Improving the Effective Potential
— Multi-Mass-Scale Case —

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Previously proposed procedure for improving the effective potential by using renormalization
group equation (RGE) is generalized so as to be applicable to any system containing several different
mass scales. If one knows $L$-loop effective potential and $(L+1)$-loop RGE coefficient functions, this
procedure gives an improved potential which satisfies the RGE and contains all of the leading,
next-to-leading, ..., and $L$-th-to-leading log terms. Our procedure here also clarifies how naturally
the so-called effective field theory can be incorporated in the RGE in MS scheme.

§ 1. Introduction

In a previous paper,[1] we have presented a procedure for improving the effective
potential so as to satisfy the renormalization group equation (RGE). By knowing
$L$-loop effective potential and $(L+1)$-loop RGE coefficient functions, the procedure
gives an improved potential which contains all of the leading, next-to-leading, ..., and
$L$-th-to-leading log terms. However, its applicability was restricted to systems which
possess essentially a single mass scale.

The purpose of this paper is to generalize the procedure so as to be applicable to
any system possessing multi-mass-scales. The main idea is to make use of the
decoupling theorem.[2] By this theorem, it is made sufficient to treat essentially a
single log factor at any scale of field strength, since all the heavy particles (heavier
than that scale) decouple and all the light particles (lighter than that scale) yield
essentially the same log factors. In other words, we treat effective field theory[3–5] in
each interval between mass thresholds, in which heavier particles decouple and lighter
particles may be regarded as massless. So the problem of improving effective
potential reduces to that for a single mass scale system and our previous procedure
becomes applicable. A novel recognition in this context is that the RGE's of those
effective field theories, apparently different interval by interval, are in fact the same
one. This guarantees that we are solving the same RGE for the effective potential,
using different effective field theories depending on the field scale.

To explain our procedure, we consider the following Yukawa model which is
probably the simplest system possessing two mass scales:

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 + \bar{\psi}(i \gamma - g \phi)\psi - hm^4, \quad (1.1)$$

where $\phi$ is a single component massive real scalar field and $\psi=(\phi_1, \ldots, \phi_N)^T$ is an
$N$-component massless Dirac spinor field. [We take the Dirac field to be of $N$-
component simply because the factor $N$ may play the role of a tracer of the fermion
Note that the masslessness of the fermion is protected by the invariance under 'chiral-parity' transformation: $\phi \rightarrow -\phi, \psi \rightarrow \exp(i\gamma_5\pi/2)\phi$. The last term $hm^4$ in the Lagrangian (1.1) is the vacuum energy term which is usually omitted but, as noted in the previous paper, becomes relevant to us in the calculation of the effective potential in the mass-independent renormalization scheme.

§ 2. Structure of the effective potential

The effective potential satisfies the RGE:

$$\mathcal{D} V(\phi, m^2, g^2, \lambda, \mu) = 0,$$

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta_\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} - \gamma \frac{\partial}{\partial \lambda} - \gamma \frac{\partial}{\partial \phi} + \beta h \frac{\partial}{\partial h}. \tag{2.1}$$

The solution is well-known:

$$V(\phi, m^2, g^2, \lambda, \mu) = V(\bar{\phi}(t), \bar{m}^2(t), \bar{g}^2(t), \bar{\lambda}(t), \bar{h}(t); e^{2\mu^2}), \tag{2.2}$$

where $\bar{g}^2, \bar{\lambda}, \bar{m}^2, \bar{\phi}$ and $\bar{h}$ are running parameters whose $t$-dependence is determined by

$$\frac{d\bar{g}^2(t)}{dt} = \beta_\phi(\bar{g}^2(t), \bar{\lambda}(t)), \quad \frac{d\bar{\lambda}(t)}{dt} = \beta_\lambda(\bar{g}^2(t), \bar{\lambda}(t)), \tag{2.3}$$

$$\frac{d\bar{X}(t)}{dt} = -\gamma X(\bar{g}^2(t), \bar{\lambda}(t)) \bar{X}(t) \quad \text{for} \quad X = \phi, m^2, \tag{2.4}$$

with the boundary conditions that they reduce to the unbarred parameters at $t = 0$. Note that the vacuum-energy parameter $h$ can affect only the evolution of itself.

The solution (2.3) gives full information of RGE: As a result of the fact that RGE is a first order differential equation, the effective potential is determined once its function form is known at a certain value of $t$. So, to derive useful information from RGE, we need to know the function of effective potential at a certain value of $t$, a 'boundary' function.

To see the logarithm structure of the effective potential, let us first write the quantum Lagrangian in the following form by rescaling the fields by a factor $g$:

$$\mathcal{L} = \frac{1}{g^4} \left[ \frac{1}{2} (\partial (g\phi))^2 - \frac{1}{2} m^2 (g\phi)^2 - \frac{1}{4!} \left( \frac{\lambda}{g^2} \right) (g\phi)^4 + (g\bar{\psi})(i\partial -(g\Phi))(g\psi) - g h m^4 \right]. \quad (2.5)$$

