A new mathematical problem—the Cauchy problem involving noncommutative quantities—is encountered in solving higher-order operator equations of nonabelian gauge theories and quantum gravity. In order to make this problem analyzable, a new mathematical technique is introduced, and its effectiveness is demonstrated by showing that the Heaviside-Mikusinski operational calculus is extended to the case involving noncommutative quantities. In a simple nonlinear field-theoretical model, it is explicitly shown how systematically its operator solution is obtained by this technique.

§ 1. Introduction

In a series of our papers, we are developing a new method of solving the operator formalism of gauge theories and quantum gravity in the Heisenberg picture, in such a way that both manifest covariance and local (more precisely, BRS) symmetry are respected in each order. We have already succeeded in solving the zeroth-order approximations of nonabelian gauge theory and quantum Einstein gravity, which have been shown to be essentially equivalent to the nonabelian BF theory and the two-dimensional quantum gravity, respectively.

Our next task is to proceed to considering higher orders. In order to find the operator solution, we have to solve inhomogeneously linear partial differential equations for field operators. In Ref. 1), we have made some preliminary consideration on quantum Einstein gravity, but we have encountered ambiguity in expressing the operator solution, because, in the canonical formalism, initial data are given not in terms of field operators but in terms of equal-time (anti)commutation relations. Hence we should replace equations for fields by those for (anti)commutators. In quantum electrodynamics, we have explicitly carried out this program. We have encountered no trouble there because of the linearity with respect to fields in the homogeneous part of the equation. This is, however, the speciality of the abelian gauge theory. In general, we encounter the nonlinearity with respect to fields, and it is not easy to express a commutator involving a nonlinear function of fields in terms of field commutators because of operator ordering problem. The purpose of the present paper is to clarify how to manage this problem.

The present paper is organized as follows. In § 2, we describe our operator ordering problem encountered. In § 3, we propose a new mathematical technique to deal with this problem. In § 4, we demonstrate the effectiveness of our technique by extending the Heaviside-Mikusinski operational calculus to the ordinary differential...
§ 2. New mathematical problem encountered

In solving higher-order operator equations of quantum nonabelian gauge theory or quantum gravity, we encounter a new mathematical problem. In order to clarify the essence of the problem, we describe the operator equation in a generic form.

Let $A(x)$ be a representative of the nonabelian gauge field or the gravitational field. The field equation for $A(x)$ may be written as

$$f(A(x)) = aJ(x),$$  \hspace{1cm} (2.1)

where $a$ is an expansion parameter ($a=g^2$ for gauge theory and $a=\kappa$ for quantum gravity), $f(A)$ is a nonlinear function of $A(x)$ (and its derivatives) involving no $a^*$ and $J(x)$ stands for the source function expressible in terms of $A(x)$ and other fields.

In order to solve (2.1) iteratively, we expand all fields in powers of $a$; for example,

$$A(x) = A^{(0)}(x) + aA^{(1)}(x) + a^2 A^{(2)}(x) + \cdots.$$  \hspace{1cm} (2.2)

Here we know that

$$[A^{(0)}(x), A^{(0)}(y)] = 0,$$  \hspace{1cm} (2.3)

though $A^{(0)}(x)$ is not a $c$-number. Substituting (2.2), etc., into (2.1), we obtain

$$[f(A(x))]^{(N)} = [J(x)]^{(N-1)}.$$  \hspace{1cm} (2.4)

Since we have already solved the zeroth order explicitly, we may inductively assume that all $j$th-order quantities ($0 \leq j \leq N-1$) are known. Since we postulate that any product of field operators at the same spacetime point is uniquely defined, (2.4) is rewritten as

$$f'([A^{(0)}(x)], A^{(N)}(x)) = F(x),$$  \hspace{1cm} (2.5)

where $F(x)$ is a known operator.

