=0$ associated with the classical motion.

This situation is analogous to the case that we encountered for Berry’s phase or canonical phase. We have derived the Berry phase with the use of the time-dependent variational principle in a previous paper. In that paper, we have derived Berry’s phase which includes dynamical corrections in closed systems. In the adiabatic limit, the usual Berry phase is reproduced and the geometric nature becomes apparent.

In this squeezed coherent state approach, it is understood that the classical motion $q(t)$ plays the role of an external parameter. Furthermore, although the phase $\Gamma$ originally includes the contributions of the higher order in $\hbar$, this phase $\Gamma$ apparently displays the geometric aspect in the limit of semi-classical approximation in our framework. Therefore, one can say that $\hbar$ is playing a role of an “adiabatic parameter” in our squeezed coherent state approach to include the quantal fluctuations. The argument is also discussed in the context of the coherent state path integral method. If we take into account the higher order terms of $\hbar$, the geometric aspect may be made unclear.

From Eqs. (3.33) and (3.37), we obtain

$$\oint C p dq = 2\pi \hbar \left( n + \frac{\nu}{2} \right). \quad n, \nu: \text{integer} \quad (3.38)$$

Here, $\nu$ is nothing but a so-called “Maslov phase” and is equal to twice the Maslov index.

As a simple illustrative example, we consider the harmonic oscillator Hamiltonian:

$$\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \omega^2 \hat{Q}^2 \quad (3.39)$$

Then, the expectation value and the Wigner transform are given by

$$H = \hat{H}(q, p) + \hbar \left\{ \frac{G}{2} \omega^2 + \frac{1}{2} \left( \frac{1}{4G} + 4GH^2 \right) \right\},$$

$$H_w(q, p) \equiv H_c = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 \quad (3.40)$$

The classical equations of motion are written as

$$\dot{q} = \frac{\partial H_w}{\partial p} = p \quad (3.41a)$$

$$\dot{p} = - \frac{\partial H_w}{\partial q} = - \omega^2 q \quad (3.41b)$$

The solutions of the above equations of motion are easily obtained as

$$q = q_0 \cos \omega t + \frac{p_0}{\omega} \sin \omega t \quad (3.42a)$$
\[ p = -q_0 \omega \sin \omega t + p_0 \cos \omega t \, . \]  

(3.42b)

Therefore, one can easily evaluate the elements of the monodromy matrix in Eq. (3.15):

\[ A_{qq} = \frac{\partial q}{\partial q_0} = \cos \omega t \, , \]

(3.43a)

\[ B_{qp} = \frac{\partial q}{\partial p_0} = \frac{1}{\omega} \sin \omega t \, , \]

(3.43b)

\[ C_{pq} = \frac{\partial p}{\partial q_0} = -\omega \sin \omega t \, , \]

(3.43c)

\[ D_{pp} = \frac{\partial p}{\partial p_0} = \cos \omega t \, . \]

(3.43d)

The initial condition for \( G(t) \) is determined with the aid of the “Least Quantal Effects”:

\[ \left| G_0 - \frac{1}{2} \omega^2 + \frac{1}{8} G_0 \right| : \text{Minimal} \, , \]

(3.44)

that is

\[ G_0 = \frac{1}{2\omega} \, . \]

(3.45)

From Eqs. (3.25a) and (3.25b), we can evaluate the classical image of the quantal fluctuations:

\[ G = \frac{1}{2\omega} \, , \quad \Pi = 0 \, . \]

(3.46)

As is mentioned previously, from Eq. (3.34), we can calculate the “geometric phase” \( \Gamma \)

\[ \Gamma = \int_0^{t_0} dt \frac{\dot{A} B - A \dot{B}}{4G} \]

\[ = \frac{\omega}{2} \int_0^{(2\pi/\omega)} dt (-\sin^2 \omega t - \cos^2 \omega t) \, . \]

(3.47)

Thus, winding number \( \nu \) is determined:

\[ \nu = 1 \, . \]

(3.48)

Therefore, the standard WKB quantization rule is reproduced by taking the semi-classical limit in the present framework of the TDVP with the squeezed coherent states.

\[ \oint pdq = 2\pi \hbar \left( n + \frac{1}{2} \right) \, . \]

(3.49)
The energy of the system should be evaluated in terms of the classical Hamiltonian $H_0(=H_w)$.

