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Approximate insightful ODE solutions

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An approximate solution, as I heard often from Professor Middlebrook,[1] is *better* than an exact solution. An ideal approximate solution would also share important features of the exact solution yet not become messy.

An Euler solution to an ordinary differential equation (ODE) can succeed on the first count. Compared to an exact solution, a crude Euler solution can fit on the back of an envelope while giving physical insight. As an illustration, here's the ODE for the fall speed, v, of a rain drop

$$\frac{dv}{dt} = g - kv^2, \tag{1}$$

where k contains the air density and the drop's drag coefficient, mass, and size.

The drop's terminal speed is $v_{\text{term}} = \sqrt{g/k}$, which gives the system a time constant $\tau = v_{\text{term}}/g$. With the extreme time step $\Delta t = \tau$, the crude-Euler solution is just two straight lines (Fig. 1). From them, we see how the true $v(t)$ starts, tangent to the first slanted line, and see that it bends smoothly underneath the enclosure. However, how much does it bend? For example, what fraction of v_{term} has the drop reached after one time constant?

To find out, we could use Euler with a smaller step size, say $\Delta t = \tau/2$. We get more algebra and arithmetic (predicting a fraction of 7/8) but without much more insight.

This brute-force approach also ignores a mathematical intuition: that local information about a function makes a wobbly pedestal from which to launch a prediction.[2] My favorite, even if the extreme, example is the function $f(x) = e^{-1/x^2}$. All its derivatives are zero at $x = 0$, making the quintessential local analysis, the Taylor series, zero everywhere. Yet the function still manages to get off the ground. A less extreme example is $f(x) = \tan x$. Based on its first three derivatives at zero ($f'(0) = 1, f''(0) = 0$, and $f'''(0) = 2$), who could guess that it has diverged by $x = \pi/2$?

The brute-force approach also ignores what we know physically: that the drop's acceleration, dv/dt, cannot drop abruptly at $t = \tau$ (when it drops from g to 0). Forces are produced by physical systems, which cannot reconfigure themselves infinitely fast. So, neither forces nor accelerations should change abruptly.

Let us fix these mathematical and physical infelicities. To avoid clutter, let us work with the nondimensionalized version of the ODE, Eq. (1). It is

$$\underbrace{\bar{v}'}_{d\bar{v}/d\bar{t}} = 1 - \bar{v}^2, \tag{2}$$

where the nondimensionalized speed and time are $\bar{v} \equiv v/v_{\text{term}}$ and $\bar{t} \equiv t/\tau$, respectively. With these variables, the problematic lumped $\bar{v}'$ is a simple step function (Fig. 2).

Knowing that no physical quantity changes so abruptly, we can improve our lumping approximation by making $\bar{v}'$ continuous and, to avoid overcomplicating it, also piecewise straight (Fig. 3).

This revision makes $\bar{v}(1)$, the area under the $\bar{v}'$ triangle, equal to 1/2. Then, we get a revised $\bar{v}$ picture incorporating what we know so far: $\bar{v}(0) = 0$, $\bar{v}(1) = 1/2$, $\bar{v}'(0) = 1$, and $\bar{v}'(1) = 0$ (Fig. 4).

However, this figure contradicts itself. The nondimensionalized ODE, Eq. (2), requires that $\bar{v}'(1) = 1 - \bar{v}^2(1)$, which is now 3/4. However, the figure shows $\bar{v}'$ reaching zero. Fortunately, this conflict can be resolved through iteration.

(1) Make $\bar{v}'$ descend linearly from $\bar{v}'(0) = 1$ to the newly calculated $\bar{v}'(1) = 3/4$ (instead of 0).
(2) Integrate the revised $\bar{v}'$ shape (a trapezoid) to revise $\bar{v}(1)$.

