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Discussion

Discussion: "A Paradox in Sliding Contact Problems With Friction" (Adams, G. G., Barber, J. R., Ciavarella, M., and Rice, J. R., 2005, ASME J. Appl. Mech., 72, pp. 450–452)

D. A. Hills[1]
e-mail: david.hills@eng.ox.ac.uk

A. Sackfield

C. M. Churchman

Department of Engineering Science,
University of Oxford,
Parks Road,
Oxford OX1 3PJ, UK

In this interesting paper, the authors address an anomaly which arises when a rigid, square-ended block is pressed against a linear elastic half plane and slid along. The authors note that, within the framework of linear elasticity, the singularity in the contact pressure, and hence shearing traction, produces, adjacent to the edges, regimes in which the implied local relative slip direction dominates the rigid-body sliding velocity, and hence produces a violation of the Coulomb friction law. They seek to resolve the paradox by appealing to a more sophisticated strain definition. All of this is within the context of a quasistatic formulation. The authors recognize, of course, that in any real problem the paradox is unlikely to arise because of (a) the finite strength of the contact giving rise to a yield zone, and (b) the absence of an atomically sharp corner at the contact edge where there is, in all probability, a finite edge radius. Here, we wish to address these issues quantitatively, and so demonstrate that it is unlikely that the paradox described, though interesting, will have any bearing in a real contact. [DOI: 10.1115/1.2201886]

1 Basic Formulation

The punch is moving at velocity U_0 in the positive x direction, relative to the half plane (with the coordinate system (x,y) moving with it), and the surface normal displacement, $v(x)$, and tangential displacement, $u(x)$, are given by

$$\frac{1}{A}\frac{du}{dx} = -\frac{1}{\pi}\int_{\text{contact}} \frac{q(\xi)d\xi}{x-\xi} - \beta p(x) \tag{1}$$

$$\frac{1}{A}\frac{dv}{dx} = \frac{1}{\pi}\int_{\text{contact}} \frac{p(\xi)d\xi}{x-\xi} - \beta q(x) \tag{2}$$

where

$$A = \frac{1-\nu}{\mu}, \quad \beta = \frac{1-2\nu}{2(1-\nu)} \tag{3}$$

ν being the Poisson's ratio and μ the modulus of rigidity of the half plane. The slip velocity of particles on the punch surface relative to particles on the half-plane surface, $U(x)$, is given in the original paper by

$$U \equiv \frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = -U_0\frac{du}{dx} \tag{4}$$

As the contact is sliding the tractions are related everywhere by $q(x)=fp(x)$, where f is the coefficient of friction, so that Eq. (1) becomes

$$\frac{du}{dx} = -\frac{fA}{\pi}\int_{\text{contact}} \frac{p(\xi)d\xi}{x-\xi} - A\beta p(x) \tag{5}$$

and from Eq. (2) we have

$$\frac{1}{\pi}\int_{\text{contact}} \frac{p(\xi)d\xi}{x-\xi} = \frac{1}{A}\frac{dv}{dx} + \beta f p(x) \tag{6}$$

Therefore, the surface normal displacements and surface tangential displacements are given by

$$\frac{du}{dx} = -f\frac{dv}{dx} - A\beta(1+f^2)p(x) \tag{7}$$

and hence the relative slip velocity is

$$U = U_0\left[f\frac{dv}{dx} + A\beta(1+f^2)p(x)\right] \tag{8}$$

The slip direction is reversed when $U/U_o > 1$.

2 Asymptotic Representation

Because the region of apparent reverse slip is so small the problem can conveniently be readdressed using an asymptote which gives the problem additional simplicity and applicability. Suppose that a rigid quarter plane is pressed onto the elastic half plane and slid in the positive x direction, with friction. The local contact pressure, $p(s)$, and shearing traction, $q(s)$, may be written in the form [1]

$$\frac{q(s)}{f} = p(s) = K_N s^{\lambda-1} \tag{9}$$

where K_N is a generalized stress intensity factor and s is a coordinate measured from the contact edge. The exponent, λ, is given by the characteristic equation

[1]Author to whom correspondence should be addressed.

$$\tan \pi\lambda = \frac{1}{f\beta}, \quad 0 < \lambda < 1 \tag{10}$$

We now turn our attention to the finite, flat-ended rigid punch, of half-width a, to which a normal load, P, is applied, together with a force sufficient to cause sliding in the positive x direction. This develops a contact pressure, $p(x)$, [2] given by

$$p(x) = \frac{P}{a\pi}\sin \pi\lambda\left(1+\frac{x}{a}\right)^{\lambda-1}\left(1-\frac{x}{a}\right)^{-\lambda} \tag{11}$$

such that $p(x)$ is positive in compression. If we apply the change of coordinate $s/a=1+x/a$ we get

$$p(s) = \frac{P}{a\pi}\sin \pi\lambda(s/a)^{\lambda-1}[2-(s/a)]^{-\lambda} \tag{12}$$