Then, to compute the effective potential $V(\phi)$, we make the field shift $\Phi \rightarrow \Phi + \phi$ and regard $g\Phi$ and $g\psi$ as our basic quantum fields. In this form the parameters characterizing the theory are only the scalar and fermion masses $M_s$ and $M_f$ (in the presence of scalar background $\phi$),
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\( M^2_b = \frac{1}{2} \lambda \phi^2 + m^2, \quad M_F = g \phi, \) \hspace{1cm} (2.6)

the cubic coupling \((\lambda/g^2)M_F\), the quartic coupling \(\lambda/g^2\), and \(g^2\) (aside from the vacuum-energy term). Moreover the last parameter \(g^2\) is no longer the Yukawa coupling constant but an overall factor in front of the action just like Planck constant \(\hbar\). Then, it is clear that the \(L\)-loop level contribution to the effective potential has the following form:

\[
V^{(L)} = (g^2)^{L-1} M_b^4 \left[ \text{function in } \ln \frac{M_F^2}{\mu^2}, \ln \frac{M_b^2}{\mu^2}, \frac{\lambda}{M_b^2}, \frac{g^2}{g^2} \right]. \hspace{1cm} (2.7)
\]

We have two logarithm factors \(\ln(M_F^2/\mu^2)\) and \(\ln(M_b^2/\mu^2)\) in this two mass scale system. For the purpose of the leading-log series expansion below, we express the latter log as \(\ln(M_F^2/\mu^2) = \ln(M_b^2/\mu^2) + \ln(M_b^2/M_F^2)\), and introduce the following variables:

\[
s = g^2 \ln \frac{M_b^2}{\mu^2}, \quad u = g^2 \ln \frac{M_b^2}{M_F^2}, \quad x = \frac{M_F^2}{M_b^2}, \quad y = \frac{\lambda}{g^2}, \quad z = g^2 \hbar \frac{m^4}{M_b^4}. \hspace{1cm} (2.8)
\]

Since we know that the logarithms appear only up to \(L\)-th power at the \(L\)-loop level, the \(L\)-loop contribution (2.7) takes the form

\[
V^{(L)} = g^{-2} M_b^4 \sum_{l=0}^L \sum_{k=0}^{L-l} (g^2)^{L-(l+k)} v^{(L)}_{l,k}(x, y) s^l u^k, \hspace{1cm} (2.9)
\]

so that the full effective potential has the form:

\[
V = g^{-2} M_b^4 \sum_{l=0}^\infty \sum_{k=0}^{L-l} (g^2)^{L-(l+k)} v^{(L)}_{l,k}(x, y) s^l u^k. \hspace{1cm} (2.10)
\]

Just as in the previous paper,1) this form of expansion (2.10) of the effective potential in powers of \(g^2\) gives a leading-log series expansion: namely, the functions \(f_0, f_1, \cdots\) correspond to the leading, next-to-leading, \(\cdots\), log terms, respectively. So the explicit \(g^2\) factors, which appear when the expression is written in terms of variables \(s, u, x, y, z\) and \(g^2\), show the order in this leading-log series expansion. We refer to the term proportional to \((g^2)^{L-1}\) in \(V\) as \(L\)-th-to-leading log term.

The second equation in (2.10) tells us that the \(l\)-th-to-leading log function \(f_l\) at \(s=0\) in particular is given by

\[
f_l(s=0, u, x, y) = \sum_{l=0}^\infty \sum_{k=0}^{L-l} v^{(L)}_{l,k}(x, y) s^l u^k. \hspace{1cm} (2.11)
\]

Namely, the information of the \((l+k)\)-loop level potential \(V^{(L-l+k)}\) determines the \(u^k\) term of the function \(f_l(s=0, u, x, y)\), i.e., the \((g^2)^{L-1} u^k\) term in \(V|_{s=0}\). Therefore if we restrict ourselves to the region of \(\phi\) in which \(u\) is as small as an \(O(g^2)\) quantity (in the sense of leading-log series expansion), i.e., to the region.
\[
\ln \frac{M^2_\phi}{M^2_e} \leq O(1) \rightarrow g^2 \phi^2 \approx m^2, \tag{2·12}
\]

then the L-loop potential \( V_L = V^{(0)} + V^{(1)} + \cdots + V^{(L)} \) at \( s=0 \) already gives the effective potential 'exact' up to L-th-to-leading log order:

\[
V|_{s=0} = V_L|_{s=0} + O(g^{2L}). \tag{2·13}
\]

That is, in such a region of \( \phi \), we can use the function \( V_L|_{s=0} \) as a 'boundary' function required on the RHS of the solution (2·3) of RGE. Therefore, with the L-loop potential \( V_L \) at hand, the effective potential satisfying the RGE can be given by

\[
V(\phi, m^2, g^2, \lambda, h, \mu^2) = V_L(\bar{s}(t), \bar{m}^2(t), \bar{g}^2(t), \bar{\lambda}(t), \bar{h}(t); e^{2t}\mu^2)|_{\bar{s}(t)=0} \tag{2·14}
\]

with \( \bar{s}(t) \) being the \( s \) variable at 'time' \( t \):

\[
\bar{s}(t) = g^2(t) \ln \frac{M^2_\phi(t)}{e^{2t}\mu^2}, \quad \bar{M}^2_\phi(t) = \bar{g}^2(t) \bar{\phi}^2(t). \tag{2·15}
\]

The barred quantities in the solution (2·14) should of course be evaluated at \( t \) satisfying \( \bar{s}(t) = 0 \).

