Nuclear matter in the generator coordinate space is studied introducing the tetrahedral $\alpha$-clusters in $\alpha$-cluster matter. Whether the phase transition from $\alpha$-cluster matter to nuclear matter takes place as the density of nucleons increases is investigated using two kinds of effective inter-nucleon forces. Though the physical quantities strongly depend upon the properties of the effective inter-nucleon force, the phase transition or the state close to the phase transition occurs in the nuclear interior. On the other hand, the surface never shows such a phase transition, and $\alpha$-cluster matter is always more stable than nuclear matter. The calculated results thus indicate that $\alpha$ clusters float on the surface of nuclear matter.

§ 1. Introduction

In our previous paper,\(^1\) we showed that, when a simple lattice structure in the generator coordinate (GC) space established by Brink and Bloch\(^2\) is adopted as a method of $\alpha$-cluster matter, convergent analytical formulae for the binding energy per $\alpha$ cluster can be obtained. This is one of the microscopic models which takes full account of the Pauli principle among all nucleons and effective inter-nucleon forces simultaneously. Three-dimensional $\alpha$-cluster matter problem can be reduced to one-dimensional one even if a system comprising the infinite number of $\alpha$ clusters is considered. In other words, indispensable information about all of the kernels in the GC space, which is directly related to the binding energy, can be written in separated formulae with respect to each direction. This is because the overlap matrix in the GC space has the Kronecker product form in the respective orthogonal directions, and this leads to the Kronecker product form for its inverse matrix.

The purpose of the present paper is not only to improve this theoretical framework by introducing the spatial distribution of each $\alpha$ cluster considered as a tetrahedral shape of four nucleons, but also to show the calculated results using two kinds of effective inter-nucleon forces. Matter with a uniform nucleon density is easily constructed when the spatial distribution follows some systematic rule. Let us explain this rule intuitively. First of all, we need to keep in mind the fact that a face-centered cubic ($fcc$) lattice is made of four kinds of independent simple lattices. Then, the neighbouring four lattice-points belonging to different simple lattices just become the vertices of a tetrahedron. It is possible to make the respective simple lattices correspond to four kinds of nucleons in the $\alpha$ clusters. This system not only

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consists of infinite numbers of tetrahedral $\alpha$-clusters arranged in order but also with a uniform nucleon density for the close packed configuration. Besides, such fcc structure of nucleons is expected to show the most stable binding energy per nucleon. We naturally call the uniform nucleon density-configuration nuclear matter in the GC space. On the contrary, when all the tetrahedral $\alpha$-clusters shrink to point-like ones in the GC space at the same time, we call this system $\alpha$-cluster matter.

Now, we need common theoretical foundation in order to study the transition process from $\alpha$-cluster matter to nuclear matter as a function of the distance parameter between neighbouring $\alpha$-clusters or nucleons. Brink and Castro\(^3\) have estimated critical nucleon density corresponding to the phase transition from $\alpha$-cluster matter to nuclear matter, by comparing $\alpha$-cluster matter in the GC space with nuclear matter using the plane wave description. However, it is very difficult to compare two different kinds of models, since $\alpha$-cluster matter in the GC space is a sort of static model in spite of the correct treatment of the composite nature of $\alpha$ clusters.

As mentioned in our previous paper,\(^4\) the model adopted in the present study enables us to treat infinite matter and semi-infinite matter with a surface at the same time within a common framework in the GC space. Therefore, we can easily evaluate those surface effects of matter that are directly connected with the surface term in Weizaecker mass formula. Besides, the incompressibility of matter must be expressed in terms of two variables namely the distance parameter between neighbouring particles and the size parameter of the single nucleon wave function. The former variable is directly related to the nucleon density, and the latter concerns the breathing mode of the $\alpha$ cluster. From energetic consideration we find that the size parameter varies so as to prevent the nucleon density from changing; thus the eigenvalue containing a dominant component with respect to the distance parameter for the Hessian matrix is the incompressibility of matter, (see Eq. (11) in Ref. 4)).

