Quantum Einstein Gravity as a Differential Geometry. II

— An Interpretation of Quantum Einstein-Hilbert Action

Based on Intrinsic Orthosymplectic Super-Symmetry —

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Quantum Einstein gravity by Nakanishi yields a large symmetry, $OSp(2D; 2D)(D \geq 3)$. In a new framework, the quantum Einstein gravity is uniquely reconstructed on the basis of the intrinsic $OSp$-symmetry and the nonlinear $\xi$-field realization by Abe. Since the intrinsic $OSp$-covariant "derivation" is defined so that the nonlinearity of the symmetry transformation causes no trouble, the "conventional" Einstein-Hilbert action is derived as a bilinear form on a $C^\infty$-module as proposed in the previous paper. The essential difference between the classical Einstein gravity and the quantum Einstein one is clearly shown through this framework.

§ 1. Introduction

Quantum Einstein gravity (QEG)\(^1\) by Nakanishi is a fundamental quantum-gravitational theory which was constructed on the keynote of the guarantee of the physical S-matrix unitarity. This keynote brings us a remarkable symmetry, $OSp(2D; 2D)(D \geq 3)$. This symmetry should be called a quantum symmetry. The intrinsic $OSp(2D; 2D)$-symmetry, not being an operator symmetry, is its intrinsic part. In this sense, we may well say that the intrinsic $OSp$-symmetry is a 'quantum symmetry' with single quotation marks.*\(^1\) Since QEG is totally a quantum-theoretical version of Einstein's general theory of relativity, it might be also said that QEG is a quantized theory of the geometry of gravity. In QEG, it seems that Einstein's gravitational theory is harmonized with quantum field theory. The most important viewpoint of QEG lies in that the "space-time" is not described with a Riemannian geometry as in the classical theories. Therefore the "conventional" $g_{\mu \nu}$ and $R_{\mu \nu}$ in QEG do not stand for a metric and a Ricci curvature in the classical sense, respectively. In QEG, we cannot conceive the same space-time structure with the one as in the general theory of relativity. But, it does not necessarily mean that we cannot characterize the geometrical structure of QEG. Indeed, the indirect hint of the geometrical characterization has been implicitly shown in the nonlinear $\xi$-field realization theory\(^2\) by Abe and Nakanishi. A generalization\(^3\) of the theory stated by Abe suggests the orientation of the characterization.

In the previous paper,\(^4\) we formally clarified the differential geometrical structure of the gauge-fixing term plus the FP-ghost one in QEG on the basis of the fundamental symmetry, intrinsic $OSp(2D; 2D)$. It was formulated as a bilinear form $\xi$ on $E$-module: $\Gamma(A_D, E) \equiv \Gamma(E)$ and we called the element of $\Gamma(E)$ 'quantum vielbein' field. Here the "space-time" $A_D$ stands for an affine space $R^D \times GL(D)/GL(D)$ and $E$\(^*\)

\(^*\) The 'quantum' here refers to the general framework of quantum theory, but not explicit quantization.
\( \equiv (S \times \mathcal{F}_1) \otimes T(A_0) \) does for the tensor product bundle, where \( S \) and \( \mathcal{F}_1 \) are a spin bundle and a Clifford subalgebra, respectively. The 'quantum'-gravitational field \( (g^{\mu \nu}) \) was introduced as a composite field of the 'quantum vielbein's through the bilinearity of \( \xi \). The reason why we employed a spin bundle lies in Einstein’s gravitational theory: \( SO(1, D-1) \) is the structural group in the tangent bundle \( T(M_D) \) over a \( D \)-dimensional Lorentzian manifold \( M_D \).

The consistency of the theory now requires that the Einstein-Hilbert action — we should say 'quantum' Einstein-Hilbert action — also should be formulated as a bilinear form on \( \Gamma(\mathcal{E}) \) under the following first principle as in Ref. 4):

“The theory should be manifestly intrinsic \( OSp(2D; 2D) \)-covariant”.