Our solution (2·14) is 'exact' only up to L-th-to-leading log order and only in the region \( g^2 \phi^2 \geq m^2 \). However, even with the approximate boundary function, RGE is satisfied exactly if the runnings of the barred quantities are solved exactly, of course. To satisfy also the RGE only up to the L-th-to-leading log order, it is sufficient to solve the runnings of the parameters \( \bar{g}^2/g^2, \bar{\lambda}/\lambda, \bar{\phi}/\phi, \bar{m}^2/m^2 \) and \( \bar{h}/h \) up to L-th power in \( g^2 \) in the sense of leading-log series expansion, and for this order of accuracy, the \((L+1)\)-loop RGE coefficient functions \( \beta_\phi, \beta_\lambda, \gamma_\phi, \gamma_m, \) etc. just give enough information. This point was already explained in detail in the previous paper. Thus, as far as the region \( g^2 \phi^2 \geq m^2 \) is concerned, with L-loop effective potential and \((L+1)\)-loop RGE coefficient functions, we can obtain an RGE improved effective potential which is exact up to L-th-to-leading log order.

§ 3. Leading-log effective potential in the region \( g^2 \phi^2 \geq m^2 \)

Before explaining how to obtain the effective potential valid in the complementary region \( g^2 \phi^2 \leq m^2 \), let us give an explicit expression of the effective potential which is obtained by the procedure up to here and exact in the leading log order in the region \( g^2 \phi^2 \geq m^2 \).

The one-loop effective potential \( V_1 = V^{(0)} + V^{(1)} \) is given by the usual formula in the MS scheme as

\[
V_1 = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 + hm^4 + \frac{1}{64\pi^2} \left[ M^4_s \left( \ln \frac{M^2_\phi}{\mu^2} - \frac{3}{2} \right) - 4NM^4_s \left( \ln \frac{M^2_\phi}{\mu^2} - \frac{3}{2} \right) \right]. \tag{3·1}
\]

The coefficient functions in RGE are found at the one-loop order as

\[
\beta_s = -\frac{1}{16\pi^2} (3\lambda^2 + 8N_\lambda g^2 - 48N_\phi g^4) = \beta_{s1} \lambda^2 + \beta_{s2} \lambda g^2 + \beta_{s3} \phi g^4,
\]
If one calculates only $\beta_\sigma$ and $\gamma$ among these, all the others can in fact be found immediately by substituting the one-loop effective potential (3.1) into the RGE (2.1) itself.

For solving the RG running equations of the barred quantities analytically, it is more convenient to switch to use the $\bar{s} = \bar{s}(t)$ in place of the 'time' $t$ and to regard the barred quantities as functions of $\bar{s}$, as was performed in the previous paper. Then the barred quantities needed in the solution (2.14) are obtained simply by setting $\bar{s} = 0$ there.

Noting that

$$\frac{d \bar{s}}{dt} = \bar{\beta}_\sigma = \left( \frac{\beta_\sigma}{\bar{g}^2} \right) \bar{s} - 2 \bar{g}^2 + (\bar{\beta}_\sigma - 2 \bar{g}^2 \bar{\gamma})$$

$$= (\beta_\sigma \bar{s} - 2) \bar{g}^2 + O(\bar{g}^4),$$

the running of the coupling $\bar{g}^2$ with respect to $\bar{s}$ is determined to the leading-log order by

$$\left( \beta_\sigma \bar{s} - 2 \right) \bar{g}^2 \frac{d \bar{g}^2}{d \bar{s}} = \beta_\sigma \bar{g}^4.$$ (3.4)

This is integrated from $\bar{s} = s$ to $\bar{s} = 0$ to yield

$$\bar{g}^2 = g^2 \left( 1 - \frac{\beta_\sigma}{2} s \right)^{-1}.$$ (3.5)

Running of $\bar{\phi}$ is found quite similarly to be

$$\bar{\phi} = \phi \left( 1 - \frac{\beta_\sigma}{2} s \right)^{\gamma/\beta_\sigma}.$$ (3.6)

Running of the coupling $\bar{\lambda}$ is a bit more complicated: the RGE for the ratio $\bar{y} = \bar{\lambda}/\bar{g}^2$ reads in this order

$$\left( \beta_\sigma \bar{s} - 2 \right) \frac{d \bar{y}}{d \bar{s}} = \left[ \beta_{\lambda\lambda} \bar{y}^2 + (\beta_{\lambda\sigma} - \beta_\sigma) \bar{y} + \beta_{\lambda\sigma} \right].$$ (3.7)

Denoting the two roots of quadratic equation $\beta_{\lambda\lambda} \bar{y}^2 + (\beta_{\lambda\sigma} - \beta_\sigma) \bar{y} + \beta_{\lambda\sigma} = 0$ by $a$ and $b$, this is integrated to give

$$\bar{y} = a (y - b) (g^2/\bar{g}^2)^{a(\beta_{\lambda\lambda}/\beta_\sigma)} - b (y - a) (g^2/\bar{g}^2)^{b(\beta_{\lambda\lambda}/\beta_\sigma)} (y - a) (g^2/\bar{g}^2)^{\delta(\beta_{\lambda\lambda}/\beta_\sigma)}.$$

(3.8)
so that we obtain

\[ \bar{\lambda} = g^2 \frac{a(\lambda - b g^2)(g^2 / \bar{g}^2)^{a(b_{11}/\beta_{11})-1} - b(\lambda - a g^2)(g^2 / \bar{g}^2)^{b(b_{11}/\beta_{11})-1}}{(\lambda - b g^2)(g^2 / \bar{g}^2)^{a(b_{11}/\beta_{11})} - (\lambda - a g^2)(g^2 / \bar{g}^2)^{b(b_{11}/\beta_{11})}}. \] 

(3.9)