We adopt two kinds of effective inter-nucleon forces as mentioned above, which have completely different properties from each other but guarantee the saturation properties of traditional nuclear matter. The first is the Brink-Boeker force\(^5\) which gives the saturation properties of traditional nuclear matter by introducing the strong Majorana exchange mixture in the short range force. Another is the Skyrme force\(^6\) with a $\delta$-type three-body interaction corresponding to the density-dependent effects of nucleons which prevent traditional nuclear matter from collapsing. However, the Skyrme force does not give a good size parameter for the $\alpha$ particle but reproduces roughly its binding energy. On the other hand, all of usual two-body effective nuclear forces employed in the study of light nuclei such as Volkov force\(^7\) cannot prevent nuclear matter from collapsing. Of course, in the present study, we must switch off the Coulomb interaction in order to avoid making nuclear matter and $\alpha$-cluster matter explode. However, it is interesting to investigate the destabilization and eventually the explosion of finite matter, namely real nuclei, due to the Coulomb interaction. In § 2, the model and the related formalism are given. Section 3 describes and discusses the results and the final section contains brief concluding remarks.
§ 2. Model and the related formalism

We start with the total wave function of the $4N$ system ($N$ is an integer including infinity):

$$\Psi = \frac{1}{\sqrt{(4N)!}} \mathcal{A} \prod_{p=1}^{4N} [\phi_i^{(p)}, \ldots, \phi_N^{(p)}] \otimes \chi_p],$$

(1)

where $\mathcal{A}$ is the antisymmetrizer operator for all nucleons. The spatial wave function, $\phi_i^{(p)}$, of the $i$-th nucleon belonging to the $p$-th spin-isospin state, $\chi_p$, is expressed relative to the spatial parameter $R/P)$ by

$$\phi_i^{(p)} = (\pi b^2)^{-3/4} \exp \left\{ -\frac{1}{2b^2}(r-R_i^{(p)})^2 \right\},$$

(2)

where $b$ is the size parameter of the $(0s)$ h.o. wave function with $\hbar\omega = \hbar^2/(Mb^2)$ ($M$ is nucleon masses). The parameter $R_i^{(p)}$ is the GC of the $i$-th nucleon in the spin-isospin state $\chi_p$. Here, if the labels, $p$, differ from each other, the GCs are also different due to the spatial distribution of nucleons in the $\alpha$ clusters although they have the same label, $i$, corresponding to the various $\alpha$ clusters. The dynamical variable $r$ of the nucleon is integrated over the infinite space of the remaining parameters, $R_1^{(p)}, \ldots, R_N^{(p)}$.

The normalization kernel is given by

$$\mathcal{N}(R_1^{(1)}, \ldots, R_N^{(1)}, \ldots, R_1^{(4)}, \ldots, R_N^{(4)}) = \langle \Psi | \Psi \rangle = \prod_{p=1}^{4N} (\det(B_i^{(p)})),$$

(3)

where $B_i^{(p)}$ is the overlap matrix element

$$B_i^{(p)} = \langle \phi_i^{(p)} | \phi_i^{(p)} \rangle = \exp \left\{ -\frac{1}{4b^2}(R_i^{(p)} - R_i^{(p)})^2 \right\}. $$

(4)

Each $R_i^{(p)}$ is a mesh point in the independent $p$-th simple lattice in the GC space which has the same distance parameter between neighbouring mesh points. In this case, since four simple lattices have the same structure, all the determinants, $\det(B_i^{(p)})$, are taken to have the same value. Though the normalization kernel obviously converges to zero when $N$ goes to infinity, the inverse matrix elements, $(B_i^{(p)})^{-1}$, never diverge except for a zero-interval mesh; the information about the analytical formulae for the inverse $B$-matrix has been already given in our previous paper.\(^1\) There was shown that the formula for $B^{-1}$ separates in three dimensions like the Kronecker product formula. Since the other matrix elements with any operator similarly have the separated form for the respective dimensions in the GC space due to the h.o. wave function, the resultant kernels can be written as a simple summation expression of the separated formula; namely, all of the kernels in the three-dimensional configuration are reduced to those of a one-dimensional configuration (linear chain of nucleons). In

\(^1\) In Ref. 1 we adopted the parameter $\nu = 1/(2b^2)$ as the oscillator parameter of a single nucleon wave function to give an elegant formula for the kernels. In the present paper, we use the parameter $b$ to facilitate a simple discussion of the physical quantities.
general, the interaction between the different simple lattices does not take place through the $B^{-1}$ owing to the Pauli principle but comes out only by effective inter-nucleon forces.

