A method of obtaining a velocity-depth envelope from wide-angle seismic data

Rakesh Mithal and John B. Diebold Lamont-Doherty Geological Observatory of Columbia University, Palisades, New York 10964-0190, USA

Accepted 1986 November 7. Received 1986 October 30, in original form 1986 March 18

Summary. Due to the non-uniqueness of traveltime inversion of seismic data, it is more appropriate to determine a velocity-depth ($v-z$) envelope, rather than just a $v-z$ function. Several methods of obtaining a $v-z$ envelope by extremal inversion have been proposed, all of which invert the data primarily from either $x-p$, or $t-p$, or both domains. These extremal inversion methods may be divided into two groups: linear extremal and non-linear extremal. There is some debate whether the linearized perturbation techniques should be applied to the inherently non-linear problem of traveltime inversion. We have obtained a $v-z$ envelope by extremal inversion in $t-p$ with the constraint that the inversion paths also satisfy $x-p$ observations. Thus we use data jointly in $t-p$ and $x-p$, and yet avoid the linearity assumptions.

This joint, non-linear extremal inversion method has been applied to obtain a $v-z$ envelope down to a depth of about 30 km in the Baltimore Canyon trough using $x-t$ data from an Expanding Spread Profile acquired during the LASE project. We have found that the area enclosed by the $v-z$ envelope is reduced by about 15 per cent using $x-p$ control on the $t-p$ inversion paths, compared to the inversion without $x-p$ control.

Key words: extremal inversion, traveltime inversion

Introduction

Traveltime inversion of seismic refraction data is a non-unique and non-linear process. The non-uniqueness exists even for a 'perfect' data set due to the fact that it has a finite range of offsets, and is recorded at discrete offsets and arrival times. This aspect of non-uniqueness has been described in detail by Backus & Gilbert (1967, 1968, 1970), and summarized by Aki & Richards (1980). The analysis of 'real' observational data is complicated by additional sources of non-uniqueness: errors in the data, the bandwidth of the observed data, lateral variations in the Earth's structure, rough layer-boundaries, and the method used for velocity analysis. The bandwidth limitation is a function of the source and receiver characteristics, and of attenuation. The resulting pulse widening creates an uncertainty in picking arrival
times. Topographic variations on any interface will result in the smearing of arrivals reflected from or refracted through it. The method used for velocity analysis can also be a source of non-uniqueness of the resulting velocity function; as Healy (1963) has shown, a number of velocity functions can match refraction data if only the first arrivals are used.

Considering these effects, it is not expected that any single solution will fit the data exactly. Therefore, it is appropriate to determine a velocity depth \((v-z)\) envelope of solutions. This envelope should contain the complete range of \(v-z\) models which are consistent with the given data set. One \(v-z\) function obtained by inversion of the most probable picks can be viewed as the optimal solution, and the \(v-z\) envelope can be considered as the confidence limits of this optimal solution.

Inversion of seismic \((x-t)\) data to obtain a velocity-depth \((v-z)\) function is a non-linear process. The non-linearity can be reduced by analysing the data in the domain of offset and ray parameter \((x-p)\) and/or in the domain of intercept time and ray parameter \((T-p)\). The non-linearity is not completely removed because the integral equations relating \(x-p\) or \(T-p\) to \(z-p\) are still non-linear (Kennett 1976).

Inversion of seismic data in the \(x-p\) domain was proposed independently by Herglotz (1907) and Bateman (1910). It was further simplified by improvements in the formulation of Wiechert (1910).