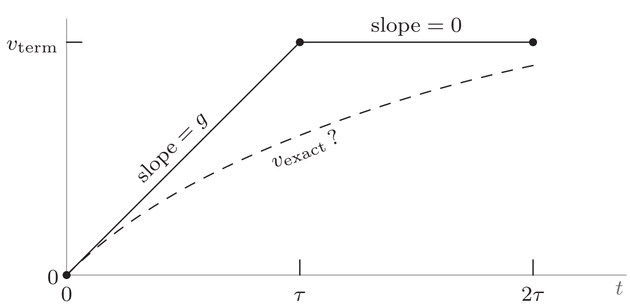

Fig. 1. Fall speed using the extreme lumping approximation where the Euler step is the time constant (τ). The true curve must share the slope at $t = 0$ and lie below the lumped curve. But where?.

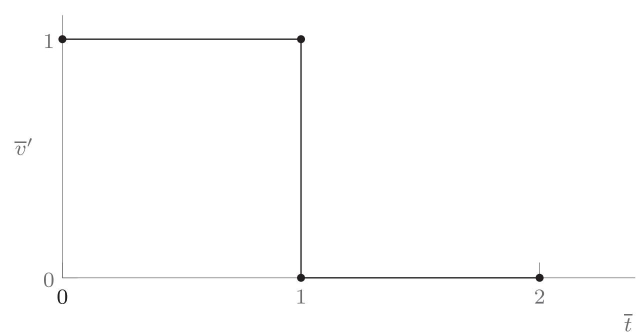

Fig. 2. Acceleration (nondimensionalized) using the extreme lumping approximation that produced Fig. 1.

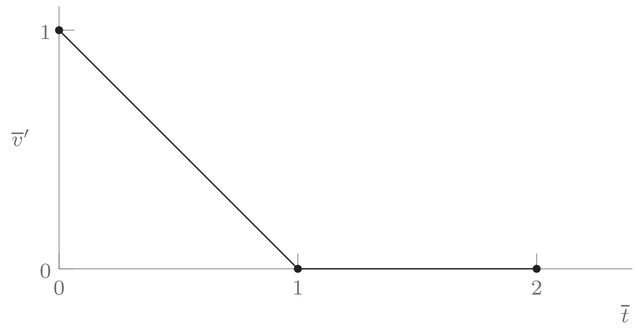

Fig. 3. Smoother acceleration (nondimensionalized). It is still lumped, but not so discontinuously.

(3) Put $\bar{v}(1)$ into the ODE to revise $\bar{v}'(1)$, and go to step 1 with the new $\bar{v}'(1)$ replacing the 3/4 there.

This feedback loop quickly converges to a consistent $\bar{v}$ and $\bar{v}'$ over the range $\bar{t} = 0 \ldots 1$. (These $\bar{v}$ and $\bar{v}'$ curves will not be fully consistent; only the exact ODE solution could be. However, they are more consistent than were the piecewise-straight v of Fig. 1 and the $\bar{v}'$ of Fig. 2.)

We can short-circuit the loop by finding the iteration's fixed point.[3] The integration (step 2) takes the averaged slope and multiplies it by the Euler step, which is $\Delta\bar{t} = 1$, to get the new $\bar{v}(1)$

$$\bar{v}(1) = \underbrace{0}_{\bar{v}(0)} + \underbrace{1}_{\Delta\bar{t}} \cdot \underbrace{\left[\frac{\overbrace{1}^{\bar{v}'(0)} + \overbrace{\left(1-\bar{v}(1)^2\right)}^{\bar{v}'(1)}}{2}\right]}_{\text{averaged slope of } \bar{v}}. \tag{3}$$

This self-consistency constraint simplifies to

$$\bar{v}(1)^2 + 2\bar{v}(1) - 2 = 0, \tag{4}$$

whose solution is $\bar{v}(1) = \sqrt{3} - 1 \approx 0.73$. So, after one time constant, the rain drop—or, such is the magic of universal functions,[4] any falling object subject to quadratic drag—has