Next, we must explain the method of calculating the physical quantities such as the binding energy per nucleon and the incompressibility of matter and so on. The microscopic Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2M} \sum_i \nabla_i^2 - T_{\text{c.m.}} + \frac{1}{2!} \sum_{ij} V^{(2)}_{ij} + \frac{1}{3!} \sum_{ijk} V^{(3)}_{ijk},$$

where the first term is the kinetic energy operator and the second one the c.m. energy one which can be ignored for the system with the infinite number of nucleons. In this expression, the third and final terms are the effective two- and three-body inter-nucleon force operators. The energy kernel is defined by

$$\mathcal{H}(R^{(1)}, \ldots, R^{(4)}) = \langle \Psi | H | \Psi \rangle = \langle \Psi | \sum_i E_i(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}) \rangle,$$

where $E_i$ is the binding energy of $i$-th set of four nucleons, which can be divided into three parts:

$$E_i(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}) = E_i^{(4)}(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}) + E_i^{(2)}(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}) + E_i^{(3)}(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}).$$

Here, the respective terms are also given by

$$E_i^{(4)}(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}) = \frac{4}{3!} \sum_{p=1}^{4} \sum_{j} \langle \phi_i^{(p)} \chi_r | \Phi_j | \Phi_j^{(p)} \chi_r \rangle (B_{ij}^{(p)})^{-1},$$

$$E_i^{(2)}(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}) = \frac{1}{2!} \sum_{p,q} \sum_{j,k,l} \langle \phi_i^{(p)} \chi_r \phi_j^{(q)} \chi_q | V_{kl}^{(2)} | \phi_i^{(q)} \chi_r \phi_j^{(p)} \chi_q \rangle (B_{ij}^{(p)})^{-1}(B_{kl}^{(q)})^{-1}$$

and

$$E_i^{(3)}(R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}) = \frac{1}{3!} \sum_{p,q,r} \sum_{j,k,m,n} \langle \phi_i^{(p)} \chi_r \phi_j^{(q)} \chi_q \phi_k^{(r)} \chi_r | V_{lmn}^{(3)} | \phi_i^{(q)} \chi_r \phi_j^{(p)} \chi_q \phi_k^{(r)} \chi_r \rangle (B_{klm}^{(p)})^{-1}(B_{kln}^{(q)})^{-1}(B_{kmn}^{(r)})^{-1} + \text{the other 5 permutated terms}.$$
is employed. Each $E_i$ converges to a finite value although the total binding energy diverges for the system with an infinite number of nucleons.

We assume infinite simple lattices with or without a surface as a model of $\alpha$-cluster matter and nuclear matter. At first, we must prepare four kinds of simple lattices which exactly coincide with each other, and must place all sets of four nucleons on the respective meshes of the lattices. Next, we move three kinds of nucleons towards the respective centers of the planes orthogonal to each other at the same velocity relative to the one remaining nucleon (see Fig. 1). We see that four nucleons surrounded by a circle in Fig. 1 always form a tetrahedral $\alpha$-cluster. In this way, separation of the four nucleons in the $\alpha$ cluster is naturally introduced in the GC space. Since all nucleons in each lattice do the same movement, the four simple lattices separate from each other. The starting point of such a movement can be regarded as $\alpha$-cluster matter, and it becomes nuclear matter by the time the moving nucleons reach the centers of the planes. At this moment, all nucleons have the $fcc$ configuration which corresponds to a uniform nucleon density. In infinite matter without any surface, all the $E_i$ must have the same value because all the sets of four nucleons are on the same situation. This $E_i$ is simply regarded as the binding energy per nucleon and may be defined by

$$B_0(s, t, b) = \frac{1}{4} E_i,$$  \hspace{1cm} (11)

where $s$ is the mesh interval and $t$ stands for the side length of a tetrahedron. The volume term in Weizaeker mass formula is expressed by

![Image of phase transition from $\alpha$-cluster matter to nuclear matter](https://academic.oup.com/ptp/article-abstract/90/4/871/1869811)
\[ b_v = -B_v(s, t, b)_{\min} , \]  
(12)

which must satisfy simultaneously the following conditions:

\[ \frac{\partial B_v}{\partial s} = 0, \quad \frac{\partial B_v}{\partial t} = 0 \quad \text{and} \quad \frac{\partial B_v}{\partial b} = 0 . \]  
(13)

Therefore, the volume term, \( b_v \), can be obtained for nuclear matter and also for \( a \)-cluster matter. These conditions depend on whether the phase transition from \( a \)-cluster matter to nuclear matter takes place or not.