In an effort to reduce the non-linearity of the traveltime inversion problem, Johnson & Gilbert (1972) introduced a new datum \(-\tau\), and proposed an inversion scheme for \(v-z\) from \(\tau-p\) data using the Backus–Gilbert method. Inversion from \(\tau-p\) was also obtained by Bessonova et al. (1974, 1976). For discrete \(\tau-p\) observations, Diebold & Stoffa (1981) proposed the ‘\(r\)-sum’ inversion method, which was shown by Vera & Diebold (1984) to be equivalent to HBW inversion when the integral path in \(x-p\) corresponds to the elliptical \(\tau-p\) paths implicit in \(r\)-sum. Linear programming techniques have been used for \(\tau-p\) inversion by expanding the \(v-z\) function according to various basis functions (Dorman 1979; Garmany, Orcutt & Parker 1979; Orcutt 1980).

The relationship between a \(v-z\) model and its predicted \(\tau-p\) arrivals is linear, but the resulting advantages for inversion are diminished due to a corresponding loss of resolution (Müller & Alsop 1974). Therefore, the use of more than one form of data in the inversion process is desirable (Dorman & Jacobson 1981). McMechan & Wiggins (1972) have used \(x-p\) and \(x-t\) together for non-linear extremal inversion, and Orcutt (1980) used \(x-p\) and \(\tau-p\) estimates together to obtain a linear extremal inversion of observed \(x-t\) data.

Several methods of obtaining a \(v-z\) envelope have been proposed, and these may be divided into two groups: The non-linear extremal inversion methods as proposed by McMechan & Wiggins (1972); Bessonova et al. (1974, 1976); Diebold et al. (1981); and Lee & Johnson (1984); and the linear extremal inversion methods (following the linearized perturbation theory of Backus & Gilbert (1967, 1968, 1970)) as proposed by Johnson (1971), Garmany, Orcutt & Parker (1979), and Orcutt (1980).

The methods in the non-linear group avoid linearity assumptions as far as possible. Wiggins, McMechan & Toksoz (1973) indicate that the inherent non-linearity of traveltime inversion is too strong for the linearized perturbation technique to be applicable. Müller & Alsop (1974) suggest that one should continue to apply methods that take full account of the non-linearity of the inversion problem. Lee & Johnson (1984) have pointed out that while perturbation techniques may give a velocity model which is in some sense optimal there may be other models which fit the data equally as well.

On the other hand, several workers have successfully implemented linearized perturbation techniques for the traveltime inversion of seismic refraction data from the \(x-p\) and/or the \(\tau-p\) domains. Johnson (1971) proposed inversion in terms of linearized perturbation of a
best fitting model. Garmany, Orcutt & Parker (1979) introduced linear programming to simplify the application of this technique in the $\tau-p$ domain. Orcutt (1980) extended the linear programming approach to obtain inversion simultaneously from the $\tau-p$ and $x-p$ domains.

It is clear from the different opinions expressed in these papers that there is a considerable debate as to whether or not the linearized perturbation techniques are appropriate for extremal inversion of seismic refraction data.

In the following we present a non-linear method for obtaining a $v-z$ envelope by using observed data simultaneously in the $\tau-p$ and the $x-p$ domains. We shall use $x-p$ error bounds on the data to control extremal inversion paths in the $\tau-p$ domain, and in this way avoid the linearity assumption. The application of this method will be illustrated using Expanding Spread Profiles (ESP) in the Baltimore Canyon Trough acquired during the Large Aperture Seismic Experiment project (Diebold et al. 1987).

Joint non-linear extremal inversion

We present a new method of obtaining a $v-z$ envelope using error bounds of critical and postcritical arrivals in seismic data simultaneously in the $\tau-p$ and the $x-p$ domains. This method of joint extremal inversion follows the work of McMechan & Wiggins (1972), and Diebold et al. (1981). McMechan & Wiggins (1972) developed a non-linear method of extremal inversion from the $x-p$ domain with additional control by those $x-p$ points at which traveltimes are well known. Diebold et al. (1981) adapted the extremal inversion method for inversion from the $\tau-p$ domain. Both of these methods employ successive applications of exact inversion formulas (the HBW integral, or the $\tau$-sum recursion) to many different paths lying within the confidence bounds of the $x-p$ or $\tau-p$ data. McMechan & Wiggins (1972) pointed out the fact that the minimum and maximum depths for a given velocity would be produced by inversion paths crossing from one data bound to the other at the ray parameter corresponding to the velocity. The resulting velocity–depth curves are often unrealistic, in that they may contain layers with negative thicknesses. McMechan & Wiggins employed an ad hoc averaging approach to eliminate these, while Diebold et al. showed how the paths through $\tau-p$ data could be adjusted to produce realizable results. The last method produces results that are identical, or nearly so, to the basic linear programming method, with positivity constraints, but its inner workings are more accessible.