The running of the mass parameter \( \bar{m}^2 \) is also a bit complicated:

\[ (\beta_{\bar{m}} \bar{s} - 2) \frac{d \bar{m}^2}{d \bar{s}} = -(\gamma_{\bar{m}1} \bar{y} + \gamma_{\bar{m}0}) \bar{m}^2, \] 

(3.10)

which yields, by the help of (3.7),

\[ \int_{\bar{m}^2}^{\bar{m}^2} \frac{d \bar{m}^2}{\bar{m}^2} = - \int_{\bar{s}}^{\bar{s}} \frac{d \bar{s}}{\beta_{\bar{m}}} \bar{s} - 2(\gamma_{\bar{m}1} \bar{y} + \gamma_{\bar{m}0}) \]

\[ \quad = - \int_{\bar{s}}^{\bar{s}} d \bar{y} \frac{\gamma_{\bar{m}1} \bar{y} + \gamma_{\bar{m}0}}{\beta_{\bar{m}} \bar{y}^2 + (\beta_{\bar{m}} - \beta_{\bar{m}1}) \bar{y} + \beta_{\bar{m}} \bar{y}}. \] 

(3.11)

Performing the integral, we find

\[ \bar{m}^2 = m^2 \left( \frac{g^2}{\bar{g}^2} \right)^{7_{\bar{m}1}/\beta_{21}} \left( \frac{(\lambda - b g^2)(g^2 / \bar{g}^2)^{a(b_{11}/\beta_{11})} - (\lambda - a g^2)(g^2 / \bar{g}^2)^{b(b_{11}/\beta_{11})}}{a - b} \right) \]

\[ \times \left( \frac{1}{\beta_{\bar{m}}} \right)^{2_{\bar{m}1}/\beta_{11}} \] 

(3.12)

Finally, we write RGE for \( \bar{h} \bar{m}^4 \) instead of \( \bar{h} \), which reads

\[ (\beta_{\bar{m}} \bar{s} - 2) \frac{d(\bar{h} \bar{m}^4)}{d \bar{s}} = \beta_{\bar{h}1} \bar{m}^4. \] 

(3.13)

The solution to this is given by

\[ \bar{h} \bar{m}^4 = h m^4 + m^4 \frac{\beta_{\bar{h}1}}{\beta_{\bar{m}}} F\left( \frac{\bar{g}^2}{g^2}, \frac{\lambda}{g^2} \right) \] 

(3.14)

with the function defined by

\[ F(x, y) = \int_1^x dt \ t^{-2(7_{\bar{m}1}/\beta_{11} + 1)} \left( \frac{a - b}{y - b} \right)^{-2(7_{\bar{m}1}/\beta_{11})} \left( \frac{a - b}{y - a} \right)^{-2(7_{\bar{m}1}/\beta_{11})} \] 

(3.15)

As the 'boundary' function of our RGE solution (2.14), we use the one-loop effective potential \( V_{11} = V^{(0)} + V^{(1)} \) at \( s = 0 \) which is given simply by setting \( \mu^2 = g^2 \phi^2 \) directly in Eq. (3.1):

\[ V_{11}|_{s=0} = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 + h m^4 \]

\[ + \frac{1}{64 \pi^2} \left[ \left( \frac{1}{2} \lambda \phi^2 + m^2 \right)^2 \left( \ln \frac{1}{g^2 \phi^2} \frac{3}{2} - \frac{3}{2} \right) + 6N(g^2 \phi^2)^2 \right]. \] 

(3.16)

To the leading-log order, the tree potential part \( V^{(0)}|_{s=0} \) is enough. But, as explained in the previous paper,\(^1\) retaining the one-loop part \( V^{(1)}|_{s=0} \) makes the approximation better also in the region in which the log-factor \( \ln(M^2 / \mu^2) \) is not so large. Replacing all the parameters \( g^2, \lambda, m^2, \phi \) and \( h m^4 \) in (3.16) by the above-obtained barred ones (3.5), (3.6), (3.9), (3.12) and (3.14), Eq. (3.16) gives the desired effective potential
which is leading-log ‘exact’ in the region $g^2 \phi^2 \gtrsim m^2$.

§ 4. Decoupling and RGE in the region $g^2 \phi^2 \lesssim m^2$

Up to here the procedure is essentially the same as in the previous paper. Now we turn to our main task of this paper to develop a method for obtaining the effective potential valid in the complementary region $g^2 \phi^2 \lesssim m^2$. For that purpose, we should first recall what we have done in the above. We had two logarithm factors $g^2 \ln(g^2 \phi^2 / \mu^2)$ and $g^2 \ln[(\lambda \phi^2 / 2 + m^2) / \mu^2]$. We have chosen the first factor as the variable $s$ with which we summed up the leading, next-to-leading, ..., log terms, and treated the second factor also essentially as $s$ by rewriting it into

$$g^2 \ln \frac{1}{2} \lambda \phi^2 + m^2 \mu^2 = s + u.$$  (4.1)

This is all right in the region $g^2 \phi^2 \gtrsim m^2$ since $u \sim O(g^2)$ there, but becomes of course problematic for $g^2 \phi^2 \lesssim m^2$ in which $u$ becomes very large $\sim O(1)$.