On the other hand, each \( E_i \) converges to a different value in the semi-infinite matter with a surface as shown in Fig. 2, in which the set of four nucleons is labelled in the order from the surface area; that is, \( E_1 \) is just the binding energy of four nucleons in the surface and \( E_n \) is the usual binding energy of the nucleon obtained by letting \( n \) approach infinity (we call these nucleons surface-nucleons and inner nucleons, respectively). In general, surface effects of matter per nucleon can be evaluated by

\[
B_s = \frac{1}{4} \sum_{i=1}^{\infty} \{ E_n(s, t, b) - E_i(s, t, b) \} \quad (n \to \infty)
\]

\[ = \sum_{i=1}^{\infty} (B_v - B_i) . \]  
(14)

Although each \( E_i \) is a function of the parameters, \( s, t \) and \( b \) leading to the minimum energy of four nucleons in each layer, the set of parameters is chosen so as to lead to the minimum value of \( E_z(n \to \infty) \) except for the case of \( E_1 \). As mentioned in the following section, the behaviour of four nucleons in the second layer already resembles that of inner nucleons. The quantity \( B_s \) can be related to the surface term of Weizaecker mass formula as follows:

\[ b_s = -6 \cdot 4^{4/3} B_s , \]  
(15)

where the factor 6 comes from the number of surfaces due to the assumed infinite cube, and \( 4^{4/3} \) is the correction factor coming from the set of four nucleons.\(^*1\)

The parameter \( t \) varies from zero, which corresponds to that of \( a \)-cluster matter, to \( s/\sqrt{2} \) which coincides with that of nuclear matter. The incompressibility of matter related to the nucleon density must be defined by a total second order differential with respect to both variables, such as \( s \) and \( b \), around the minimum value of \( B_v \):

\(^*1\) The binding energies, \( B_v \) and \( B_s \), must not be confused with the \( B \)-matrix, and the volume term, \( b_v \), and surface term, \( b_s \), can be also distinguished from the size parameter \( b \).
where

\[
C = \left( \begin{array}{cc}
(\partial^2 B_v / \partial s^2) s^2 & (\partial^2 B_v / \partial s \partial b) s b \\
(\partial^2 B_v / \partial b \partial s) b s & (\partial^2 B_v / \partial b^2) b^2
\end{array} \right)_{s, b, t = \text{minimum}}
\]

and

\[
S = \left( \begin{array}{c}
(\partial B_v / \partial s) s \\
(\partial B_v / \partial b) b
\end{array} \right)_{s, b, t = \text{minimum}}
\]

Here the contribution coming from the parameter \( t \) is ignored due to the small effect of the off-diagonal derivatives on the binding energy. As can be seen from this expression, the incompressibility \( K_v \) is one of the eigenvalues of the Hessian matrix \( C \), corresponding to the eigenvector of which contains the dominant component related to the parameter \( s \). Substituting \( B_v \) by \( B_l \) in Eq. (16), the incompressibility of the surface region, \( K_s \), can be easily obtained.

By investigating the dependency on the parameter \( t \) over the wide range of \( s \), we can clarify the strength of \( \alpha \)-clustering and the uniformity of matter; as for \( \alpha \)-clustering, the definitions are given by

\[
U_{\alpha}^s(s, b) = \left. \frac{\partial^2 B_v}{\partial t^2} \right|_{t = 0} \quad \text{and} \quad U_{\alpha}^s(s, b) = \left. \frac{\partial^2 B_l}{\partial t^2} \right|_{t = 0},
\]

where the first expression stands for that of inner nucleons and the second one refers to the surface-nucleons; then as for uniformity of matter,

\[
U_{\alpha}^s(s, b) = \left. \frac{\partial^2 B_v}{\partial t^2} \right|_{t = \text{minimum}} \quad \text{and} \quad U_{\alpha}^s(s, b) = \left. \frac{\partial^2 B_l}{\partial t^2} \right|_{t = \text{minimum}},
\]

where these two equations have the same definitions as Eq. (19).