The inversion approach presented in this paper stems from and improves upon the work of Diebold et al. (1981). As outlined above, those workers used the constraint of non-negative layer thickness to effectively narrow $\tau-p$ confidence bounds. We now show how determination of an $x-p$ data envelope can be used to further narrow the $\tau-p$ bounds, by constraining the slope, $d\tau/dp$, of the inversion paths. Orcutt, MacKenzie & McClain (1980), and Priestly & Orcutt (1982) showed that even the most conservative restrictions upon the allowable field of arrivals in $x-p$ space could narrow inversion bounds in linear extremal inversions of $\tau-p$ data. Taking the non-linear approach, and using a somewhat more sophisticated method for defining the $x-p$ envelope, we present results which come closer, perhaps, to the inherent maximum resolution of seismic traveltime inversion.

As described by Diebold et al. (1981), the application of the extremal inversion method of McMechan & Wiggins (1972) to $\tau-p$ data requires a crossover in the integration path which is schematically shown in Fig. 1. The upper curves represent the minimum $\tau$ bounds, and the lower curves represent the maximum $\tau$ bounds. The maximal depth inversion for a given slowness, $u$, is obtained by starting from the largest ray parameter, on the minimum $\tau$ bound, and at $p = u$, crossing over to the maximum $\tau$ bound. The minimal inversion is
obtained by starting on the maximum block, again from the largest ray parameter, and crossing over to the minimum block (Fig. 1). This has to be repeated for every slowness.

The slope of the crossover in the integration path in $\tau-p$ can result in an unrealizable solution; giving either a negative thickness, which is incorrect, or a positive thickness which is inconsistent with the data in the $x-p$ domain. The inversion path may result in a negative thickness and this can be corrected within the $\tau-p$ inversion scheme as outlined by Diebold et al. (1981); but an inversion path which results in a positive thickness needs to be checked against the $x-p$ information from the data.

Minimal inversion involves choosing an integration path which crosses over from the maximum to the minimum $\tau$ block. It is easy to control the integration path in minimal inversion. In Fig. 2, the paths BC and BC' will give negative layer thicknesses. BD which corresponds to a zero layer thickness can be the first acceptable path. This path BD will inherently correspond to a certain offset, say $x_d$; and any path below BD, e.g. BE will correspond to $x_e > x_d$. The path BD will result in a broader velocity envelope, and any path below BD will result in an increasingly tighter envelope. If $x-p$ information from the data is available, then the minimum range at which the $p$ in consideration is observed, say $x_{\text{min}}$, can be used to decide the appropriate crossover in the integration path.

If in the minimal inversion $x_{\text{min}} < x_d$, or $x_{\text{min}} > x_e'$, an inconsistency between the $\tau-p$ pick and the corresponding $x-p$ pick is determined. In such a situation, the observations should be confirmed again; and if the inconsistency continues, it is assumed that the $\tau-p$ picks are more reliable than the $x-p$ constraints, because they were obtained by direct transformation. Therefore if the $x-p$ pick would result in a $\tau < \tau_d$, or $> \tau_{e'}$, we force it to be $x_d$ or $x_e'$ respectively.