As seen in Ref. 4), another symmetry employed is \( SO(1, D-1)^* \) as an internal symmetry. In the above situation, we will know the existence of an intrinsic \( OSp \)-'connection' \( (T^\mu_\nu) \) \((\mu, \nu, \rho = 0, 1, \cdots, D-1)\). Due to the nonlinear \( \xi \)-field realization by Abe, we get the concept of intrinsic “\( OSp \)-gauge” transformation in which no arbitrary function appears. In this sense, the intrinsic \( OSp(2D; 2D) \)-symmetry is a global one. Therefore the new ‘connection’ in this framework differs from such a connection as viewed in the classical gauge theories. Although we employ a “Levi-Civita connection” by analogy with a Riemannian geometry, it is not Riemann-geometrical. This is the reason why the \( R_{\mu \nu} \) is not a Ricci curvature as stated above. The theoretical structure and the logic in this theory are totally different from those in the traditional ones, notwithstanding the outward resemblance.

This paper is organized as follows: In § 2, we recapitulate the content relevant to our problem, but the keynote is fully explained. In § 3, we present a formulation of the ‘quantum’ Einstein action. Then we will again recognize the difference between the Einstein theory and the quantum Einstein one. Section 4 is devoted to the concluding remarks.

Throughout the present paper, we employ the following notation:

\[
\epsilon(X, Y) = \begin{cases} -1: & \text{if both } X \text{ and } Y \text{ obey Fermi-statistics,} \\ +1: & \text{otherwise,} \end{cases}
\]

\[
\epsilon(X) = \epsilon(X, X), \quad \sqrt{+1} = +1, \quad \sqrt{-1} = i,
\]

\[
T_{[\mu \nu]} = \frac{1}{2} (T_{\mu \nu} - T_{\nu \mu}).
\]

§ 2. Preliminaries

We recall the intrinsic \( OSp(2D; 2D) \)-symmetry and the representation theory.\(^{1,3,4}\)

It is the first principle of our theory as repeatedly asserted.

Let \( \{ \hat{d}_{xy} \} \) be the intrinsic \( OSp(2D; 2D) \)-symmetry and \( (\mathcal{V}_{(2D; 2D)}, \hat{\eta}) \) the representation space with a super-metric \( \hat{\eta} \), where the former \( 2D \) stands for the bosonic sector and the latter \( 2D \) does for the fermionic one:

\(^{*b)} \) In Ref. 4), we employed the twofold cover of \( SO(1, D-1) \). But, since the results obtained are all the same, we employ \( SO(1, D-1) \), for simplicity.
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\[ V_{(2D;2D)} = V_{(2D;0)} \oplus V_{(0;2D)}, \]

\[ \tilde{\eta}_{uv} = -\sqrt{\epsilon(U)} \tilde{\eta}(U, V) = \epsilon(U, V) \tilde{\eta}_{vu}, \]

\[ \frac{1}{2} \delta_{xy}(U) = i \sum_{r} c_{xyr}^{uv} V, \]

\[ c_{xyr}^{uv} = -\varepsilon(X, Y) [\eta(U, Y) \delta(X, W) - \epsilon(X, Y) \eta(Y, U) \delta(Y, W)], \]

\[ \sum_{r} \eta(X, Z) \tilde{\eta}(Z, Y) = \delta(X, Y). \quad (4D\text{-dimensional Kronecker delta}) \]

Here the super-metric \( \eta(X, Y) \) or its matrix-inverse \( \tilde{\eta}(X, Y) \) equals zero if \( X \) (resp. \( Y \)) is bosonic and \( Y \) (resp. \( X \)) is fermionic, in consideration of \((2\cdot1)\). Moreover, in the sequel we will be in need of the following self-evident identities in addition to the above expressions:

\[ \epsilon(XY, U) = \epsilon(X, U) \epsilon(Y, U), \]

\[ \sqrt{\epsilon(U)} \eta(U, V) = \sqrt{\epsilon(V)} \eta(U, V). \]