\section*{5. Results and discussion}

The Brink-Boeker No. 1 force\(^5\) and the Skyrme No. 2 force\(^6\) modified by Vautherin and Brink\(^7\) (hereafter abbreviated as BBI and SII forces) are adopted as reasonable effective inter-nucleon forces, which can reproduce the bulk properties of traditional nuclear matter. The BBI force has the following form:

\[
V_{\alpha}^{(B)}(s) = \sum_{\pi, \theta} \nu_{\pi} \exp \left\{ -\left( \frac{\gamma_{\pi, \theta}}{\rho_{\pi, \theta}} \right)^2 \right\} \left\{ (1 - M_{\pi}) + M_{\pi} P_{\pi, \theta} \right\},
\]

where \( P_{\pi, \theta} \) is the Majorana exchange operator expressed by \( -P_{\pi, \theta} P_{\pi, \theta} \). The force strengths \( \nu_{\pi} \), the range parameters \( \rho_{\pi, \theta} \) and the Majorana exchange mixture \( M \) are listed in Table I. The strong Majorana mixture in the short range force, as seen in Table I, guarantees the saturation properties of traditional nuclear matter. On the other hand, the SII force is a \( \delta \)-function type force without any range parameter defined as
$$V^{(2)}_{ij} = t_1(1 + \chi_0 P'_{ij}) + \frac{1}{2} t_1 \delta(r_i - r_j) k^2 + k^2 \delta(r_i - r_j) + t_3 k' \delta(r_i - r_j) k$$

and

$$V^{(3)}_{ijk} = t_3 \delta(r_i - r_j) \delta(r_j - r_k),$$

where $k$ and $k'$ are the momentum operators acting on the ket and bra wave functions, respectively. Here, we ignore the additional term, $i(\sigma_i + \sigma_j) \cdot k' \times \delta(r_i - r_j) k$, since it makes no contribution in the present study. The expression (23) is not symmetric with respect to suffix $i$ but is symmetric with respect to $j$. This is the reason why the central single nucleon wave function in the bra vector is chosen to obtain the binding energy per four nucleons in Eq. (10). All the parameters in the effective inter-nucleon forces for the SII force are also listed in Table I.

Now, let us come to the calculated results. In order to see the outlines of the features of $\alpha$-cluster matter and nuclear matter, we refer to Figs. 3 and 4 focussing upon $B_n$ and $B_i$ versus the distance parameter $s$ or $t$. On the left-hand side of each figure, the size parameter $b$ is chosen so as to give the minimum value of $B_n$ over all the $s$, $t$ and $b$ within the respective forces. For comparison, a common $b$ parameter is adopted in estimating $B_i$. On the other hand, on the right-hand sides of the figures, we show the results using the $b$ parameter which corresponds to that for minimum energy of an isolated $\alpha$ particle. We can list the features of these figures as follows:

(i) For the BBI force, $\alpha$-cluster matter is more stable than nuclear matter in all cases. However, the "$\alpha$-cluster" corresponding to the minimum energy for the inner nucleon is no longer regarded as a real $\alpha$-cluster, because the two energy curves do not cross with each other but become close around the distance parameters $s$ for the minimum energy. This "$\alpha$-cluster" may be easily broken towards a uniform density of nucleons even under a small external field.

(ii) Taking an inner part of matter using the SII force, nuclear matter becomes more stable than $\alpha$-cluster matter when the nucleon density increases and crosses some critical value. On the contrary, $\alpha$-cluster matter is stable for surface nucleons around the distance parameter corresponding to the minimum energy of matter.

(iii) If we compare the left-hand side and right-hand side of the figures, we see that the size parameter $b$ for $\alpha$-cluster matter increases from that of an isolated $\alpha$-particle

Table I. The parameters for effective inter-nucleon forces.

<table>
<thead>
<tr>
<th>BBI</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$v_n$(MeV)</td>
<td>$r_n$(fm)</td>
<td>$M_n$</td>
</tr>
<tr>
<td>1</td>
<td>389.5</td>
<td>0.7</td>
<td>-0.5290</td>
</tr>
<tr>
<td>2</td>
<td>-140.6</td>
<td>1.4</td>
<td>0.4864</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SII</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$(MeV·fm$^3$)</td>
<td>$\chi_0$</td>
<td>$\delta$(MeV·fm$^3$)</td>
<td>$\delta$(MeV·fm$^3$)</td>
<td>$\delta$(MeV·fm$^3$)</td>
</tr>
<tr>
<td>-1169.9</td>
<td>0.34</td>
<td>585.6</td>
<td>-27.1</td>
<td>9331.1</td>
</tr>
</tbody>
</table>
Figs. 3 and 4. Energy curves per nucleon versus the distance parameter between neighbouring sets of nucleons. For nuclear matter, the scales of upper sides correspond to the nucleon distance.
A. Tohsaki

Table II. Calculated results for α-cluster matter and nuclear matter.