Unfortunately, as we pointed out previously, the above way of analysis is not adequate because, in the canonical formalism, the initial data are given only in the form of equal-time (anti)commutators. Indeed, what we have to find is not the field $A(x)$ itself but the commutator between it and any field. Accordingly, (2.4) must be replaced by

$$[f(A(x)), \Phi(y)]^{(N)} = [J(x), \Phi(y)]^{(N-1)};$$  \hspace{1cm} (2.6)

where $\Phi(y)$ denotes an arbitrary field operator.

Now, if

$$[A^{(0)}(x), [A(x), \Phi(y)]^{(N)}] = 0,$$  \hspace{1cm} (2.7)

* The BRS transformation is defined so as to be independent of $a$.\(^{(1,2)}\)
then we still have

\[ f'(A^{(y)}(x))[A(x), \Phi(y)]^{(y)} = F(x, y) \]  \hspace{1cm} (2.8)

just like (2.5), where \( F(x, y) \) is a known operator. Unfortunately, (2.7) is not true generally, because \( y^\nu \) is a spacetime point different from \( x^\nu \). Thus our problem is this: How can we write down the equation for \([A(x), \Phi(y)]^{(y)}\) without assuming (2.7)?

If \( f(A) = A^n \) or \( A^{-n} \), \( n \) being a positive integer, the answer is obvious from the identities

\[ [A^n, \Phi] = \sum_{n=0}^{n-1} A^k [A, \Phi] A^{n-k-1}, \]  \hspace{1cm} (2.9)

\[ [A^{-n}, \Phi] = -\sum_{k=0}^{n-1} A^{-k-1} [A, \Phi] A^{-n+k}. \]  \hspace{1cm} (2.10)

For the case in which \( f(z) \) is an arbitrary function, which is assumed to be analytic, we have

\[ [f(A), \Phi] = \frac{1}{2\pi i} \int dz \frac{f(z)}{z-A} [A, \Phi] \frac{1}{z-A} \]  
\[ = \frac{1}{2\pi i} \int dz \frac{f(z)}{z-A} [A, \Phi] \frac{1}{z-A}. \]  \hspace{1cm} (2.11)

because of the residue theorem, where the contour is a sufficiently large anticlockwise circle. Setting

\[ f(z) = \tilde{f}(z, w)(z-w) + f(w), \]  \hspace{1cm} (2.12)

we rewrite (2.11) as

\[ [f(A), \Phi] = \frac{1}{2\pi i} \int dz \frac{\tilde{f}(z, A)[A, \Phi]}{z-A} \frac{1}{z-A} \]  
\[ + f(A) \frac{1}{2\pi i} \int dz \frac{1}{z-A} [A, \Phi] \frac{1}{z-A}. \]  \hspace{1cm} (2.13)

Then the residue theorem implies

\[ [f(A), \Phi] = \tilde{f}(A_R, A_L)[A, \Phi], \]  \hspace{1cm} (2.14)

where \( A_R \) and \( A_L \) stand for the \( A \) located in the right of \([A, \Phi] \) and the \( A \) located in the left of \([A, \Phi] \), respectively. From (2.12), we formally have

\[ \tilde{f}(A_R, A_L) = \frac{f(A_R) - f(A_L)}{A_R - A_L}, \]  \hspace{1cm} (2.15)

whence \( \tilde{f}(A_R, A_L) \) is symmetric in \( A_R \) and \( A_L \).

Since \( f(A) \) in (2.6) contains not only \( A \) but also its derivatives, we have to take the presence of differentiation into account. In this case, \( \tilde{f}(A_R, A_L) \) is not unique. For example, for \( f(A) = A \square A \), formal calculation yields
\[
\tilde{f}(A_R, A_L) = A_R \Box - A_L \Box A_R
\]
\[
= A_R \Box (A_R - A_L) + (A_R - A_L) \Box A_R
\]
\[
= A_R \Box + (\Box A_R)
\]
and also
\[
\tilde{f}(A_R, A_L) = A_R \Box + (\Box A_L).
\]
This formal calculation is correct because differentiation is irrelevant to the operator ordering problem. Indeed, we have
\[
[A \Box A, \Phi] = A[\Box A, \Phi] + [A, \Phi] \Box A
\]
\[
= \Box A \cdot [A, \Phi] + [\Box A, \Phi] A
\]
because \(A \Box A = \Box A \cdot A\).