Let \( A_{D} \to V_{(2D;2D)} \), and let the \( C^{\infty} \)-embedding stand for the 4D-dimensional supercoordinate of QEG.\(^{1}\)

\[ U(x) = \{ x^{\mu}, b_{\nu}(x), c^{\nu}(x), \bar{c}_{\sigma}(x); \mu, \nu, \rho, \sigma = 0, 1, \ldots, D-1 \}, \]

where \( x^{\mu} \) is a c-number parameter, \( b_{\nu}(x) \) is the gravitational B-field of Nakanishi, and \( c^{\nu}(x) \) and \( \bar{c}_{\sigma}(x) \) are the gravitational FP-ghosts. (For the concrete form of the super-metric \( \eta \) in terms of the 4D-dimensional supercoordinates, see Ref. 1). Let \( [\partial_{\mu}] = Z \) be the \( \xi \)-field of Abe.\(^{3,4}\) The generalized \( \xi \)-field representation is nothing but the pull-back of \( \{ \delta_{xy} \} \) by \( U \) into the frame bundle \( Fr(A_{D}) \approx A_{D} \times GL(D) \):

\[ \delta_{xy}(\partial_{\mu} U) = \partial_{\mu} \delta_{xy}(U) - \partial_{\nu} \delta_{xy}(x^{\nu}) \cdot \partial_{\sigma} U, \]

— precisely speaking, we should express the left-hand side in \( \delta_{xy}(\partial_{\mu} U)(x), x \in A_{D} \), and so on. Since the nonlinear transform \((2\cdot9)\) is rewritten as

\[ \delta_{xy}(\partial_{\mu} U) = \partial_{\nu} [\delta_{xy}(U) - \delta_{xy}(x^{\nu}) \cdot \partial_{\sigma} U] + \delta_{xy}(x^{\nu}) \cdot \partial_{\sigma} U, \]

putting \( \delta_{xy} = \delta_{xy} - \delta_{xy}(x^{\sigma}) \cdot \partial_{\sigma} \), we easily have \( [\delta_{xy}, \partial_{\mu}] = 0 \). This \( \{ \delta_{xy} \} \) is nothing but the \( OS\pi(2D; 2D) \)-symmetry in QEG. In QEG, it is represented as an operator symmetry.\(^{1}\)

Aiming at the second term of the right-hand side of \((2\cdot9)\), we inductively have the following reciprocal formula:

\[ \delta_{xy}(dx^{\mu}) = \partial_{\nu} \delta_{xy}(x^{\mu}) dx^{\sigma} id \delta_{xy}(x^{\mu}). \]

Thus we find the transformation rule for any intrinsic \( OS\pi \)-tensor. For instance,

\[ \delta_{xy}(h^{\sigma}_{\mu}) = -\partial_{\nu} \delta_{xy}(x^{\sigma}) \cdot h^{\sigma}_{\nu}, \]

and reciprocally (by the help of the formulae in Appendix A)

\[ \delta_{xy}(h^{\sigma \mu}) = \partial_{\sigma} \delta_{xy}(x^{\mu}) \cdot h^{\sigma \mu}. \]
for \( \gamma h(x) = h^{ab} \gamma_a \otimes \partial_{\mu} x \in E_x \). (This \( E_x \) is the standard fibre in \( E \) which was stated in § 1). These transformation formulae play an important role in the next section. (Hereafter the formulae in Appendix A are freely used without notice.) Making use of (2·11)~(2·13), we have

\[
\delta_{xy}(dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{D-1}) = \partial_x \delta_{xy}(x^0) d^D x \det[h_{\mu
u}] + d^D x \det[h_{\mu
u}] h^{\alpha\beta} \delta_{xy}(h_{\alpha\beta}) = 0, \tag{2·14}
\]

that is, the "volume form" \( d^D x \det[h_{\mu
u}] \) is invariant under \( \{ \delta_{xy} \} \). This is a piece of new knowledge. (It was verified by Nakanishi that the above "volume form" is invariant under the intrinsic BRS-transformation.\(^5\) In the previous paper,\(^4\) we applied this fact to our theory.) The identity (2·14) is related to the following content.