<table>
<thead>
<tr>
<th></th>
<th>BBI force</th>
<th>SII force</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α-cluster matter</td>
<td>nuclear matter</td>
</tr>
<tr>
<td></td>
<td>matter</td>
<td>surface</td>
</tr>
<tr>
<td>b(fm)</td>
<td>2.85</td>
<td>1.73</td>
</tr>
<tr>
<td>s(fm)</td>
<td>2.88</td>
<td>2.95</td>
</tr>
<tr>
<td>t(fm)</td>
<td>2.28</td>
<td></td>
</tr>
<tr>
<td>$B_1$(MeV)</td>
<td>-7.22</td>
<td>-8.34</td>
</tr>
<tr>
<td>$B_2$(MeV)</td>
<td>-10.97</td>
<td>-10.77</td>
</tr>
<tr>
<td>$B_3$(MeV)</td>
<td>-11.39</td>
<td>-11.07</td>
</tr>
<tr>
<td>$K_{sv}$ (MeV)</td>
<td>148.9</td>
<td>73.4</td>
</tr>
<tr>
<td>$U_s^a$(MeV·fm$^{-2}$)</td>
<td>5.7</td>
<td>14.9</td>
</tr>
<tr>
<td>$U_v^a$(MeV·fm$^{-2}$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U_{sv}^a$(MeV·fm$^{-2}$)</td>
<td>52.2</td>
<td>99.4</td>
</tr>
</tbody>
</table>

as the distance parameters $s$ decrease. As a result, both α-cluster matter and nuclear matter show energy minima around almost the same $b$ parameter.

In Table II, the physical quantities are listed. The first column corresponds to the data showing minimum energy for $B_v$, and the second column corresponds to the results to minimum for $B_1$. The parameter $s$ (or $t$) and $b$ used here are those leading to minima for $B_v$ and $B_1$, respectively. For comparison, we list the strength of the α-clustering for an isolated α particle in the final row. As shown in this table, the results strongly depend upon what kind of the effective inter-nucleon forces are adopted. The features are as follows:

(i) The most important point is whether the phase transition from α-cluster matter to nuclear matter takes place or not. The results for the BBI are referred only to α-matter, as the BBI force does not generate phase transition. The fact that the SII gives rise to the phase transition indicates the important role of the existence of three-body forces directly related to the nucleon density. We will discuss later the discrepancy with the conclusion of Brink and Castro.3)

(ii) As for the SII force, the phase transition occurs around 0.85ρ, where ρ is the nucleon density with the minimum energy of nuclear matter. Although the binding energy of $B_v$ for nuclear matter is close to that for α-cluster matter, the uniformity of nuclear matter is so strong that nuclear matter may be stable. On the other hand, as for the BBI force, the phase transition does not appear, but the α-clustering of α-cluster matter is vanishingly weak so as to lead to easy fluidity of nucleons without making compact α-clusters.

(iii) In the case of four surface-nucleons, the phase transition never occurs for two kinds of the effective inter-nucleon forces. Even the BBI force gives stronger α-clustering for surface-nucleons than that for inner nucleons. The α-clustering for the SII force in the surface bears comparison with that for a free α-particle.

(iv) The size parameter $b$ for matter tends towards a larger value than that of a free
$\alpha$-particle. This trend is opposite to the conclusion of our previous work,\cite{4} which was obtained by adjusting the binding energy with an infinite value for $s$ corresponding to a free $\alpha$-particle. The adjusted value is $(3/16)\hbar\omega$ per nucleon, as discussed later, which increases inverse-quadratically as the $b$ parameter decreases. Such large values play an important role for concluding the $b$-dependence of matter.

(v) The change of size parameter $b$ makes little difference to the incompressibility of matter. For the BBI force, the incompressibility of matter interior is about one half of the empirical value,\cite{9} and the incompressibility of the surface is weaker than that of the interior of matter. On the other hand, for the SII force, the incompressibility of the interior of matter is rather strong compared with the empirical value, but the surface incompressibility decreases to about one third of that for matter interior. It is this incompressibility that is affected by the existence of the density-dependent term in the effective inter-nucleon force such as three-body one.