The estimation of offset $x(= -d\tau/dp)$ from the $\tau-p$ picks is complicated by the fact that the continuous $\tau-p$ curve between any two $\tau-p$ picks is non-linear. Typically, it will be either concave downward corresponding to post-critical reflections within a constant velocity layer, or concave upward due to critical refractions within a zone of positive

---

**Figure 1.** Crossover in the integration path for extremal inversion in the $\tau-p$ domain. (a) shows crossover from the minimum to the maximum $\tau$ bound which gives the maximum thickness of the layer with velocity equal to the inverse of the ray parameter at which crossover occurs, and (b) shows crossover from the maximum to the minimum $\tau$ bound which gives the minimum thickness.
velocity gradient. Many different values of $x$ are, therefore, possible between any two $\tau-p$ picks. According to the mean value theorem of calculus there must be at least one value of $p$ between the two $\tau-p$ picks at which the gradient of the continuous $\tau-p$ curve is equal to the point-to-point linear gradient ($=\Delta \tau/\Delta p$). Therefore if we have a fine enough sampling in $p$ (i.e. several ray parameters per layer), it is appropriate to use the single $x$ obtained by the linear gradient for each $p$ interval.

Maximal inversion involves choosing an integration path which crosses over from the minimum to the maximum $\tau$ bound. This process is not as straightforward as the minimal inversion, where the zero thickness path can be easily calculated. The sudden increase in $\tau$ at the crossover path, and then the need to follow the lower bound at later ray parameters sometimes predicts negative thicknesses. In that situation, one has to go back to the crossover $p$, reduce the $\tau$; and then check the later ray parameters again. This trial and error iteration significantly increases the computational time for the maximal inversion.

The integration paths for maximal inversion are shown in Fig. 3. The path BC will give positive thickness, but it may give negative layer thicknesses for smaller ray parameters. Let BD, corresponding to a zero layer thickness for smaller ray parameter, be the first acceptable path. This path BD will inherently correspond to a certain $x$, say $x_d$; and any path above BD, e.g. BE will correspond to $x < x_d$. The path BD will result in a broader envelope and any path above BD, e.g. BE will result in an increasingly tighter envelope. From $x-p$, the maximum range at which this $p$ is observed can be used to obtain the appropriate crossover in the integration path.

If in the maximal inversion $x_{max} < x_{e'}$ or $x_{max} > x_c$, an inconsistency between the $\tau-p$ pick and the $x-p$ pick is determined. Again assuming the $\tau-p$ picks to be more reliable, if the $x-p$ pick results in a $\tau < \tau_{e'}$, or $> \tau_d$, we force it to be $x_{e'}$ or $x_d$ respectively.
Figure 3. Various possibilities of the crossover in the integration paths in $\tau-p$, their $x-p$ significance, and the resulting $u-z$ functions for typical maximal inversion.

For $N$ picks in the $\tau-p$ domain, we obtain $(N-1)$ sets of maximal and minimal $u-z$ solutions. The area occupied by these solutions defines the location of all realizable $u-z$ models. Methods have been developed to reduce the width of this area by one form or another of averaging (McMechan & Wiggins 1972; Davies & Chapman 1975). To demonstrate the effect of averaging, let us consider the maximal inversion paths shown in Fig. 4. There are seven maximal inversion paths, resulting in seven different thicknesses for each layer. Staying on the lower $\tau$ bound would give some thickness for each layer $d_{i,L}, i = 1, 7$. Similarly staying on the higher $\tau$ bound would give some other thickness for each layer $d_{i,H}, i = 1, 7$. These form the no-crossover paths of inversion. In this situation the layer of velocity $u_4(=1/p_4)$ will get the thickness $d_{4,L}$, Three times; $d_{4,H}$, Three times; and $d_{4,max}$, One time. By averaging all the seven thicknesses mentioned above, we are giving a weight of $1/7$ to the crossover path, and $6/7$ to the no-crossover paths. This would bias the result toward the no-crossover path and thereby subdue the effect of extremal inversion. Also, it would result in an envelope which excluded some of the realizable models. Therefore we have chosen not to average the $u-z$ solutions, and have obtained $u-z$ envelope which encompasses all of the possible solutions.