It should be noted that the conception of the usual WKB method is different from that of our squeezed coherent state approach. In the squeezed coherent state approach, we have kept the energy of the system in the form of the expectation value of the Hamiltonian operator, $E = \langle \Phi | \hat{H} | \Phi \rangle$. On the other hand, in the usual WKB method, the energy is taken to be the classical form $E = H_0$. Therefore, a proper comparison can be made by letting the quantal energy absorbed in the quantized phase as is described above.

In the next section, with the aim of going beyond the usual WKB approximation, the conception of the TDVP with the squeezed coherent states is developed.

§ 4. Beyond the WKB approximation

---The conception of the squeezed coherent state approach---

In the previous section, the relationship between the usual WKB approximation and the semi-classical limit in our framework of the TDVP with the squeezed coherent states has been shown. However, as for the calculation of the energy, there is some difference in the basic conceptions between our framework and the usual WKB method. The basic conception in our framework is to regard the expectation value of Hamiltonian as the energy itself including higher order corrections of $\hbar$. Therefore, it is not necessary to make a certain part of Hamiltonian absorbed into the requantized phase. Thus, in our framework, the energy expectation value is nothing but Eq. (3.4) itself.

We will again display the above idea in the simplest example of the harmonic oscillator potential. In this case, the equations of motion in Eqs. (3.6) and (3.7) are decoupled from each other. Therefore, the solutions of the classical image of quantal fluctuations $G$ and $\Pi$ in Eq. (3.46) are exact ones, and Eqs. (3.30a) and (3.30b) are also exact results. Thus, we can easily calculate both of the phase factor in Eq. (3.30a) and the energy in Eq. (3.30b).

In the phase factor in Eq. (3.30a), the second term is nothing but the geometric phase as was mentioned in Eq. (3.47) previously. The third term originates from the dynamics of the classical image corresponding to the quantal fluctuations. From Eqs. (3.43a)–(3.43d), we can obtain the phase factor due to this dynamics as follows:

$$\Gamma_d = \int_0^{\tau_\text{en}} dt \left[ \frac{1}{2} \left( 2G_0 \dot{A} + \frac{\dot{B}}{G_0} \right) \right]$$

$$= \pi . \quad (4.1)$$

This phase factor exactly cancels the geometric phase factor $\Gamma$ is Eq. (3.47). Therefore, Eq. (3.30a) is reduced to

$$\int_0^{\tau_\text{en}} dt \langle \Phi(t) | i\hbar \frac{\partial}{\partial t} | \Phi(t) \rangle = \int_\mathcal{C} pdq \quad (4.2)$$

and Eq. (3.38) to...
Fig. 2. The phase space contour of $G(t)-\Pi(t)$ is shown by solving a set of coupled equations numerically in the case of Eckart potential, $V(Q) = -U_0 / \cosh^2 \alpha Q$. Here, we set the parameters $U_0 = 1$ and $\alpha = 0.1$. The initial values of $q(t)$ and $p(t)$ are taken to be 0.

Fig. 3. The energies are shown in the case of Eckart potential. "This Case" represents the time-developed energy calculated numerically in our squeezed coherent state approach. The initial values of classical parts are taken as $q_0 = p_0 = 0$. "WKB" and "Exact" represent the energies obtained by the usual WKB approximation and the exact eigenvalue of the ground state, respectively.

$$\int_c p dq = 2\pi \hbar n.$$ \hspace{1cm} (4.3)

The initial conditions for $q(t)$ and $p(t)$ are governed by this relation in Eq. (4.3). Since we are interested in the least quantal effect, that is, the minimal quantum fluctuations, then the initial values of $q(t)$ and $p(t)$ are taken as $q_0 = p_0 = 0$ corresponding to the ground state.

The fluctuation part of the energy expectation value in Eq. (3.28) is easily calculated. The result is

$$H_{\text{fl}} = \frac{\omega}{2}.$$ \hspace{1cm} (4.4)
Therefore, the total energy is Eq. (3.26) is given by

\[ H = \frac{1}{2} p_o^2 + \frac{1}{2} \omega q_o^2 + \frac{\hbar \omega}{2}. \]  

(4.5)

With the aid of the requantization procedure in Eq. (4.3), the energy in Eq. (4.5) is expressed as \( H = \hbar \omega (n + 1/2) \). This relation is an exact one.