How can we calculate the effective potential, or the ‘boundary’ function, in the region $g^2 \phi^2 \lesssim m^2$? The key to this question is to note that it is the low-energy region. Physically speaking, any heavy particle, here $\phi$ with mass $m$, must have decoupled already in such a low-energy region and the running of the parameters such as couplings, masses and so on should be governed solely by the effective low-energy theory containing no heavy particles. Namely all the heavy particle loop contributions can be hidden in the redefinition of the low-energy theory parameters. This is the wisdom of effective field theory approach. If we do so, we have only one mass scale $M^2 = g^2 \phi^2$ in the low-energy theory and so will not encounter such a problematic variable like $u = \ln[(\lambda \phi^2 / 2 + m^2) / g^2 \phi^2]$, which appeared owing to the presence of two different mass scales.

Let us spell out about this in some different way. In the low-energy region $g^2 \phi^2 \lesssim m^2$, the above rewriting (4·1) of the second log factor into $s + u$ is clearly inadequate. Instead, the following expansion in $\phi^2 / m^2$ becomes good:

$$g^2 \ln \frac{1}{2} \lambda \phi^2 + m^2 \mu^2 = g^2 \ln \frac{m^2}{\mu^2} + g^2 \ln \left( 1 + \frac{\nu}{2} \frac{g^2 \phi^2}{m^2} \right)$$

$$= g^2 \ln \frac{m^2}{\mu^2} + g^2 \left[ \frac{\nu}{2} \frac{g^2 \phi^2}{m^2} - \frac{\nu^2}{8} \left( \frac{g^2 \phi^2}{m^2} \right)^2 + \ldots \right]. \quad (4.2)$$

In the region $g^2 \phi^2 \lesssim m^2$, while the first term $g^2 \ln(m^2 / \mu^2)$ becomes large $\sim O(1)$ for $m^2 \gg \mu^2$ (= $g^2 \phi^2$ in boundary function), the remaining terms can be regarded as small, $\sim O(g^2)$ (actually $O(g^2 / 16 \pi^2)$), in the sense of leading-log series expansion. Therefore now the log-factors which we have to take account of are $g^2 \ln(m^2 / \mu^2)$ and $s = g^2 \ln(g^2 \phi^2 / \mu^2)$. [Note that we have the same form of logarithm expansion as Eqs. (2·9) and (2·10) even if we take the second log factor (4·1) itself as the variable $u$.]

If we must treat these two log factors simultaneously, we would still have essentially the same difficulty as before. But fortunately, the decoupling theorem, or more
basically the renormalization theory itself, guarantees that all the powers of the former log factor, \([\ln(m^2/\mu^2)]^p\), like any positive power terms in \(m^2\), can be absorbed into redefinitions of the coupling constants and mass parameters in the low-energy effective field theory. So if we use those low-energy parameters, the explicitly appearing log-factor is only \(s\), and hence we can apply our original method for improving the effective potential with no problems. After performing the expansion (4.2) and renormalizing the log factor \(\ln(m^2/\mu^2)\) into low-energy parameters, we can use the \(L\)-loop potential evaluated at \(s=0\) as a boundary function suitable for the region \(g^2 \phi^2 \approx m^2\).

This reasoning also explains the point that this approximation is valid only in the region \(g^2 \phi^2 \approx m^2\) and so really complementary to the previous method: when \(g^2 \phi^2\) becomes larger than \(m^2\), the second and higher terms in the expansion (4.2) become non-negligible and make the original log-factor again, which should be summed up equally as \(s\) when \(g^2 \phi^2 \gg m^2\). It should be noted that our procedure of expanding the second log factor works well even at \(g^2 \phi^2 = m^2\). Recall that the second terms in the expansion (4.2) are small not because they are of higher order in \(g^2 \phi^2/m^2\), but because they are of higher order in the leading log series expansion. The difference

\[
g^2 \ln \left( \frac{1}{2} \frac{\lambda \phi^2 + m^2}{m^2} \right) = g^2 \ln \left( \frac{1}{2} \frac{\lambda \phi^2 + m^2}{\mu^2} \right) - g^2 \ln \left( \frac{m^2}{\mu^2} \right) = g^2 \ln \left( 1 + \frac{g^2 \phi^2}{2 m^2} \right) \tag{4.3}\]

remains of \(O(g^3)\) at \(g^2 \phi^2 = m^2\). This implies that (some power of) the log factor (4.3) contained in the \(L\)-loop part \(V^{(L)}(L \geq 1)\) affects, if any, the boundary function precisely at \(L\)-th-to-leading log order. So the approximation does not become bad even at \(g^2 \phi^2 = m^2\) if we retain all the terms in \(V_{\phi^2=0}\), as we shall explicitly do in the next section, instead of neglecting the corrections of \(O(g^2 \phi^2/m^2)\). With this understanding, our procedure works well in the region \(g^2 \phi^2 \approx m^2\).

However, a point may still seem to remain unclear: if we use the low-energy effective field theory for the region \(g^2 \phi^2 \approx m^2\) while keeping to use original theory in \(g^2 \phi^2 \ll m^2\), then what is the relation between the two theories? In particular, in this context of RGE improvement of the effective potential, what is the relation between the RGE's of the two theories?