The comparison of various $u-z$ envelopes has been quantified by considering the area within each envelope. For each velocity $u_i, i = 1, n$; the extremal inversion gives us $z_{max,i}$ and $z_{min,i}$. The $u-z$ envelope is made up of $(n-1)$ trapezoids, and the area can be computed by

$$A = dz_1(v_2 - v_1)/2 + \sum_{i=2}^{n-1} dz_i(v_{i+1} - v_{i-1})/2 + dz_n(v_n - v_{n-1})/2,$$

where $dz_i = z_{max,i} - z_{min,i}$. 

Downloaded from https://academic.oup.com/gji/article-abstract/89/3/965/626963 by guest on 13 April 2019
ALL THE MAXIMAL INVERSION PATHS IN $\tau-p$

Figure 4. Typical inversion paths in $\tau-p$ showing the relative weight of the maximal thickness obtained by crossover in $\tau-p$, compared to the normal thickness obtained by remaining on either the lower or the higher $\tau$ bound. This demonstrates how increasing the $p$-range for averaging the extremal solutions will reduce the width of the resulting $v-z$ envelope.

Application to field data

We demonstrate the applicability of this technique of joint, non-linear extremal inversion on an Expanding Spread Profile (ESP) in the Baltimore Canyon Trough, acquired during the Large Aperture Seismic Experiment (LASE) project. The flow diagram of the inversion process is shown in Fig. 5. The LASE study area is shown in Fig. 6. The data were acquired once using airguns, and again using explosives as the seismic source. The airgun ESP data are shown in Fig. 7. The pre-critical reflections can be traced almost continuously, but the refractions as first arrivals fade out beyond about 20 km offset. The explosive ESP data are shown in Fig. 8. The pre-critical reflections cannot be traced at all, but the refractions as first arrivals can be seen out to 90 km offset. Note that the first arrivals at larger offsets (between 30 km and 90 km) interfere with the later arrivals coming shortly thereafter, and with the bubble pulses. Also, there are topographic effects on the arrival times which can be clearly seen in the first arrivals between 5 km and 25 km offsets. Due to these reasons we will later see a large uncertainty in the intercept times, resulting in a wide $v-z$ envelope at
higher velocities. The details of the acquisition and basic processing of LASE data have been described by Diebold et al. (1987).

Our method of joint extremal inversion requires determination of \( \tau_{\text{min}}, \tau_{\text{max}}, x_{\text{min}} \) and \( x_{\text{max}} \) for each ray parameter corresponding to all the principal arrivals in the observed data. The results depend on the accuracy of the required information, not on how this information had been obtained. There are different methods of obtaining this information, e.g. error bounds in \( \tau \) may be obtained either by the method of parallelograms (Bessonova, Fishman & Sitnikova 1970) or by the envelope amplitude method (Diebold et al. 1981). In the following we describe the methods that we have used in this study to obtain the \( \tau \) and \( x \) error bounds from the observed \( x-t \) data.

**DATA TRANSFORMATION: \( x-t \) TO \( \tau-p \)**

Seismic data can be directly mapped to the \( \tau-p \) domain by slant stacking (Stoffa et al. 1981; Phinney, Chowdhury & Frazer 1981). Given complete \( x-t \) data, slant stacking provides complete data in the \( \tau-p \) domain which can be used to obtain accurate \( \tau \) bounds of the principal arrivals in the observed data. The choice of the various ray parameters for which to map the observed data is arbitrary. The range of \( p \)'s normally chosen is from 0 to \( p_{\text{max}} = 1/u_{\text{min}} \), and the ray parameter spacing \( dp \) can be either equal or unequal. The appropriate value of \( dp \) depends on the bandwidth of the data, and on \( dx \), the offset spacing.
between the observed seismograms. \(\tau-p\) transforms obtained by slant stacking can be severely contaminated by aliasing, particularly when strong arrivals, such as the seafloor reflection, interface with weaker arrivals. Stoffa et al. (1981) showed that this aliasing can be overcome by filtering the slant stack traces having low semblance. This process is most effective when the semblance, a measure of coherency, is used to window-out aliased energy within slant stacks of the data taken in successive subarrays.