In Ref. 4), we derived the gauge-fixing term plus the FP-ghost one in QEG on the basis of the intrinsic \( OSP(2D; 2D) \)-symmetry and the fibre bundle theory. By considering \( A_0 \rightarrow V_{(2D; 2D)} \) and employing the following associated fibre bundle with the spin bundle \( S \)

\[
SO(1, D-1) \rightarrow S \times A_0 \xi_1 \quad (\xi_1: \text{Clifford subalgebra})
\]

\[
\downarrow \pi_1
\]

\[
A_0
\]

we naturally got a bilinear form \( \tilde{\xi} \) on \( \Gamma(E) \) in § 1:

\[
\tilde{\xi}(h, h) = \int_{A_0} d^D x \det[h_{\mu
u}] \frac{1}{2} h^{ab} h^{\alpha\beta} \frac{1}{D} \langle \gamma_a, \gamma_b \rangle \tilde{\xi}(\partial_\mu, \partial_\nu),
\]

where

\[
\langle \gamma_a, \gamma_b \rangle \equiv \text{Tr}(\gamma_a \gamma_b) = D\eta_{ab}
\]

with \( \{ \gamma_a, \gamma_b \} = 2\eta_{ab}, \) \( [\gamma_{ab}] = \text{diag}(+1, -1, \ldots, -1) \), and

\[
\tilde{\xi}(\partial_\mu, \partial_\nu) \equiv -\sum_{a,b} \sqrt{e(U, V)} \bar{\eta}(U, V) \partial_\mu U \cdot \partial_\nu V
\]

\[
= \tilde{\xi}(\partial_\nu, \partial_\mu).
\]

When we put \( (h^{ab} h_{\alpha\beta})(x) = g^{\mu\nu}(x) \) and \( \tilde{\xi}(\partial_\mu, \partial_\nu) = \tilde{\xi}_{\mu\nu}(x) \), (2·16) becomes

\[
\tilde{\xi}(h, h) = \int_{A_0} d^D x \det[h_{\mu
u}] \frac{1}{2} g^{\mu\nu} \tilde{\xi}_{\mu\nu}
\]

\[
= \int_{A_0} d^D x \det[h_{\mu
u}](\mathcal{L}_{\text{CF}} + \mathcal{L}_{\text{FP}}).
\]

As seen from (2·18), \( \tilde{\xi}_{\mu\nu}(x) \) is a symmetric \( GL(D) \)-tensor: \( \tilde{\xi}_{\mu\nu}(x) = dx^\mu \otimes dx^\nu \). The \( GL(D) \)-tensor \( \tilde{\xi}_{\mu\nu}(x) \) — more rigorously speaking it is an intrinsic \( OSP \)-covariant tensor as is shown below — is formally regarded as a 'metric' on \( Fr(A_0) \). Here, whether the \( GL(D) \)-tensor \( \tilde{\xi}_{\mu\nu}(x) \) is an intrinsic \( OSP \)-tensor or not, we must investigate.
An intrinsic $OSp$-tensor automatically implies to be a $GL(D)$-tensor, but the converse does not generally hold. (This is caused by the fact that the nonlinear $\xi$-field realization theory of the intrinsic $OSp(2D; 2D)$-symmetry implicitly contains the information of the general coordinate transformation, in fact, the infinitesimal general coordinate transformation disguised "itself" as the intrinsic BRS "derivation". Therefore we need to check whether the nonlinearity of the $\xi$-field realization causes troubles or not, by explicit calculation.)

By (2·9), we obtain

$$\delta_{XY}(\xi_{\mu\nu}) = -\partial_\nu \delta_{XY}(x^\sigma) \cdot \xi_{\sigma\nu} - \partial_\nu \delta_{XY}(x^\sigma) \cdot \xi_{\mu\sigma} \quad (2·20)$$

after some calculation (Appendix B). Thus we understand that the nonlinearity causes no trouble. This equation (2·20) is very important for the argument in the next section.