In order to secure the uniqueness of \(I(A_R, A_L)\), therefore, we require \(\tilde{f}(A_R, A_L)\) to be symmetric in \(A_R\) and \(A_L\), that is,
\[
\tilde{f}(A_R, A_L) = \tilde{f}(A_L, A_R).
\]
For example, for \(f(A) = A \Box A\), we set
\[
\tilde{f}(A_R, A_L) = \frac{1}{2} [(A_R + A_L) \Box + (\Box A_R) + (\Box A_L)].
\]
Note that \(\tilde{f}(A_R, A_L) = 0\) is symmetric in \(A_R\) and \(A_L\).

Thus (2·6) is put into the form
\[
\tilde{f}(A^{(0)}_R(x), A^{(0)}_L(x))[A(x), \Phi(y)]^{(\nu)} = F(x, y)
\]
instead of (2·8).

§ 3. New mathematical technique

Our next task is to solve (2·21). Since we have been able to write it in closed form by introducing \(A^{(0)}_R\) and \(A^{(0)}_L\) instead of \(A^{(0)}\), it is preferable to keep this form in solving (2·21). To deal with this kind of problem, we introduce a new mathematical technique in this section.

To begin with, we consider the following problem. Let \(A\), \(X\) and \(F\) be noncommutative quantities, where \(A^2 = 1\) and \(A\) and \(F\) are known, and the \(a_{ij}\)'s be \(c\)-numbers. Our problem is how to solve
\[
a_{00}X + a_{01}XA + a_{10}AX + a_{11}AXA = F
\]
with respect to \(X\). Evidently, the solution is expressed as
\[
X = b_{00}F + b_{01}FA + b_{10}AF + b_{11}AFA,
\]
the \(b_{ij}\)'s being \(c\)-numbers. Substituting (3·2) into (3·1) and using \(A^2 = 1\), we obtain the
simultaneous linear equations for $b_{ij}$,

$$
\begin{align*}
    &a_{00}b_{00} + a_{01}b_{01} + a_{10}b_{10} + a_{11}b_{11} = 1, \\
    &a_{01}b_{00} + a_{00}b_{01} + a_{11}b_{10} + a_{10}b_{11} = 0, \\
    &a_{10}b_{00} + a_{11}b_{01} + a_{00}b_{10} + a_{01}b_{11} = 0, \\
    &a_{11}b_{00} + a_{10}b_{01} + a_{01}b_{10} + a_{00}b_{11} = 0.
\end{align*}
$$

(3·3)

We note that (3·3) is independent of $F$. Solving (3·3) with respect to $b_{ij}$, we obtain the solution explicitly.

Now, (3·3) can be rewritten into the following form. Let $[A_R, A_L] = 0$ and $A_R^2 = A_L^2 = 1$. Then (3·3) is equivalent to

$$
(a_{00} + a_{01}A_R + a_{10}A_L + a_{11}A_R A_L)(b_{00} + b_{01}A_R + b_{10}A_L + b_{11}A_R A_L) = 1.
$$

(3·4)

Hence it is legitimate to rewrite (3·1) as

$$
(a_{00} + a_{01}A_R + a_{10}A_L + a_{11}A_R A_L)X = F
$$

and solve it in the form

$$
X = (a_{00} + a_{01}A_R + a_{10}A_L + a_{11}A_R A_L)^{-1}F.
$$

(3·6)

The right-hand side of (3·6) may be calculated by rationalizing its denominator.

It is straightforward to extend the above analysis to the case in which $A^n = 1$ instead of $A^2 = 1$. By making $n$ arbitrarily large, we can expel the effect of $A^n = 1$ from any finite order. Hence, taking $n \to \infty$, we see that our way of calculation, in which $A_R$ and $A_L$ are regarded as if they were independent quantities, is valid in the sense of the formal power series. For example, (3·1) without any condition like $A^2 = 1$ is solved as (3·6) without $A_L^2 = A_R^2 = 1$, which is formally defined by the power series expansion.