Fortunately this has a very simple and natural answer: The differential operator \(\partial\) in the RGE is in fact unique; namely the apparently different \(\partial\) operators in both theories are the same.\(^4\)

In order to show this explicitly, let us come back to our Yukawa theory. There the heavy particle is \(\phi\) with mass \(m\). The \(\phi\)'s one-loop contribution to the effective potential was given before as \((M_\phi^4/64 \pi^2)\ln(M_\phi^2/\mu^2) - (3/2))\) with \(M_\phi^2 = (1/2) \lambda \phi^2 + m^2\), which is expanded in \(\lambda \phi^2/m^2\) to yield

\[
\frac{1}{64 \pi^2} \left[ m^2 \left( \ln \frac{m^2}{\mu^2} - \frac{3}{2} \right) + \lambda m^2 \phi^4 \left( \ln \frac{m^2}{\mu^2} - 1 \right) + \frac{\lambda^2 \phi^4}{4} \ln \frac{m^2}{\mu^2} \right] + O \left( \frac{\lambda \phi^2}{m^2} \right) \times \phi^4. \tag{4.4}\]

\(^4\) This fact may be regarded as a corollary of the decoupling theorem. Essentially the same statement can be found in earlier references.\(^7,8\) We note that our emphasis here is on its importance in the context of summing up the large logarithms.
As expected, all the effects of the heavy particle-loop are to shift the low-energy theory parameters, the vacuum energy $\hat{h}m^4$, mass parameter $\tilde{m}^2$ and coupling constant $\tilde{\lambda}$, aside from the non-renormalizable type higher power terms in $\phi$ all of which are suppressed by powers of $\phi^2/m^2$. This thus implies that the low-energy theory has the following mass $\tilde{m}$, coupling $\tilde{\lambda}$ and vacuum-energy $\hat{h}m^4$ parameters:

$$\tilde{\lambda} = \lambda + \frac{1}{16\pi^2} \frac{3\lambda^2}{2} \ln \frac{m^2}{\mu^2},$$

$$\tilde{m}^2 = m^2 + \frac{1}{16\pi^2} \frac{\lambda m^2}{2} \left( \ln \frac{m^2}{\mu^2} - 1 \right),$$

$$\tilde{h}m^4 = hm^4 + \frac{1}{16\pi^2} \frac{m^4}{4} \left( \ln \frac{m^2}{\mu^2} - \frac{3}{2} \right). \quad (4.5)$$

These are of course relations valid at one-loop level, and the higher-loop corrections give contributions of higher power terms in $\ln(m^2/\mu^2)$. If we were discussing effective action $\Gamma[\phi, \phi', \phi'']$ instead of effective potential, which contains Yukawa term $-g\hat{\phi}\phi\phi'$ in the tree part, we would find that Yukawa coupling $g$ is also shifted as

$$\tilde{g}^2 = g^2 + \frac{1}{16\pi^2} g^4 \left( 3\ln \frac{m^2}{\mu^2} - \frac{5}{2} \right). \quad (4.6)$$

The scalar field $\phi$ remains the same, i.e., $\tilde{\phi} = \phi$, as a special situation at the one-loop order of this model; the $\phi$'s one-loop diagram does not contribute to the wave-function renormalization of $\phi$. We now rewrite the RG differential operator $\mathcal{D}$ in (2.2) in terms of these new parameters of the low-energy theory:

$$\mathcal{D} = (\mathcal{D} \mu) \frac{\partial}{\partial \mu} + (\mathcal{D} \tilde{\lambda}) \frac{\partial}{\partial \tilde{\lambda}} + (\mathcal{D} \tilde{m}^2) \frac{\partial}{\partial \tilde{m}^2} + (\mathcal{D} \tilde{\phi}) \frac{\partial}{\partial \tilde{\phi}} + (\mathcal{D} \tilde{h}) \frac{\partial}{\partial \tilde{h}},$$

$$= \mu \frac{\partial}{\partial \mu} + \tilde{\beta}_\lambda \frac{\partial}{\partial \tilde{\lambda}} + \tilde{\beta}_m \frac{\partial}{\partial \tilde{m}^2} - \tilde{\gamma}_m \tilde{m}^2 \frac{\partial}{\partial \tilde{m}^2} - \tilde{\gamma}_\phi \frac{\partial}{\partial \tilde{\phi}} + \tilde{\beta}_h \frac{\partial}{\partial \tilde{h}}, \quad (4.7)$$

where using the one-loop relations (4.5) and (4.6) we have

$$\tilde{\beta}_\lambda = \mathcal{D} \tilde{\lambda} = -\frac{1}{16\pi^2} (8N\lambda g^2 - 48Ng^4) + O(h^2),$$

$$\tilde{\beta}_m = \mathcal{D} \tilde{m}^2 = \frac{1}{16\pi^2} 4Ng^4 + O(h^2),$$

$$\tilde{\gamma}_m = -\mathcal{D} \ln \tilde{m}^2 = -\frac{1}{16\pi^2} 4Ng^4 + O(h^2),$$

$$\tilde{\beta}_h = -\mathcal{D} \tilde{h} = 2h \tilde{\gamma}_m + O(h^2), \quad (4.8)$$

and $\tilde{\gamma} = \gamma = 2Ng^2/16\pi^2 + O(h^2)$. Note that one can freely replace the parameters here on the RHS's of (4.8) by the tilded ones since the difference is of $O(h^2)$. We immediately notice here that these are nothing but the $\beta$ and $\gamma$ functions in the low-energy

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*) Even if we discuss only the effective potential, we can find this shift (4.6) of the Yukawa coupling $g$. This is found, however, by computing two-loop contributions since $g$ appears in the effective potential only from the one-loop level.
effective field theory in which the running is governed solely by the light particle (here $\phi$) loop effects, as is clearly seen from the fact that they are all now proportional to $N$, the number of fermion species. This result may sound as a matter of course. It is important, however, here in the context of improving the effective potential that the renormalization group $\mathcal{D}$ operator is the same one between the low-energy effective theory and the original theory. So even if we solve the RGE in the low-energy effective field theory, it is guaranteed that we are solving the same RGE as in the original theory simply by using different set of parameters. Therefore it is also trivial that the solutions obtained in those two ways agree with each other at least around $g^2\phi^2 \sim m^2$ where the approximations adopted in the two methods are both valid. The parameters should be matched via the relations like (4·5). Since the relations contain powers of the log factor ($\lambda$ or $g^3\ln(m^2/\mu^2)$), the parameter matching between the low energy theory and the original theory has to be done at a renormalization point $\mu$ around $\mu \sim m$. Otherwise the unknown higher loop corrections may become large.