In general, the procedure to obtain the energy approximately in our framework is as follows. First, we analytically or numerically solve the self-consistent equations in Eqs. (3.6)~(3.7) or (2.33)~(2.34) under the initial conditions in Eqs. (2.36) and (2.37). Secondly, we calculate the energy expectation value which includes the higher order effect of \( \hbar \) than the WKB approximation. Since we are interested in the least quantal effect, it is plausible to set the initial values of \( q \) and \( p \) to be \( q_o = p_o = 0 \) similar to the case of the harmonic oscillator Hamiltonian. For example, in the case of Eckart potential, \( V(Q) = -U_o/\cosh^2 aQ \), in which we set the parameters \( U_o = 1 \) and \( a = 0.1 \), Fig. 2 shows the \( G(t) - \Pi(t) \) motion under the initial conditions \( q_o = p_o = 0 \). Therefore, the energy thus obtained corresponds to the ground state one. In the \( G(t) - \Pi(t) \) phase space, a closed orbit is depicted. Thus, the effect of the quantal fluctuations is automatically generated through \( G(t) - \Pi(t) \) motion under the initial condition in Eqs. (2.36) and (2.37). In Fig. 3, we compare the energy expectation value, calculated numerically in our framework, with the exact energy eigenvalue and the usual WKB energy. It can be seen that our treatment gives a fairly good result owing to the incorporation of the higher order terms in \( \hbar \).

§ 5. Summary

In this paper, we have shown that, when the time-dependent variational principle with the squeezed coherent states is applied to a simple boson system, the Maslov phase originating from the lowest-order quantum fluctuations can be interpreted as a geometric or canonical phase analogous to Berry’s phase in the adiabatic approximation. This idea that the Maslov phase occurring in the semi-classical quantization rule is an example of Berry’s phase has originally been pointed out by Littlejohn and related works have been developed by Littlejohn and his collaborators. In the present time-dependent variational approach with the squeezed coherent states, the higher order contributions of \( \hbar \) than the WKB approximation are taken into account, so that one can only reproduce the Maslov phase by taking a semi-classical limit in this framework. However, the conceptions of the usual WKB method and the squeezed coherent state approach are different. In the standard WKB method, the energy of a system under consideration is kept in the classical form which does not include terms of order \( \hbar \). On the other hand, in our squeezed coherent state approach, the expectation value of energy, \( \langle \Phi | \hat{H} | \Phi \rangle \), already includes the terms depending on \( \hbar \). Therefore, it is necessary to recombine these terms with the quantized phase, in order to carry out the proper comparison between the usual WKB approximation and the semi-classical limit in the squeezed coherent state approach.

The quantized phase in bound state problems consists of two different parts in the semi-classical limit of the squeezed coherent state approach. One is the Bohr-
Sommerfeld phase, $\int c \, dq$, and the other corresponds to the Maslov phase. It is interesting to see that the latter one has the aspect of a geometric or canonical phase, realized as a winding number around the origin of the complex phase space, and the classical orbit is playing a role of external parameters. In this approach, we can avoid the complexities related to the turning points in the usual semi-classical theories, such as the consideration of the crossing of the Stokes lines in the connection formula of wave functions, and also the consideration of caustics in the path-integral method up to the second order of quantum fluctuations. The main task to obtain the Maslov phase is reduced to the counting of the winding numbers in our approach. Although a simplest case of the one-dimensional boson system is only treated at this stage, it is expected that a generalization to a multi-dimensional system can easily be carried out.

It has also been demonstrated in this paper that, under the conception of our squeezed coherent state approach, the energy expectation value of a ground state is well reproduced owing to the inclusion of the higher order effects in $\hbar$. This fact certainly implies that the time-dependent variational approach with the squeezed coherent states is a possible way to go beyond the WKB approximation.

Regarding the geometric nature of quantization rule, various versions of semi-classical approximations in the Bargmann and the generalized coherent-state representations have been developed. It is known that the semi-classical quantization rules (Bohr-Sommerfeld, Maslov or Einstein-Brillouin-Keller rules) are attributed to the analytic property of the Bargmann or complex phase space. Also, the relevance of squeezed and “generalized” squeezed states in the semi-classical limit for general time-dependent Hamiltonians is discussed in connection with applications to the quantum chaology with the aim of using it in that field. Rapid progress in this field has attracted renewed interests in semi-classical approximations, and the quantum-classical correspondence is also one of the exciting fields of physics.

The extensions of the time-dependent variational approach with the squeezed coherent states to more general systems with many degrees of freedom are of great interest. Much progress may be expected in an attempt to describe the quantum fluctuations of quantal systems in terms of classical mechanics.

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