Hereafter we assume that our formal calculus is meaningful. In order to solve (2·21), we introduce the $D$-dimensional q-number Pauli-Jordan $D$-function defined by

$$
\tilde{f}(A_R^{(0)}, A_L^{(0)}) = \mathcal{D}(x, y; A_R^{(0)}, A_L^{(0)}) = 0,
$$

(3·7)

together with appropriate Cauchy data at $x^0 = y^0$, which involve no $A_R^{(0)}$ and $A_L^{(0)}$ explicitly, so that

$$
\mathcal{D}(x, y; A_R^{(0)}, A_L^{(0)}) = \mathcal{D}(x, y; A_L^{(0)}, A_R^{(0)}).
$$

(3·8)

Then (2·21) is solved as

$$
[A(x), \Phi(y)]^{(\omega)} = - \int d^D u \epsilon(x, y; u) \mathcal{D}(x, u; A_R^{(0)}, A_L^{(0)}) F(u, y)
$$

+ Cauchy-data terms

(3·9)

with $\epsilon(x, y; u) = \theta(x^0 - u^0) - \theta(y^0 - u^0)$. Here Cauchy-data terms are expressed in terms of the equal-time commutators $[A(x), \Phi(y)]^{(\omega)}|_0$ and $[\partial_0 A(x), \Phi(y)]^{(\omega)}|_0$ together with $\mathcal{D}$, where $|_0$ means setting $x^0 = y^0$; the explicit expressions can be easily written down by calculating $\partial_0^x \partial_0^y \mathcal{D}|_0$ explicitly.
§ 4. Extension of the operational calculus

In this section, we demonstrate that our new mathematical technique introduced in § 3 is really useful for solving the differential equations involving noncommutative quantities. We discuss an extension of the Heaviside-Mikusinski operational calculus\(^{10}\) for constant-coefficient ordinary differential equations.

The problem which we consider is to solve an ordinary differential equation

\[ P(D)X(t) = F(t), \quad (D = d/dt) \quad (4\cdot1) \]

under given initial data. Here

\[ P(z) = \sum_{k=0}^{n} a_k z^{n-k}, \quad (4\cdot2) \]

\[ a_k = a_k(A_R, A_L) \] being a polynomial in \( A_R \) and \( A_L \). As in the previous sections, \( A_R \) (\( A_L \)) stands for the \( A \) placed in the right (left) of \( X(t) \); \( A \) is, in general, noncommutative with \( X(t) \) and \( F(t) \). In this section, however, we assume that \( A \) is independent of \( t \). “\( A \)” may represent a number of operators if all of them mutually commute, but we forbid having noncommutative quantities in \( a_k \).

Our proposal is to apply the Heaviside-Mikusinski operational calculus to \((4\cdot1)\) by supposing that \( A_R \) and \( A_L \) are two independent, mutually commutative, quantities. Then what we have to show is that the final result is expressible in terms of \( A \), that is, we do not encounter in it such quantities as \( \sqrt{A_R + A_L} \).

We first consider the case in which

\[ a_0 = 1, \quad (4\cdot3) \]

\[ X(0) = DX(0) = \cdots = D^{n-1}X(0) = 0. \quad (4\cdot4) \]

We set

\[ P(z) = (z-a_1)(z-a_2)\cdots(z-a_n); \quad (4\cdot5) \]

then, in the generic case\(^*\) we can write

\[ \frac{1}{P(z)} = \sum_{k=1}^{n} h_k z^{-a_k}, \quad (4\cdot6) \]

where \( a_k \) and \( h_k \) are, in general, irrational functions of the \( a_i \)’s. According to the operational calculus, the solution to \((4\cdot1)\) under the initial data \((4\cdot4)\) is expressed as

\[ X(t) = \frac{1}{(D-a_1)\cdots(D-a_n)} F(t) \]

\[ = \sum_{k=1}^{n} h_k \frac{1}{D-a_k} F(t) \]