§ 5. Leading-log effective potential in the region $g^2\phi^2 \lesssim m^2$

We are now ready to demonstrate the procedure for obtaining the effective potential in the complementary region $g^2\phi^2 \lesssim m^2$ by explicit computations to the leading-log order. Now the Yukawa coupling is $\bar{g}$, so we use $\bar{s} = \bar{g}^2\ln(\bar{g}^2\bar{\phi}/\mu^2)$ in place of $s$, and the one-loop potential $V_1$ in (3·1) with $M_F$ replaced by $M_\psi = \bar{g}\bar{\phi}$, although the differences are of next-to-leading log (or two-loop) order. The boundary function is given by the one-loop effective potential $V_1$ at $\mu^2 = \bar{g}^2\bar{\phi}$ (i.e., $\bar{s} = 0$). But, from the $\phi$’s one-loop contribution, we should subtract the first three terms up to $\phi^4$ in (4·4) since they are absorbed in the redefinitions of the parameters, $h \to \bar{h}$, $m^2 \to \bar{m}^2$ and $\lambda \to \bar{\lambda}$. The leading-log effective potential is obtained by replacing the parameters there by the barred ones and so is given by

$$V = \frac{1}{2} m^2 \bar{\phi}^2 + \frac{1}{4!} \bar{\phi}^4 + \bar{h} \bar{m}^4 + \frac{1}{64\pi^2} \left[ m^2 G \left( \frac{\bar{\lambda} \bar{\phi}^2}{2 \bar{m}^2} \right) + 6N \bar{g}^2 \bar{\phi}^2 \bar{s}^2 \right], \quad (5·1)$$

where the $G$ term denotes the rest contribution (higher than $\phi^4$) of the $\phi$-loop,

$$G(x) = (1 + x)^2 \ln(1 + x) - x - \frac{3}{2} x^2. \quad (5·2)$$

Although this contribution is at most of $O(g^2)$ in the region $g^2\phi^2 \lesssim m^2$, we retained it since it makes better the matching of the present effective potential with the previous one around $g^2\phi^2 \sim m^2$.

Since the RG running equation for barred quantities was already solved in the full theory before, we can find the solution in this case simply by substituting $\bar{\beta}_h = \gamma_{\bar{m}1} = \bar{\beta}_{\bar{g}1} = 0$ and $\beta_{\phi1} = 4N/16\pi^2 \equiv \bar{\beta}_{\phi1}$ in the previous solutions. Therefore the barred quantities in our potential (5·1) are found to be

$$\bar{g}^2 = \bar{g}^2 \left( 1 - \frac{\bar{\beta}_{\phi1}}{2} \bar{s} \right)^{-1}, \quad \bar{\phi} = \bar{\phi} \left( 1 - \frac{\bar{\beta}_{\phi1}}{2} \bar{s} \right)^{\gamma_{\bar{m}1} / \bar{\beta}_{\phi1}},$$

where $\gamma_{\bar{m}1}$ and $\bar{\beta}_{\phi1}$ are the appropriate running coefficients.
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\[ \bar{\lambda} - a \bar{g}^2 = (\bar{\lambda} - a \bar{g}^2) \left(1 - \frac{\bar{g}}{2} \bar{s}\right)\bar{s}^{\frac{3}{2}} = \frac{a \bar{g}^2}{\bar{g} - \bar{g}_0} \]

\[ \bar{m}^2 = \bar{m}^2 \left(1 - \frac{\bar{g}}{2} \bar{s}\right)^{\frac{3}{2}} \bar{s}^{\frac{1}{2}}, \quad \bar{\bar{m}}^4 = \bar{\bar{m}}^4 \]

Now we match the (unbarred) tilded parameters (which are also running and functions of renormalization point \( \mu \)) with the untilded ones by choosing the renormalization point \( \mu = m \). Then the relations (4.5) and (4.6) give (aside from \( \phi = \phi \))

\[ \bar{\bar{m}}^4 = m^4 - \frac{1}{16\pi^2} \frac{3m^4}{8} \]

The agreement of this effective potential (5.1) by low-energy effective theory with that obtained previously in the original theory in the region \( \phi \sim m \) will be clear from the derivation. Since both the potentials satisfy the RGE they are \( \mu \)-independent \( (\frac{dV}{d\mu} = 0) \). So we can compare them choosing \( \mu = m \). Then at \( \phi \approx \bar{\phi} = m \) the parameters \( \bar{s} = \bar{\bar{s}} \) equal zero and hence all the barred quantities reduce to the unbarred ones in both expressions. Namely, both the potentials reduce to the "boundary functions" in each scheme. But they coincide with each other (up to two-loop quantities) by construction under identification (5.4).