The LASE ESP4 data were transformed to the \(\tau-p\) domain by slant stacking as described by Stoffa et al. (1981). Fig. 9 shows the \(\tau-p\) transforms of ESP 4A and 4E. For ESP 4A, all the \(x-t\) data from 0 to 42 km were used, without any \(x-t\) windowing. For ESP 4E, 38–80 km of \(x-t\) data were used with an \(x-t\) window such as to retain only those arrivals which were observed earlier than the arrival of the high-energy seafloor reflections. The slant stacking for both the airgun and the explosive ESPs was done with a time-sampling interval of 8 ms. The data were band-pass filtered to include frequencies from 6 Hz to 20 Hz with roll-offs of 3 Hz and 5 Hz at the low and high ends, respectively. The lengths of the subarrays were computed as \(dx = 3 \text{ km} + 0.2 \times x_0\), where \(x_0\) (km) is the mid-point of the sub-
Figure 7. Expanding Spread Profile 4A acquired using airgun sources, during the Large Aperture Seismic Experiment (LASE).
Figure 8. Expanding Spread Profile 4E acquired using explosive sources, during the Large Aperture Seismic Experiment (LASE).
array, and * represents multiplication. Each sub-array was designed so as to overlap 50 per cent of the previous subarray. The slant stacking was done with a constant ray parameter spacing of 0.005 sec km⁻¹. The slant stack alone (without any semblance filter) was used for the airgun ESP data, and only the semblance values were used for the explosive ESP data. The reason for this is that the arrivals from deeper layers, as seen in the explosive ESP, are weak and therefore the slant stacks are also weak, but the slant semblances, which are dependent on the coherency of the arrivals and not on their amplitudes, are better defined.

The principal arrivals from shallow structure can be seen in the transform of the airgun ESP, and from deep structure in the explosive ESP (Fig. 9). The \( \tau - p \) picks of all the principal arrivals, together with their error bounds are also shown in Fig. 9.

The \( \tau - p \) transform of ESP 4A (Fig. 9) shows well-defined principal arrivals. For each \( p \), the intercept time at which the envelope amplitude reaches its maximum defines the \( \tau - p \) pick, and the intercept times on both sides of the \( \tau \) pick at which the envelope amplitudes fall below 14 per cent of the maximum define the error bounds (Diebold et al. 1981). These \( \tau \) bounds define 95 per cent confidence of occurrence of the arrival, assuming the envelope amplitude follows a Gaussian curve. The \( \tau - p \) transform of ESP 4E (Fig. 9) is, however, different. The principal arrivals here are not well defined, but are scattered in wide zones having several complex envelope maxima which do not look like the bubble pulse. In this case, the intercept time at which the highest amplitude peak exists defines the \( \tau - p \) pick, and the error bounds include all the complex envelope peaks which could be a principal arrival. These principal arrivals in the \( \tau - p \) transform of ESP 4E were reflected from the deeper structure and have travelled through large lateral distances. The reasons for this scatter in the intercept times may be either topographic variations in the seafloor, and/or strong lateral variations in the velocity structure.

\( x - t \) TO \( x - p \)

Kennett (1981) noted that in general it is more difficult to get good quality \( x - p \) information from observed \( x - t \) data than it is to get \( \tau - p \) information, and therefore we infer that a direct transformation of \( x - t \) data to the \( x - p \) domain would be still more difficult. McMechan (1983) has obtained \( x - p \) images from synthetic \( x - t \) seismograms by time integration of localized slant stacks. However, his data did not contain any multiples. With real data, at each offset \( x \) it is necessary to identify and eliminate the multiples in the \( t - p \) domain before integrating over time to obtain the \( x - p \) transform. Since we require error bounds, and not a complete transformation, we have elected to use a manual approach, which is applied during, and in parallel with, the slant stacking process, see Fig. 5.