\(^*\) The case in which some of the \( a_i \)’s are equal is unnecessary to be discussed separately, because it is realized as a limit of the generic case.
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We apply the residue theorem to (4·7) and make use of (4·6):

$$X(t) = \int_0^t dt' \frac{1}{2\pi i} \oint\sum_{k=1}^n h_k e^{z(t-t')} \frac{e^{z(t-t')}}{z-a_k} F(t')$$

where the contour is a sufficiently large anticlockwise circle.

If we expand $e^{z(t-t')}$ in (4·8), we encounter integrals

$$b_m = \frac{1}{2\pi i} \oint dz \frac{z^{m+n-1}}{P(z)}$$

for $m \geq -n+1$.

Transforming the integration variable $z$ to $\zeta=1/z$, we have

$$b_m = \frac{1}{2\pi i} \oint d\zeta \frac{\zeta^{m-1}}{\zeta^n P(1/\zeta)},$$

where the contour is a very small anticlockwise circle around $\zeta=0$. Note that the denominator

$$\zeta^n P(1/\zeta) = \sum_{k=0}^n a_k \zeta^k$$

is nonvanishing at $\zeta=0$. Hence the residue theorem yields

$$b_m = 0 \quad \text{for} \quad m < 0,$$

$$b_0 = 1, \quad b_1 = -a_1, \quad b_2 = a_1^2 - a_2, \quad \text{etc.}$$

In general, we have the following recurrence formula:

$$\sum_{k=0}^{\min(m,n)} a_k b_{m-k} = \delta_{m0} \quad \text{for} \quad m \geq 0.$$  \hspace{1cm} (4·14)

Indeed,

$$\sum_{k=0}^{\min(m,n)} a_k b_{m-k} = \frac{1}{2\pi i} \oint d\zeta \frac{\sum_{k=0}^{\min(m,n)} a_k \zeta^{m+k-1}}{\sum_{k=0}^n a_k \zeta^k}$$

$$= \frac{1}{2\pi i} \oint d\zeta \frac{\sum_{k=0}^n a_k \zeta^{m+k-1}}{\sum_{k=0}^n a_k \zeta^k}$$

$$= \frac{1}{2\pi i} \oint d\zeta \frac{\zeta^{m-1}}{\sum_{k=0}^n a_k \zeta^k}$$

$$= \delta_{m0}.$$  \hspace{1cm} (4·15)

where, in the second equality, we have used the fact that when $m < n$ the terms $\zeta^{m+k-1}$ ($m+1 \leq k \leq n$) have no pole. Since the coefficient of $b_m$ in (4·14) is $a_0 = 1$, we find that $b_m$ is a polynomial in $a_1, a_2, \ldots, a_{\min(m,n)}$.

Now, we rewrite (4·8) as
\[ X(t) = \int_0^t dt' \sum_{n=0}^{\infty} \frac{b_m}{(m+n-1)!} (t-t')^{n+n-1} F(t') , \] (4·16)

where \( b_m = b_m(A_R, A_L) \) is a polynomial in \( A_R \) and \( A_L \). Hence it is easy to reexpress (4·16) in terms of \( A \).

In the case of general initial data, we have only to replace \( F(t) \) by

\[ F(t) + \sum_{k=0}^{n-1} c_k \delta^{(k)}(t) \] (4·17)
in the above, where

\[ c_{n-1-k} \equiv \sum_{j=0}^{k} a_{k-j} X(0) . \] (4·18)

Indeed, we can then show that

\[ \frac{1}{P(D)} \sum_{k=0}^{n-1} c_k \delta^{(k)}(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} D^k X(0) + O(t^n) \] (4·19)

by making use of (4·14).