Making use of (2·14) and (2·20), we see that the bilinear form $\tilde{\xi}$ in (2·19) is intrinsic $OSp$-invariant.

Although discussion lacks sequence, since we need a connection in order to characterize the 'quantum' Einstein action, we make a little digression on the connection theory related to our work.

Let $\tilde{\omega}$ be an Ehresmannian-spin-connection form on $S$. Let $s_U (U \subset A_0; U$ is an open set) be a local cross-section of $S$. Then the $so(1, D-1)$-valued 1-form on $A_0$ is given by the pull-back of $\tilde{\omega}$ by $s_U$:

$$\tilde{\omega}^o s_U^* = \omega_{uv} \equiv \frac{1}{2} \dot{S}_{ab} \omega_{uv}^{ab} \quad (2·21)$$

where $\dot{S}_{ab} \equiv (1/4)[\dot{\gamma}_a, \dot{\gamma}_b]$.

The spin connection $\omega$ causes $SO(1, D-1)$-covariant exterior derivation $D(\omega)$ such that

$$D(\omega) : \Gamma(E) \rightarrow \Gamma(E \otimes T^*(A_0)) \quad (2·22)$$

$$h \quad \mapsto \quad dh + \kappa [\omega, h]$$

where $\kappa$ is a non-zero constant. In a concrete form, for $h(x) = \gamma^\nu \otimes \partial_\mu |_x \in E_x (\gamma^\nu \equiv \tilde{\gamma}_a h^{av})$

$$(D_\mu \gamma^\nu)(x) = \left( \partial_\mu \gamma^\nu + \frac{\kappa}{2} [\dot{S}_{ab}, \dot{\gamma}_c] \omega_{ab}^{\mu\nu} h^{cv} \right)(x)$$

$$= \tilde{\gamma}_a (\partial_\mu h^{av} + \kappa \omega^{ab}_\mu h_{\nu}^v)(x) \quad (2·23)$$

In (2·22), the derivative $D(\omega) h$ is, of course, covariant under the following cocycle condition:

$$\omega_{\nu} = \frac{1}{\kappa} f^{-1} df + Ad(f^{-1}) \omega f \quad (2·24)$$

where $f|_{U \cap V}: A_0 \rightarrow SO(1, D-1)$ with $U \cap V \neq \phi$.

But $D(\omega) h$ is not $OSp(2D; 2D)$-covariant. As will be seen in the next section, we
need the intrinsic ‘connection’ for OSp-covariance of the theory. And the spin connection (ω) is expressed in terms of the ‘quantum vielbein’ (h), as we will understand from (3·10) below.

§ 3. Formulation of the ‘quantum’ Einstein action

First, we investigate the transformation of \( D_\mu h^{\alpha \nu} \) under \( \{ \delta_{XY} \} \), intrinsic \( OSp(2D; 2D) \)-symmetry. (Hereafter the cumbersome “intrinsic” and “(2D; 2D)” are omitted.) From the property of \( \delta_{XY} \), we obtain

\[
\delta_{XY}(D_\mu h^{\alpha \nu}) = - \partial_\mu \delta_{XY}(x^\sigma) \cdot D_\sigma h^{\alpha \nu} + \partial_\sigma \delta_{XY}(x^\nu) \cdot D_\mu h^{\alpha \sigma} + \partial_\mu \partial_\sigma \delta_{XY}(x^\nu) \cdot h^{\alpha \sigma} . \tag{3·1}
\]