We form a sub-array from \( x' - dx \) to \( x' + dx \), slant stack the data to the sub-array midpoint, \( x' \), by \( t' = t - p(x - x') \) and assign the observed ray parameters \( p - dp \) to \( p + dp \) to the range \( x' \). Here arises an uncertainty: If we increase the size of the subarray, the uncertainty in \( x \) increases; and by decreasing the size of the subarray, uncertainty in \( p \) increases such that \( \delta x \cdot \delta p = \) constant. Therefore, if the data has highly coherent arrivals, we can keep \( dx \) small; but for less coherent arrivals, e.g. from deep layers and arriving at larger offsets, we have to keep \( dx \) large enough to be able to resolve them.

We have formed 9 subarrays in ESP 4A from 1 km to 35 km, and 14 sub-arrays in ESP 4E from 5 km to 90 km. The \( t' - p \) transform of a sub-array is examined to determine the minimum and maximum ray parameters (PMIN, PMAX), and the offsets (XMIN, XMAX) are obviously known. In this study, we determined the ray parameter bounds by the somewhat subjective method of manual picking based on the visual appearance of semblance plots such as Fig. 10(a). Since semblance has been measured, a more objective method to be used in the
Figure 9. (a) $\tau-p$ transform of airgun ESP 4A using 0 to 42 km offsets and no time window, and summing the slant stacks of various overlapping subarrays having apertures increasing with the offset. (b) $\tau-p$ transform of explosive ESP 4E using 38–80 km offsets and a time window such as to retain arrivals only up to the direct wave, and slant stacking in the same way as in (a). (c) $\tau-p$ picks with the error bounds from ESP 4A and ESP 4E together.
X-P BOUNDS FROM THE FIELD DATA

Figure 10(a).
future would entail setting a semblance threshold and picking automatically. Let us define the term ‘x−p transform’ to be this process of obtaining the x−p bounds from the observed data. The t−p plots of two of the sub-arrays of ESP 4E are shown in Fig. 10(a). The t−p plot of the 62−80 km subarray shows a high semblance for p’s between 0.12 sec km⁻¹ and 0.14 sec km⁻¹. The other subarray from 38 km to 50 km shows high values of semblance for p’s between 0.135 sec km⁻¹ and 0.185 sec km⁻¹. The error bounds of the x−p picks (dp = 0.005 sec km⁻¹) of the principal arrivals from all the sub-arrays of both ESPs are shown in Fig. 10(b).

**RESULTS**

The τ−p picks of the principal arrivals and their error bounds, shown in Fig. 9, are used as the starting point. Every inversion path in τ−p has a unique x−p curve associated with it — let us call this ‘x−p computed from τ−p’. These must be differentiated from the x−p picks obtained from the observed data, such as shown in Fig. 10(b). Fig. 11 shows the x−p curves computed from all the extremal inversion paths using the τ−p bounds (shown in Fig. 9), with and without the control from x−p bounds [shown by broken lines in Fig. 11, are the same as in Fig. 10(b)]. It is clear that the x−p bounds in the data are more constrained than
Figure 11. $X-p$ curves computed from the extremal inversion paths using the data $r-p$ bounds (shown in Fig. 9). (a) Without $x-p$ control, and (b) with control from the data $x-p$ bounds (shown in Fig. 10b, and by broken lines in this figure).
Figure 12. Extremal inversion paths using the $\tau$--$p$ bounds (Fig. 9) without and with control from the $x$--$p$ bounds (Fig. 10b). For clarity, this figure shows only some of the extremal inversion paths (for $p < 0.2$ sec km$^{-1}$). The $x$--$p$ control has significantly changed the inversion paths in $\tau$--$p$, resulting in a tighter $v$--$z$ envelope.
Figure 13. Outer envelope of all the velocity-depth functions obtained by $x-p$ control of the extremal inversion paths in $\tau-p$, compared to the one obtained without $x-p$ control.
the $x-p$ curves obtained from the $\tau-p$ extremal inversion paths without the $x$-control [for example, in Fig. 11(a) compare $x$ at $0.15 < p < 0.20$, and $0.30 < p < 0.40$ sec km$^{-1}$]. This means that inclusion of the $x-p$ data should be useful. The $x-p$ curves obtained from $\tau-p$ extremal inversion paths with $x$-control [Fig. 11(b)] should ideally have their bounds exactly the same as the bounds of the $x-p$ picks (shown by broken lines in this figure). The comparison shows that they are similar for most of the ray parameters. There are, however, some differences (e.g. around $p = 0.45$ sec km$^{-1}$) which are due to the inconsistency between the $\tau-p$ picks and the $x-p$ picks at these ray parameters. Fig. 12 shows the effect of $x$-control on $\tau$-sum inversion paths and the resulting $v-z$ solutions. We can see that $x-p$ control has been effective in maneuvering the extremal inversion paths, and in resulting more constrained $v-z$ solutions.