$$\simeq \frac{p_o}{\pi}\sin \pi\lambda\left(\frac{s}{2a}\right)^{\lambda-1}[1+\lambda(s/a)/2+\cdots] \tag{13}$$

where we have introduced an average pressure $p_o=P/2a$, and hence

$$K_N = \frac{(2a)^{1-\lambda}p_o}{\pi}\sin \pi\lambda \tag{14}$$

We now apply the relationship for the slip velocity $U(x)$ to this solution where, of course, $dv/dx=0$, and the region where the implied slip direction is reversed is when

$$A\beta(1+f^2)K_N s^{\lambda-1} > 1 \tag{15}$$

i.e., over a region $s<e$ where

$$e = [A\beta K_N(1+f^2)]^{1/(1-\lambda)} \tag{16}$$

Therefore, for this specific geometry, when we employ the calibration for K_N, the reversal in slip direction occurs over a region $s<e$ where

$$\frac{e}{a} = 2\left[\frac{1-2\nu}{2\pi}\frac{p_o}{\mu}(1+f^2)\sin \pi\lambda\right]^{1/(1-\lambda)} \tag{17}$$

Thus, the phenomenon presented in the original paper occurs whenever the half plane has a finite compressiblity ($\nu<1/2$), even if no shearing tractions arise ($f=0$), but clearly the region of violation increases in size with (a) friction, (b) reduced Poisson effect, (c) dimensionless contact pressure (p_o/μ). The most extreme values one might expect to encounter in practice might be a Poisson's ratio of 0.2, a coefficient of friction of 0.8, and a mean contact pressure of $p_o/\mu=0.001$, giving $\lambda=0.41$, so that

$$\frac{e}{a} = 7.1\times 10^{-7} \tag{18}$$

which is itself tiny, and readily swamped by local plasticity or the effects of rounding.

3 Local Plasticity

We turn now to the question of envelopment of the region of reverse slip by plasticity. As the complete stress field associated with the asymptote is known, through the Muskhelishvili potential, it is straightforward to establish an estimate of the size of the edge plastic zone, simply by seeing where the yield condition is exceeded, in the spirit of the usual fracture mechanics crack-tip plasticity correction.

The Muskhelishvili potential for a rigid punch on a half plane is given by

$$\Phi(z) = \frac{(1-if)}{2}K_N z^{\lambda-1} \tag{19}$$

where $z=s+iy$ is a complex coordinate in the half plane (and $i=\sqrt{-1}$). The second invariant of the stress tensor as described by von Mises' equivalent stress is (along the interface, $y=0$)

$$\sigma_e = |K_N|s^{\lambda-1}\sqrt{3f^2+(1-\nu)^2} \tag{20}$$

and yield is expected to occur when $\sigma_e=\sigma_Y=\sqrt{3}\tau_Y$. This condition is satisfied over a region $s<r_p$, where r_p is

$$r_p = \left(\frac{|K_N|}{\sigma_Y}\right)^{1/(1-\lambda)}[3f^2+(1-\nu)^2]^{1/2(1-\lambda)} \tag{21}$$

and again using the calibration for K_N from the finite geometry

$$\frac{r_p}{a} = 2\left(\frac{p_o}{\sigma_Y}\frac{\sin \pi\lambda}{\pi}\right)^{1/(1-\lambda)}[3f^2+(1-\nu)^2]^{1/2(1-\lambda)} \tag{22}$$

which for $\nu=0.2, f=0.8$, and $\sigma_Y/\mu=3\times 10^{-3}$ (hence $p_o/\sigma_Y=\frac{1}{3}$) gives

$$\frac{r_p}{a} = 0.0934 \tag{23}$$

Thus, for this contact pressure, there will be plasticity over a region roughly 10^5 times larger than the zone over which the paradox occurs.

4 Effect of Rounding

To probe the effect of edge rounding we consider sliding contact between another rigid semi-infinite punch pressed onto the half plane and sliding. This time, contact is assumed to extend from $-d<x<\infty$, and the surface displacement gradient is defined by

$$\frac{dv}{dx} = -\frac{1}{R}x, \quad -d\le x\le 0 \tag{24}$$

$$=0, \quad x>0 \tag{25}$$

i.e., the indenter has the form of a parabolic arc of equivalent radius R to the left of the origin, and is flat to the right of the origin. This profile is substituted into integral Eqs. (1) and (2) and solved, giving a pressure along the interface of

$$p(x) = \frac{d\sin^2\lambda\pi}{AR\pi}(s/d)^{\lambda}\left[\frac{1}{1-\lambda}+\frac{1-(s/d)}{\lambda}{}_2F_1(1,\lambda;1+\lambda;(s/d))\right] \quad (s/d)<1 \tag{26}$$

$$=\frac{d\sin^2\lambda\pi}{AR\pi(1-\lambda)(2-\lambda)}(s/d)^{\lambda-1}{}_2F_1\left(1,1-\lambda;3-\lambda;\frac{1}{(s/d)}\right) \quad (s/d)>1 \tag{27}$$

where $s=x+d$ and ${}_2F_1(.)$ is a standard hypergeometric function. We note that, when $(s/d)\gg 1$, $p(x)\sim s^{\lambda-1}$, so that the asymptotic form given by Eq. (27) applies, and calibration shows that the generalized stress intensity factor is given by

$$K_N = \frac{\sin^2\lambda\pi d^{2-\lambda}}{AR\pi(1-\lambda)(2-\lambda)} \tag{28}$$