This is obviously not OSp-covariant. In order to offset the absence of OSp-covariance, we must introduce a new ‘connection’ \( (\Gamma^{\mu \nu}_\alpha) \) such that

\[
\delta_{XY}(D_\mu h^{\alpha \nu} + \Gamma^{\nu \rho \sigma} h^{\alpha \rho}) = - \partial_\mu \delta_{XY}(x^\sigma) \cdot (D_\sigma h^{\alpha \nu} + \Gamma^{\nu \rho \sigma} h^{\alpha \rho}) + \partial_\sigma \delta_{XY}(x^\nu) \cdot (D_\mu h^{\alpha \sigma} + \Gamma^{\sigma \rho \mu} h^{\alpha \rho}) . \tag{3·2}
\]

Here we require that “the ‘connection’ should be expressed in terms of the fundamental fields and their derivatives of the first order only”. (This reason is stated later.) In Eq. (3·2), since the operand \( (D_\mu h^{\alpha \nu} + \Gamma^{\nu \rho \sigma} h^{\alpha \rho}) \) is an OSp-tensor, we may put

\[
D_\mu h^{\alpha \nu} + \Gamma^{\nu \rho \sigma} h^{\alpha \rho} = \Phi^{\nu \rho \sigma} h^{\alpha \rho} . \tag{3·3}
\]

Then Eq. (3·2) may be also regarded as a homogeneous “OSp-differential” equation for \( \Phi^{\nu \rho \sigma} h^{\alpha \rho} \). On the other hand, from (3·1) and (3·2), we obtain the following “OSp-gauge” transformation:

\[
\delta_{XY}(\Gamma^{\nu \rho \sigma}) = - \partial_\mu \delta_{XY}(x^\sigma) \cdot \Gamma^{\nu \rho \sigma} - \partial_\sigma \delta_{XY}(x^\sigma) \cdot \Gamma^{\nu \rho \sigma} + \partial_\rho \delta_{XY}(x^\nu) \cdot \Gamma^{\sigma \rho \mu} - \partial_\rho \partial_\mu \delta_{XY}(x^\sigma) . \tag{3·4}
\]

Note that no arbitrary infinitesimal function appears in (3·4). In this sense, the above transformation is a global transformation. On the analogy of a Riemannian geometry we decompose the ‘connection’ \( (\Gamma^{\nu \rho \sigma}) \) into the “Levi-Civita part” and the remainder:*

\[
\Gamma^{\mu \nu \sigma} = \left\{ \begin{array}{c} \mu \\ \nu \\ \sigma \end{array} \right\} + \psi^{\mu \nu \sigma} , \tag{3·5}
\]

where

\[
\left\{ \begin{array}{c} \mu \\ \nu \\ \sigma \end{array} \right\} = \left\{ \begin{array}{c} \mu \\ \nu \\ \sigma \end{array} \right\} = \frac{1}{2} g^{\rho \sigma} (\partial_\sigma g_{\mu \nu} + \partial_\nu g_{\rho \sigma} - \partial_\rho g_{\mu \nu}) . \tag{3·6}
\]

From (3·5), without loss of generality, Eq. (3·3) may be reformed as

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* On these points, the author is grateful to Professor Nakanishi and Dr. Abe for many discussions and valuable suggestions which facilitated the present work.
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\[ D_{\mu}h^{av} + \left\{ \nu \right\}_{\mu\sigma} h^{a\sigma} = \Phi^{\nu}_{\mu\rho} h^{a\rho}. \]  \hspace{1cm} (3.7)

(Hereafter we use the following expressions \( D_{\mu}h^{av} + \left\{ \nu \right\}_{\mu\sigma} h^{a\sigma} = \nabla_{\mu}h^{av} + \kappa \omega^{ab}_{\mu} h^{v}_{b} \equiv D_{\mu}h^{av} \), according to circumstances.) The 'connection' \( \left\{ \nu \right\}_{\mu\sigma} \) satisfies

\[ \delta_{XY} \left( \left\{ \nu \right\}_{\mu\sigma} \right) = -\partial_{\nu}\delta_{XY}(x^{\nu}) \cdot \left\{ \nu \right\}_{\mu\rho} - \partial_{\rho}\delta_{XY}(x^{\rho}) \cdot \left\{ \nu \right\}_{\mu\sigma} + \partial_{\sigma}\delta_{XY}(x^{\sigma}) \cdot \left\{ \rho \right\}_{\mu\sigma} - \partial_{\mu}\partial_{\sigma}\delta_{XY}(x^{\nu}) \]