Finally, we consider the case in which \( a_0 \neq 1 \) but \( a_0 \) is invertible. This case is reduced to the case \( a_0 = 1 \) by dividing (4·1) by \( a_0 \). Then, of course, the \( m \)-th term in the final result (4·16) contains \( a_0^{-m-1} \), which should be expanded in powers of \( A_R \) and \( A_L \) to rewrite it in terms of \( A \).

Although our reasoning is not rigorous, we can confirm the validity of our result in some simple examples by direct calculation.

§ 5. Application to a simple nonlinear model

In this section, we apply our method proposed in the previous sections to a simple nonlinear (nonunitary) model, which has some structural similarity to the nonabelian gauge theory and quantum Einstein gravity in finding the operator solution in the covariant operator formalism.

The Lagrangian density of our model is defined by

\[ \mathcal{L} = \delta'' B \cdot \partial \partial A - BA^2 + \frac{a}{2} B^2 , \] (5·1)

where \( a \) is an expansion parameter. If \( a = 0 \), (5·1) is nothing but a one-loop model,\(^1\)) which is exactly solvable both in the operator formalism and in the conventional perturbative method.

The field equations and the equal-time commutation relations derived from (5·1) are as follows:

\[ \Box A + A^2 = aB , \] (5·2)

\[ (\Box + 2A) B = 0 ; \] (5·3)

\(^1\) This name is due to the fact that only tree and one-loop graphs are encountered in the conventional perturbation theory.
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\[ [A(x), A(y)]_0 = [A(x), \hat{A}(y)]_0 = 0, \quad (5.4) \]
\[ [B(x), B(y)]_0 = [B(x), \hat{B}(y)]_0 = 0, \quad (5.5) \]
\[ [A(x), B(y)]_0 = 0, \quad (5.6) \]
\[ [\hat{A}(x), B(y)]_0 = -[A(x), \hat{B}(y)]_0 = -i\delta^3(x - y), \quad (5.7) \]

where \( |_0 \) denotes equal-time.

We construct the operator solution of the model from (5·2)~(5·7) in the expansion in powers of \( a \) by applying the method presented in §§2 and 3.

First, we write down the Cauchy problem for the full (multiple) commutators. From (2·14), (2·15), (5·2) and (5·3), we have

\[ (\Box + A_R + A_L)\Phi[A(x), \Phi(y)] = a[B(x), \Phi(y)], \quad (5.8) \]
\[ (\Box + A_R + A_L)\Psi[A(x), \Phi(y)] + (B_R + B_L)\Phi[A(x), \Phi(y)] = 0, \quad (5.9) \]

where \( \Phi = A, B \), with the Cauchy data (5·4)~(5·7). For multiple commutators, it is straightforward to generalize (5·8) and (5·9) as follows:

\[ (\Box + A_R + A_L)\Phi[A(x), \Phi(y), \Psi_1(z_1), \ldots, \Psi_n(z_n)]_0 \]
\[ + \sum_{n=1}^n \sum_{P_n} \left( [A(x), \Psi_{j_1}(z_{i_1}), \ldots, \Psi_m(z_{i_m})]_0 \right) \]
\[ + (R\to L)\Phi[A(x), \Phi(y), \Psi_{j_1}(z_{i_1}), \ldots, \Psi_{n-m}(z_{i_{n-m}})]_0 \]
\[ = a[B(x), \Phi(y), \Psi_1(z_1), \ldots, \Psi_n(z_n)]_0, \quad (5.10) \]
\[ (\Box + A_R + A_L)\Psi[B(x), \Phi(y), \Psi_1(z_1), \ldots, \Psi_n(z_n)]_0 \]
\[ + (B_R + B_L)\Phi[B(x), \Phi(y), \Psi_1(z_1), \ldots, \Psi_n(z_n)]_0 \]
\[ + \sum_{n=1}^n \sum_{P_n} \left( [B(x), \Psi_{j_1}(z_{i_1}), \ldots, \Psi_m(z_{i_m})]_0 \right) \]
\[ + (R\to L)\Psi[B(x), \Phi(y), \Psi_{j_1}(z_{i_1}), \ldots, \Psi_{n-m}(z_{i_{n-m}})]_0 + (A\leftrightarrow B) = 0, \quad (5.11) \]

where \( \mathcal{P}_n \) is a set of partitions of indices \( \{1, \ldots, n\} \) into \( \{i_1 < \cdots < i_m\} \) and \( \{i_1 < \cdots < i_{n-m}\} \), and \( [a_1, a_2, a_3, \ldots, a_n]_0 \) stands for the standard-form \((n-1)\)-ple commutator \( \cdots [[[a_1, a_2], a_3], \ldots, a_n] \).