This scaling factor serves both to provide a connection between the applied load (K_N) and the extent of contact in the radiused portion, d, and it enables us to write the contact pressure as

$$\frac{d^{1-\lambda}p(s)}{K_N} = (2-\lambda)(s/d)^{\lambda}\left[1+\frac{(1-\lambda)}{\lambda}(1-(s/d)){}_2F_1(1,\lambda;1+\lambda;(s/d))\right] \quad (s/d)<1 \tag{29}$$

$$=(s/d)^{\lambda-1}{}_2F_1\left(1,1-\lambda;3-\lambda;\frac{1}{(s/d)}\right) \quad (s/d)>1 \quad (30)$$

whereas the relative slip displacement is given by

$$\frac{U}{U_0}=\frac{d}{R}\Psi\left(\frac{s}{d},f,\beta\right) \quad (31)$$

$$\Psi\left(\frac{s}{d},f,\beta\right)=f\left[1-\left(\frac{s}{d}\right)\right]+\frac{\beta(1+f^2)}{\pi(1+f^2\beta^2)}\left(\frac{s}{d}\right)^{\lambda}\left[\frac{1}{(1-\lambda)}+\frac{1}{\lambda}\left[1-\left(\frac{s}{d}\right)\right]{}_2F_1\left(1,\lambda;1+\lambda;\left(\frac{s}{d}\right)\right)\right] \quad (32)$$

and, employing the contact law for this problem,

$$d=\left(\frac{K_N AR\pi(1-\lambda)(2-\lambda)}{\sin^2\lambda\pi}\right)^{1/(2-\lambda)} \quad (33)$$

gives

$$\frac{U}{U_0}=(K_N AR^{\lambda-1})^{1/2-\lambda}\left(\frac{\pi(1-\lambda)(2-\lambda)}{\sin^2\lambda\pi}\right)^{1/(2-\lambda)}\Psi\left(\frac{s}{d},f,\beta\right) \quad (34)$$

$$=\left(\frac{2a}{R}\right)^{(1-\lambda)/(2-\lambda)}\left(\frac{p_o}{\mu}(1-\nu)\right)^{1/(2-\lambda)}\left(\frac{(1-\lambda)(2-\lambda)}{\sin\lambda\pi}\right)^{1/(2-\lambda)}\times\Psi\left(\frac{s}{d},f,\beta\right) \quad (35)$$

Therefore there is a violation in slip direction when $U/U_0>1$, i.e., when

$$\left(\frac{R}{2a}\right)^{\lambda-1}\left(\frac{p_o}{\mu}(1-\nu)\right)>\left(\frac{\sin\lambda\pi}{(1-\lambda)(2-\lambda)}\right)\left[\Psi\left(\frac{s}{d},f,\beta\right)\right]^{\lambda-2} \quad (36)$$

The violation is more likely to occur for cases when $R/2a$ is very small and the load p_o/μ is relatively high. However, unlike the perfectly sharp solution, the function $\Psi(s/d,f,\beta)$ does not exhibit a monotonically increasing value as $s\to 0$ but shows a maximum. This cannot be found analytically, but has been found numerically for the extreme example case used earlier ($f=0.8$, $\nu=0.2$) so that the maximum of $\Psi(s/d,f,\beta)$ occurs at $s/d=0.133$ and is of magnitude 1.002. Therefore, using the same load as before, $p_o/\mu=0.001$, we find that the paradox will occur if

$$\frac{R}{2a}<5.9\times 10^{-6} \quad (37)$$

If we have a contact of total width 20 mm then in order to *avoid* the paradox occurring we require $R>120$ nm. This is such a tiny radius that for all contacts of practical importance the paradox will not exist.

5 Concluding Remarks

The region of reverse slip violating the friction law has been quantified and shown to be extremely small. Physical boundaries for its envelopment by plasticity or absence through local rounding have also been explicitly found. It is very probable that plasticity will produce the greater effect.

References

[1] Comninou, M., 1976, "Stress Singularity at a Sharp Edge in Contact Problems With Friction," ZAMP, **27**, pp. 493–499.

[2] Hills, D. A., Nowell, D., and Sackfield, A., 1993, *Mechanics of Elastic Contacts*, Butterworth-Heinmann, Oxford, UK.