The $v-z$ envelope has been obtained using all the extremal $v-z$ solutions (not the averaged $v-z$ solutions). Fig. 13 shows three $v-z$ envelopes corresponding to no-crossover, extremal inversion — no $x$-control, and extremal inversion — with $x$-control. To compare various $v-z$ envelopes, the area within each of them has been computed using the equation described earlier. It has been found that the area of the $v-z$ envelope without $x-p$ control is 24.6 km$^2$ sec$^{-1}$, but with $x-p$ control this is reduced to 20.8 km$^2$ sec$^{-1}$ corresponding to 15 per cent reduction in the area. This means that by $x-p$ control of the extremal inversion in the $\tau-p$ domain, the resulting $v-z$ envelope has been tightened by 15 per cent, compared to the $v-z$ envelope obtained without the $x-p$ control.

The main advantage of $x-p$ control on extremal $\tau-p$ inversion of LASE ESP4 data has been found to be in the $14 < z < 20$ km depth range (Fig. 13) where the optimal solution obtained by $\tau$-sum inversion shows velocity gradient structure. The additional $x-p$ control has helped reduce the large $v-z$ uncertainties which had originated from the large $\tau$ uncertainties [Fig. 9(b)] inherited from the explosive ESP data. At greater depths, there is an important 7.2 km s$^{-1}$ reflector — there has been no improvement in the uncertainty of its depth.

**Conclusion**

We have been able to obtain the error bounds in the $\tau-p$ and $x-p$ domains from observed seismic $x-t$ data. The data used were Expanding Spread Profiles (ESP), acquired using air-guns and also explosives as sources, during the Large Aperture Seismic Experiment (LASE). Reliable $\tau-p$ and $x-p$ transformations were possible because of the high density of seismograms in the airgun ESP, and the large source size (50 lb.) in the explosive ESP.

We have further developed a method of obtaining a velocity–depth envelope by simultaneously utilizing the error bounds in both domains. This method avoids the linearity assumption, and achieves extremal inversion by providing control, from the $x-p$ domain, on the extremal inversion paths in the $\tau-p$ domain. Application of this method on LASE ESP 4 shows that the area of the $v-z$ envelope obtained by $x-p$ control is 15 per cent smaller than the area of the $v-z$ envelope obtained without the $x-p$ control.

**Acknowledgments**

This work was supported by NSF grants OCE 79-22884 and OCE 83-19518. We thank Dr John Mutter, Professor Paul Richards, and Dr G. M. Purdy for their constructive reviews. Lamont–Doherty Geological Observatory contribution no. 4107.
References


Bateman, H., 1910. The solution of the integral equation which connects the velocity and propagation of an earthquake in the interior of the earth with the times which the disturbance takes to travel to different stations on the earth’s surface, Phil. Mag., 19, 576–587.