Next, we consider the expansion in powers of \( a \). For the zeroth-order approximation, from (5·8) and (5·9), we have

\[ (\Box + A_R + A_L)^{(0)}[A(x), A(y)]_0^{(0)} = 0, \quad (5.12) \]
\[ (\Box + A_R + A_L)^{(0)}[A(x), B(y)]_0^{(0)} = 0, \quad (5.13) \]
\[ (\Box + A_R + A_L)^{(0)}[B(x), B(y)]_0^{(0)} = -(B_R + B_L)^{(0)}[A(x), B(y)]_0^{(0)}. \quad (5.14) \]

Under the postulate of the unique solvability of the Cauchy problem, (5·4) and (5·12) yield

\[ [A(x), A(y)]_0^{(0)} = 0. \quad (5.15) \]
Of course, as done previously, \( (5 \cdot 15) \) is also obtained as a formal expansion in powers of \( x^0 - y^0 \) from \( (5 \cdot 2) \) with \( a=0 \) and \( (5 \cdot 4) \).

To construct other commutators, we define the following \( q \)-number Pauli-Jordan \( D \)-function:

\[
(\square + A_R + A_L)^{(0)}(x, y) = 0,
\]

\[
D^{(0)}(x, y)|_0 = 0, \quad \partial_y D^{(0)}(x, y) = -\delta(x - y).
\]

Then, the unique solvability of the Cauchy problem implies

\[
[A(x), B(y)]^{(0)} = iD^{(0)}(x, y),
\]

\[
[B(x), B(y)]^{(0)} = i\int d^4 u \epsilon(x, y; u)D^{(0)}(x, u)(B_R + B_L)^{(0)}uD^{(0)}(u, y),
\]

where use has been made of \( (3 \cdot 9) \). Setting \( \Phi = B \) and \( \Psi_1 = A \) in \( (5 \cdot 10) \) for \( n=1 \) and using \( (5 \cdot 15) \), we obtain

\[
(\square + A_R + A_L)^{(0)}[[A(x), B(y), A(z)]_0^{(0)} = 0,
\]

which yields

\[
[D^{(0)}(x, y), A^{(0)}(z)] = 0.
\]

Likewise from \( (5 \cdot 10) \) for \( n=1 \) and \( \Phi = \Psi_1 = B \), we have

\[
(\square + A_R + A_L)^{(0)}[[A(x), B(y), B(z)]_0^{(0)}
\]

\[
= ([A(x), B(z)]^{(0)}_R + [A(x), B(z)]^{(0)}_L)[A(x), B(y)]^{(0)}
\]

then we obtain

\[
[D^{(0)}(x, y), B^{(0)}(z)]
\]

\[
= -i\int d^4 u \epsilon(x, y; u)D^{(0)}(x, u)(D^{(0)}(u, y)D^{(0)}(u, z) + (y \leftrightarrow z)).
\]

Here the ordering of the \( D^{(0)} \)'s causes no problem because we can also show that

\[
[D^{(0)}(x, y), D^{(0)}(z, w)] = 0.
\]

In this way, we have obtained the zeroth-order approximation.