§ 6. More general systems

We have explained our procedure by using the simplest example of Yukawa model. However the method described here is quite general and indeed applicable to any complicated systems. We conclude this paper by adding some explanations how we can improve the effective potential of multi-scalar fields applying the above procedure. For illustration let us consider the case of the two scalar field potential \( V(\phi_1, \phi_2) \), in a general system consisting of many particles which couple to those two scalar fields. In such a system, we would have typically the following three types of logarithm factors:

\[ s_1 = g_1^2 \ln \frac{g_1^2 \phi_1^2}{\mu^2}, \quad s_2 = g_2^2 \ln \frac{g_2^2 \phi_2^2}{\mu^2}, \quad s_3 = \lambda_3 \ln \frac{\lambda_3 \phi_1^2 + \lambda_2 \phi_2^2 + m^2}{\mu^2} \]

where \( s_1 \)- and \( s_2 \)-type factors come from fermion loops and \( s_3 \)-type from scalar loops.**

[The minimal supersymmetric standard model is similar to this example, where \( \phi_1 \) and \( \phi_2 \) correspond to the two Higgs doublets and \( m \) to the supersymmetry breaking scale.] Let us call the particles which produce the \( s_i \)-type log factors the \( i \)-type particles. We consider in any case that the coupling constants are of the same order,

\[ (\text{Our procedure described here should be compared with a more complicated method proposed by Einhorn and Jones}) \]

\[ \text{in which they introduce two renormalization points for } \phi_1 \text{ and } \phi_2. \]

\[ \text{[Here we are not claiming complete generality. We are just trying to explain a typical procedure which will be applicable to more general systems.]} \]
To treat the $s_1$- and $s_2$-type log factors well, we first have to separate the $(\phi_1^2, \phi_2^2)$ plane into two regions, $\phi_1^2 \lesssim \phi_2^2$ and $\phi_1^2 \gtrsim \phi_2^2$, and derive the effective potential in the two regions separately. Now consider the first region $\phi_1^2 \lesssim \phi_2^2$. [Second region is obtained in the same way by exchanging the role of $\phi_1$ and $\phi_2$.] Then we may regard the two scalar field potential $V(\phi_1, \phi_2)$ as a single scalar field potential $V_{s_2}(\phi_1)$ of $\phi_1$ with a parameter $\phi_2$. There the type-2 and type-3 particles can be viewed as particles carrying ($\phi_2$-dependent) masses $g_{zz}^2 \phi_2^2$ and $\lambda_0 \phi_2^2 + m^2$, respectively. Both of them are heavy in this region $\phi_1^2 \lesssim \phi_2^2$, and therefore decouple. So, in this region, the log factors which we have to treat explicitly in the effective potential are only $s_1$-type. The $s_2$ and $s_3$ log factors are taken into account simply by redefining the coupling constants (and masses) in the low-energy theory. A care, however, may be necessary in this redefinition: if $\lambda_0 \phi_2^2 \ll m^2$, the redefinition cannot be done in a single step. If we would do it in one step, the relation between an original coupling constant, write $\lambda$ generically, and its counterpart $\tilde{\lambda}$ in the low-energy effective theory would take the form like

$$\tilde{\lambda}(\mu) = \lambda(\mu) + c_2(\mu) \ln \frac{g_{zz}^2 \phi_2^2}{\mu^2} + c_3(\mu) \ln \frac{\lambda_0 \phi_2^2 + m^2}{\mu^2}$$  \hspace{1cm} (6.2)

at the one-loop level with certain coefficients $c_2$ and $c_3$ quadratic in the coupling constants $\lambda$. The last log comes from the expansion of the $s_3$-type log factors and the second log from $s_2$. The higher loop contributions give higher power terms of these two log factors. If $\lambda_0 \phi_2^2 \ll m^2$, then $g_{zz}^2 \phi_2^2 \ll \lambda_0 \phi_2^2 + m^2$, so that those two log factors cannot be made small simultaneously whatever renormalization point $\mu$ is chosen. This means that we can find no reliable relation between $\lambda$ and $\tilde{\lambda}$ unless we calculate all the higher loop contributions. Of course we know how to avoid this difficulty. As in the usual treatment of effective field theory in the presence of multi-threshold, we should do the redefinition in two steps: First, as we come down to the scale $\mu^2 \sim \lambda_0 \phi_2^2 + m^2$, we switch to an intermediate energy effective field theory in which only the type-3 particles decouple and the coupling $\lambda$ is shifted to

$$\tilde{\lambda}(\mu) = \lambda(\mu) + c_3(\lambda) \ln \frac{\lambda_0 \phi_2^2 + m^2}{\mu^2}.$$  \hspace{1cm} (6.3)

Next, at the scale $\mu^2 \sim g_{zz}^2 \phi_2^2$, we switch to the low-energy effective field theory in which the type-2 particles also decouple and the coupling $\tilde{\lambda}$ is shifted into

$$\tilde{\lambda}(\mu) = \tilde{\lambda}(\mu) + c_2(\lambda) \ln \frac{g_{zz}^2 \phi_2^2}{\mu^2}.$$  \hspace{1cm} (6.4)

Then we have a single log factor in each step of the coupling redefinition and so can find a reliable connection condition of the coupling constants. At the final stage, the effective potential is written in terms of the $\tilde{\lambda}$ coupling constants and contain only the $s_1$-type log factors explicitly, so that it can easily be improved by RGE.

References