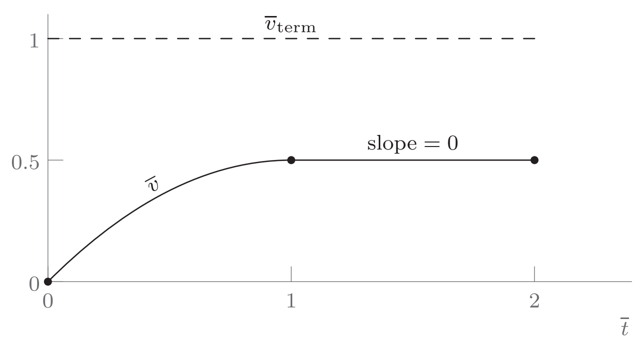

Fig. 4. Fall speed (nondimensionalized) from integrating the smoother acceleration of Fig. 3.

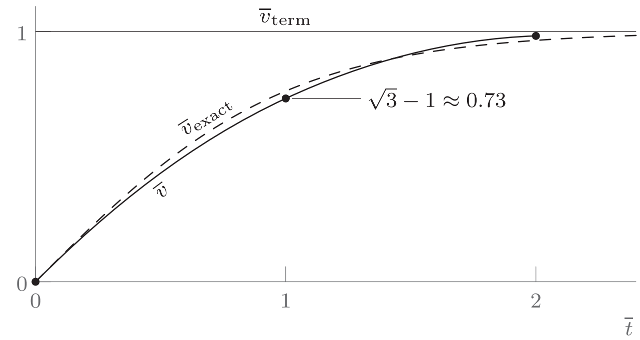

Fig. 5. Self-consistent fall speed (nondimensionalized). This $\bar{v}$ curve has been extended to include the self-consistent lumping solution for $\bar{v}(2)$. The curve is close to the exact solution (dashed), which is $\bar{v} = \tanh\bar{t}$ and, at $\bar{t} = 1$, is approximately 0.76.

reached approximately 73% of its terminal speed. The exact ODE solution, $\bar{v}_{\text{exact}} = \tanh\bar{t}$, gives the fraction as approximately 76%. In between, our approximated $\bar{v}$ curve is a parabola piece that also closely matches the exact solution (Fig. 5).

You can test drive this self-consistent lumping method on the exponential-decay ODE, $dy/dx = -y$. Even with the huge Euler step of $\Delta x = 1$, the result is reasonable: $y = 3^{-x}$ for integer values of x, with parabola pieces fitting smoothly in between. For this ODE, the method, even with such a huge step, is equivalent $e \approx 3$, which might even be too accurate for the back of an envelope.

Sanjoy Mahajan is interested in the art of approximation and physics education and has taught varying subsets of physics, mathematics, electrical engineering, and mechanical engineering at MIT, the African Institute for Mathematical Sciences, and the University of Cambridge. He is the author of Street-Fighting Mathematics *(MIT Press, 2010),* The Art of Insight in Science and Engineering *(MIT Press, 2014), and* A Student's Guide to Newton's Laws of Motion *(Cambridge University Press, 2020).*

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

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[1]Sanjoy Mahajan, "Fleeing from the quadratic formula," Am. J. Phys. **87**, 332–334 (2019).

[2]"It is very hard to predict, especially the future." This saying has often been credited to Niels Bohr (who perhaps learned it as a Danish proverb): Stanislaw M. Ulam, *Adventures of a Mathematician* (Charles Scribner, New York, 1976), p. 286, cited in Quote Investigator, "It's difficult to make predictions, especially about the future" (2021).

[3]The resulting $\bar{v}$ is similar to the parabolic trial functions discussed in J. R. Acton and P. T. Squire, *Solving Equations with Physical Understanding* (Adam Hilger, Bristol, UK, 1985), Chap. 5. This outstanding book is unfortunately out of print. Paging Dover Publications!

[4]Sanjoy Mahajan, "Universal functions," Am. J. Phys. **90**, 253–255 (2022).