(vi) The fact that the parameters $s$ leading to the minimum energies are almost the same for the surface and the interior of matter for any forces indicates the existence of the most approachable distance between neighbouring sets of four nucleons. If this distance does not change in the fcc configuration, the nucleon density can increase up to $\sqrt{2}\rho$; where $\rho$ is the nucleon density for a simple lattice structure. Since, for the BBI and the SII forces, parameters $s$ are 2.88 fm and 3.23 fm, the ranges of the nucleon density are arranged in $0.17\,\text{fm}^{-3}\leq\rho\leq0.24\,\text{fm}^{-3}$ and $0.12\,\text{fm}^{-3}\leq\rho\leq0.17\,\text{fm}^{-3}$, respectively, where $\rho$ is the nucleon density in possible configurations of nucleons.

In Table III, the calculated results are compared with the empirical values from the Weizaecker mass formula. The binding energy for surface-nucleons, $B_t$, is shown in the right column, and the other $B_i$ are taken for the left column in Table II. The volume term is smaller than the empirical value; on the other hand, the surface term is larger than the empirical value. Especially, we can see the strong force dependence for the surface term. This result shows that it is necessary to investigate the effective inter-nucleon force focusing on the surface term in addition to the volume term in the Weizaecker mass formula. The energy deficit in the volume term compared with the empirical value may be related to the following discussion.

In spite of the exact treatment of the compositeness of the constituent particles in matter, the present model is just a static model in the GC space. All of the physical quantities are discussed only through adiabatic energy curves in variational parameters such as $s$, $t$ and $b$. Therefore, the binding energies per nucleon in $\alpha$-cluster matter and nuclear matter converge to $(3/4)\hbar\omega$ and $(1/4)[E_\alpha+(3/4)\hbar\omega]$, respectively, by letting $s$ approach infinity, where $E_\alpha$ is the total binding energy of a free $\alpha$ particle.

| Table III. Comparison of calculated results with traditional nuclear matter and empirical values. |
|---------------------------------|---------------------------------|---------------------------------|
|                                | Calculated results              | Traditional nuclear matter      | empirical values                |
|                                | BBI    | SII    | BBI    | SII    |                                  |
| $b_\alpha$(MeV)                | 11.4   | 10.2   | 15.7   | 16.0   | 15.56                           |
| $b_\beta$(MeV)                 | 36.9   | 22.4   |        |        | 17.23                           |
| $K_\alpha$(MeV)                | 148.9  | 379.9  | 184.0  | 342.0  | $\sim$300                      |
This \((3/16)\hbar \omega\) in \(a\)-cluster matter was employed as the adjusted energy in our paper.\(^4\) The \((3/4)\hbar \omega\) just corresponds to the zero-point oscillation energy of one particle, which must be removed in a dynamical treatment for the relative motion among all nucleons in matter. We need to consider how much such an unphysical quantity is mixed in the binding energy for matter with a finite distance parameter \(s\). Of course, this vanishes at \(s=0\); namely, it converges to zero due to \([(3/4)\hbar \omega]/N (N \to \infty)\). Brink and Castro\(^3\) insisted without any criterion that this amount can be ignored in the region of \((1/3)\rho\), which is about \(0.04\) fm\(^{-3}\). Let us verify this point. Certainly, this amount is expected to be related to the strength of overlap between neighbouring nucleons. For example, it may be reasonable that the deviation of the diagonal part of \(B^{-1}\)-matrix from unity is a measure of the amount of overlap. This is because \(B^{-1}\) behaves like a diffraction pattern with respect to \(|i-j|\). As mentioned in our previous report,\(^1\) we have shown that this quantity is approximately proportional to \((Q_0^-)^{-6}\), where

\[
Q_0^- = \prod_{n=1}^{\infty} \left[ 1 - \exp\left(-\frac{n}{2} \left(\frac{s}{b}\right)^2\right) \right],
\]

which is shown in Fig. 5. Note that this function coming from the Pauli principle behaves like the Fermi distribution. Of course, we cannot be sure of such a relation between the two functions at the present work.