As for the higher-order approximations, it is straightforward to apply \( (3 \cdot 9) \) to \( (5 \cdot 8) \) and \( (5 \cdot 9) \). Since all the higher-order Cauchy data are trivial, we obtain

\[
[A(x), \Phi(y)]^{(N)} = -\int d^4 u \epsilon(x, y; u)D^{(0)}(x, y)[[B(u), \Phi(y)]^{(N-1)}
\]

\[
- \sum_{k=0}^{N-1}((A_R + A_L)^{(N-k)}u[A(u), \Phi(y)]^{(k)}),
\]

where \( \Phi = A, B, \) and

\[
[B(x), B(y)]^{(N)} = -\int d^4 u \epsilon(x, y; u)D^{(0)}(x, u)[((B_R + B_L)^{u}[A(u), B(y)])^{(N)}
\]
As seen from (5·25) with $\Phi = A$, $[A(x), A(y)]^{(N)}$ ($N \geq 1$) is no longer vanishing in contrast with (5·15). Likewise the multiple commutators consisting of the $A$'s only are nonvanishing in general. We can, however, show the following interesting multiple commutativity:

\[
[A(x), A(y), A(z_1), \ldots, A(z_n)]^{(n)} = 0 \quad \text{for } n \geq N,
\]

\[
[B(x), A(y), A(z_1), \ldots, A(z_n)]^{(n)} = 0 \quad \text{for } n \geq N + 1,
\]

The proof is easily done by induction on $N$. For $N = 0$, (5·27) and (5·28) reduce to (5·15) and (5·21), respectively. As for $N \geq 1$, the $N$-th order of (5·11) reduces to

\[
(\Box + A_R + A_L)^{(N)} [A(x), A(y), A(z_1), \ldots, A(z_n)]^{(N)} = 0 \quad \text{for } n \geq N
\]

by the assumption of induction. This result implies (5·27) owing to the unique solvability of the Cauchy problem. Then, from the $N$-th order of (5·11), the assumption of induction gives

\[
(\Box + A_R + A_L)^{(N)} [B(x), A(y), A(z_1), \ldots, A(z_n)]^{(N)} = 0 \quad \text{for } n \geq N + 1,
\]

which leads to (5·28). Thus the proof is accomplished.

Also for other nonvanishing multiple commutators, we have no problem in principle to apply (3·9) to (5·10) and (5·11), whence we omit writing the results here.

Finally, we emphasize that the proof of (5·27) and (5·28) is greatly indebted to the method proposed in the present paper for its extreme simplicity. Indeed, if we adhered to the traditional method of calculation, it would be quite laborious to show (5·27) and (5·28) even for small values of $N$.

\section*{§ 6. Discussion}

In the present paper, we have discussed how to analyze the Cauchy problem involving noncommutative quantities, which is encountered in solving the operator differential equation for a full commutator between fields. We have introduced a new mathematical technique: We define $A_R$ and $A_L$ according as $A$ stands in the right and in the left, respectively, and then carry out our calculation by regarding them as two independent, mutually commutative, ordinary operators. After solving the differential equation, we should rewrite the solution expressed in terms of $A_R$ and $A_L$ into the one expressed in terms of $A$ with the operator ordering indicated by $A_R$ and $A_L$. We have explicitly demonstrated in §4 that this procedure can be done at least for the constant-coefficient ordinary differential equations.

Although we have introduced our new technique for dealing with higher-order equations in the expansion, it is efficient also for the zeroth-order one. Previously, we have inferred the full commutativity (2·3) for nonabelian gauge theory\textsuperscript{4} and for quantum gravity\textsuperscript{1,5} by means of the Taylor expansion of $A^{(0)}(x)$ with respect to $x^0 - y^0$. This series, however, cannot be convergent, whence the full commutativity has not been a logical consequence. As shown in § 5, we can now derive the full
commutativity under the unique solvability of the Cauchy problem which is one of our fundamental postulates.

Our next task is to solve the first-order approximations to nonabelian gauge theory and to quantum gravity.

References


Note added in proof: In § 2, when \( f(A) \) involves differentiation, a manifestly symmetric expression for \( f(A_B, A_L) \) is given by

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left( \frac{A_B - A_L}{2} \right)^{2n} f^{(2n+1)} \left( \frac{A_B + L_L}{2} \right).
\]