On the Renormalization Condition of the Relativistic Wave Function at the Origin

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The renormalization conditions of the wave function at the origin are examined for the (one-body) Dirac equation and the two-body equation for equal-mass Dirac particles. It is shown that the covariance determines the unique renormalization point (momentum) for the Dirac equation, while the nonrelativistic limit of some physical amplitude is needed to define the condition for the latter equation.

The renormalization of the wave function at the origin (WFO) is indispensable for the theory of the annihilation decay of the bound system such as the quarkonium, if the system is described by using the relativistic wave function. Let us first consider the Dirac equation with the Coulomb potential which governs the system in the infinite limit of the mass $M$ of one of the constituent particles. We take up, for definiteness, a quarkonium consisting of quarks $Q$ (mass $M$) and $q$ (mass $m$). The renormalization constant $Z_1$ is determined through the vertex equation inherent to the Dirac equation (Dirac vertex equation) and the renormalization condition, for which we have assumed that the effects of the radiative correction disappear in the nonrelativistic limit and choose the threshold transition $Q$ (at rest) $\rightarrow q$ (at rest) as the renormalization point. The main purpose of the present paper is to clarify the condition leading to this renormalization point and to show that it is the unique point consistent with the covariance and the Dirac vertex equation.

Let the heavy quark $Q$ decay, in the rest frame, into the light quark $q$ transferring away a momentum $k_\mu$. Then, the 3-momentum squared of the quark $q$ is given by

$$p^2 = -\frac{1}{4M'^2} (M'^2 + m^2 - k^2)^2 - m^2.$$  \hspace{1cm} (1)

If $k_\mu$ is spacelike ($k^2 < 0$), $p^2$ becomes infinite in the limit $M \rightarrow \infty$. When $k_\mu$ is timelike let the invariant mass be $M'$. $p^2$ becomes infinite in this limit if $M'$ remains finite. $M'$ may be infinite in the limit. Now, $p^2 = k^2$ is either infinite or 0 in the covariant limit of $M' \rightarrow \infty$ which implies the limit with the (4-)velocity of the object $M'$ kept unchanged. Since the infinite limit of $p$ along with $M$ is beyond applicability of the Dirac vertex equation, we have to choose $p = 0$ as the unique renormalization point.

We have shown that the renormalization point of the Dirac WFO is fixed by the requirement of covariance. The reason for choosing the renormalization point is

* In QED, we have the Dirac equation in the infinite limit of $M$ if we take all crossed diagrams into account.\(^{3}\) In an effective theory of the quark confinement, Suura derived the equation similar to the Breit equation,\(^{3}\) which is reduced to the Dirac equation when the mass of one of the constituent becomes infinite.\(^{3}\)
somewhat different for the WFO of the equal-mass equation, which we name the
two-body Dirac equation.\textsuperscript{5)} The corresponding vertex equation is the same as for the
instantaneous Bethe-Salpeter equation, for which the renormalization of the WFO
was first developed.\textsuperscript{7)} We assumed there the renormalization point similar to the one
for the Dirac WFO, from which we cannot, however, get the renormalization constant
reproducing the correct low-energy limit of the Compton scattering. We have,
therefore, changed the renormalization condition, in Ref. 5), in order to make sure to
satisfy this low-energy theorem. As the second purpose of the present paper, we state
this condition more precisely and examine non-covariant nature of the employed
model.

To begin with, let us consider the covariant calculation of the Compton scattering
with the photon momenta $k_\mu$ and $k'_\mu$. If we take the correction to each vertex into
account, the amplitude for it is given by

$$A_{\mu}\equiv \bar{u}(p_f)(\epsilon \cdot \Gamma(p_f, p_i+k)S_F(p_i+k)\epsilon \cdot \Gamma(p_i+k, p_i))$$

$$+\epsilon \cdot \Gamma(p_f, p_i-k')S_F(p_i-k')\epsilon \cdot \Gamma(p_i-k', p_i))u(p_i),$$

where $S_F(q)$ denotes the free propagator and $u(p_i)$ and $\bar{u}(p_f)$ are the spinors of
the initial and the final quarks respectively. If we impose the charge-conjugation symme­
try, the vertex function $\Gamma(q, q')$ satisfies

$$C\Gamma(q, q')^T C^{-1}= -\Gamma(q', -q),$$

from which we can deduce the general form of the function. For the special case of
the vanishing momentum transfer, we have

$$\Gamma(q, q) = Z_1(a(q^2)\gamma_\mu + b(q^2)q_\mu + c(q^2)\gamma \cdot qq_\mu),$$

where $Z_1$ is the renormalization constant. We next take the radiation gauge in the
laboratory frame, which implies the restrictions to the polarization vectors $\epsilon_\mu$ and $\epsilon'_\mu$:

$$\epsilon \cdot k = \epsilon' \cdot k' = \epsilon \cdot p_i = \epsilon' \cdot p_i = 0.$$  

We, then, see that only the first term in the braces of Eq. (4) contributes to the
amplitude at the nonrelativistic limit $k_\mu$(incident momentum)$=0$. We thus obtain $Z_1 = a(m^2)^{-1}$.

We have assumed the instantaneous Coulomb kernel for the vertex equation in
the model based on the two-body Dirac equation. In this case, the general form (4)
is reduced to

$$\Gamma(q) = Z_1(a(q^2)\gamma + b(q^2)q + c(q^2)\gamma \cdot qq)$$

for the space components and there is no correction in the time component. The
result of the limit $k_\mu=0$ depends on the reference frame in this noncovariant model
and the nonrelativistic limit is defined in the laboratory frame $p_i=0$. We, therefore,
have $Z_1 = a(0)^{-1}$. Now if we boost the system to some non-zero value of the momentum $p_i$ we will get an amplitude which does not afford the correct Thomson cross
section. The deviation from the correct one gives a measure of the degree of
covariantness of the model. The possible direction of the Lorentz boost is restricted
by the gauge condition (5) and the fact that we cannot get the consistent time component $J_0$ in the framework of the Coulomb model. So, let us restrict ourselves to the special case of the forward scattering in the frame where $p_i$ is also parallel to $k$. We see that the Thomson amplitude is modified by the factor $Z^2 a(p_i^2)^2$ in this frame. This factor is about 0.96 at $|p_i|=0.6 m_e$ in the model of Ref. 5).*)

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4) H. Ito, Prog. Theor. Phys. 78 (1987), 978; 83 (1990), 1064E. Read $Zl(r) \approx Zl(r)^{-1}$, in the Abstract of this paper.
5) H. Ito, Prog. Theor. Phys. 84 (1990), 94.

*) The partial amplitude $\Gamma_i^{\prime}(p)$ of Ref. 5) is equal to $-8\sqrt{\lambda} a(p^2)$. 