From this figure, the overlap amounts with \((1/3)\rho\) are about 97% and 65% for the BBI and the SII forces, respectively. If we could consider the change of \(b\) parameter during a compression of matter, these rates will decrease due to the difference between the \(b\)-parameters corresponding to \(\rho\) and \((1/3)\rho\). A rough estimate of the influence on the binding energy coming from such an unphysical quantity is expected to be \(1.5 \sim 3\) MeV, which cannot be neglected. Their conclusion ignores the change of \(b\)-parameter during the pressure process of matter and also the above-mentioned uncertainty factor, which make it difficult to compare \(a\)-cluster matter with traditional nuclear matter. Therefore, we think that, instead of traditional nuclear matter, we should investigate the phase transition phenomena using two models in the same GC space.

For the same reason, the introduction of such adjusted energy as carried out in our paper,\(^4\) may lead to some wrong conclusions. When \(a\)-cluster matter and nuclear matter are compared in the GC space, these adjusted energies are different from each other. This is because the \((3/4)\hbar \omega\) is just the quantity related to the degree of freedom in an isolated particle to be considered. Anyhow, the presence of adjusted
parameters prevents us from investigating the existence of the phase transition from 
$\alpha$-cluster matter to nuclear matter.

The present results on the binding energies per nucleon are not compared with 
those of traditional nuclear matter. We can give the following two reasons: (i) As 
mentioned above, the first is the amount coming from the limitations of the present 
static model in the GC space. (ii) The second is the possibility of the existence of 
more stable configurations for matter than that considered in the present work. In 
$\alpha$-cluster matter, the face-centered cubic ($fcc$) lattice and the body-centered cubic 
($bcc$) lattice are more densely packed configurations compared with the simple lattice 
investigated by Brink and Castro$^3$ who used a finite number of $\alpha$ clusters. However, 
we never construct close packed and uniform density of nucleons introducing the 
spatial distribution of $\alpha$ cluster based on the $fcc$ and the $bcc$ structures.

§ 4. Concluding remarks

We propose a model to investigate the phase transition from $\alpha$-cluster matter to 
nuclear matter in the GC space established by Brink and Bloch.$^2$ This is one of the 
static treatments for matter which takes full account of the Pauli principle among all 
nucleons and effective inter-nucleon forces simultaneously. Four kinds of simple 
lattices, which make the respective sets of independent nucleons in $\alpha$ clusters corre­
spond, move from coinciding positions with each other to constructing the $fcc$ configuration. The starting point structure can be regarded as $\alpha$-cluster matter, and 
the structure with a uniform nucleon density is recognized as nuclear matter in the GC 
space. In the limitations of a static model, the bulk properties of matter such as the 
binding energies for the matter interior and the surface, $\alpha$-clustering in $\alpha$-cluster 
matter and uniformity in nuclear matter are investigated as follows: (i) The physical quantities strongly depend upon what kind of effective inter-nucleon 
forces are used. Especially, the existence of the density conserving term is the 
decisive factor leading to the phase transition in the matter interior. Nevertheless, 
even for the BBI force, the matter interior is close to nuclear matter, focussing upon 
the $\alpha$-clustering in $\alpha$-cluster matter. For the SII force, $0.85\rho$, where $\rho$ is the density 
leading to the minimum binding energy, is the critical density for the phase transition. (ii) For four surface-nucleons, they always form an $\alpha$ cluster with the effective 
inter-nucleon forces except in the region of extremely high density. Alpha-clustering 
for four nucleons in the surface is close to an $\alpha$ particle. (iii) The incompressibility of the matter interior is somewhat high compared with 
the empirical value for the SII force; on the other hand, it is about one-half for the BBI 
force. This also comes from the existence of the density-dependent term in effective 
inter-nucleon forces.

From these results, we can see that $\alpha$ clusters float on the surface of nuclear 
matter.

In order to extend this model, we must not only develop the dynamical treatment 
of $\alpha$-cluster matter and nuclear matter beyond the static model but also introduce the 
fcc and bcc configurations of $\alpha$-cluster matter and their separated distribution. For 
the former improvement, we think that the treatment of dielectric and magnetism in
electromagnetic field may give us some hints. The latter development needs to have an analytical formula of the kernels for the infinite system with the fcc and bcc configurations of α-cluster matter. Recent development of computational technique for the analytical calculation may enable us to carry it out even for the infinite number of α clusters.

In finite nuclei, whether these conclusions apply or not is a very interesting question. The Coulomb interaction as a basic force plays an important role in determining the structure of nuclei. We plan to investigate the behaviour of nuclear matter in the surface region by including the Coulomb interaction.

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