\hspace{1cm} (3.8)

from which (3.4) gives

\[ \delta_{XY}(\Phi^{\nu}_{\mu\sigma}) = -\partial_{\nu}\delta_{XY}(x^{\nu}) \cdot \Phi^{\nu}_{\mu\sigma} - \partial_{\sigma}\delta_{XY}(x^{\sigma}) \cdot \Phi^{\nu}_{\mu\rho} + \partial_{\rho}\delta_{XY}(x^{\rho}) \cdot \Phi^{\nu}_{\mu\sigma}. \] \hspace{1cm} (3.9)

We have only to solve Eq. (3.9) for the construction of our theory. Obviously \( \Phi^{\nu}_{\mu\sigma} \) is an \( OSp \)-tensor. It is inferred that \( \Phi^{\nu}_{\mu\sigma} \) except the trivial solution zero does not exist in the pure 'quantum' gravity in which we have only \( (b_{\nu}, c^{\nu}, \bar{c}_{\nu}) \), \( h^{av} \) and \( h^{a\nu} \) as the fundamental fields, as is shown in the following.

Next, we require the following natural postulates\(^*\) in order to determine \( \Phi^{\nu}_{\mu\sigma} \):

The \( \Phi^{\nu}_{\mu\sigma} \) is

- (0) \( OSp \)-tensorial,
- (i) Translational invariant,
- (ii) Internal local \( SO(1, D-1) \)-invariant,
- (iii) Expressible in terms of the fundamental fields and the first order derivatives only—this postulate is related to the physical S-matrix unitarity.

From (ii), the solution can depend on \( h^{av} \) and \( h^{a\nu} \) only through \( g^{av} \) and \( g_{\mu\nu} \). We intend that the solution is expressed in terms of the \( OSp \)-tensors and their contracted tensors. Since such an arbitrary multiplier as \( g^{av} \zeta_{av} \) is \( OSp \)-invariant as shown in § 2, this has no direct effect on the covariance. Now, we show by reduction to absurdity that the trivial solution \( \Phi^{\nu}_{\mu\sigma} = 0 \) only exists.

[A] In the case of the solution involving only the 'quantum vielbein'

We have only to check on the polynomial of the minimal degree in \( g^{av} \) and \( g_{\mu\nu} \). The odd-rank \( OSp \)-tensor in this problem cannot be composed of \( g^{av} \) and \( g_{\mu\nu} \) only without use of differentiation(\( \partial_{\nu} \)). But, the use of the differentiation causes some non-vanishing inhomogeneous terms by the \( \delta_{XY} \)-operation. Indeed, although we have a nontrivial \( GL(D) \)-tensor \( \Phi^{\nu}_{\mu\rho} = g^{av}_{\nu\rho} g_{av \rho} \) from (i), (ii) and (iii), then we have

\[ \delta_{XY}(\Phi^{\nu}_{\mu\rho}) = -\partial_{\nu}\delta_{XY}(x^{\nu}) \cdot \Phi^{\nu}_{\mu\sigma} - \partial_{\sigma}\delta_{XY}(x^{\sigma}) \cdot \Phi^{\nu}_{\mu\rho} + \partial_{\rho}\delta_{XY}(x^{\rho}) \cdot \Phi^{\nu}_{\mu\sigma} - \partial_{\mu}\partial_{\sigma}\delta_{XY}(x^{\nu}). \]

Similarly we have another nontrivial \( GL(D) \)-tensor \( \Phi^{\nu}_{\mu\sigma} = g^{av}_{\nu\rho} \partial_{\nu}g_{av \rho} \), but then we have

\[ \delta_{XY}(\Phi^{\nu}_{\mu\rho}) = \partial_{\nu}\delta_{XY}(x^{\nu}) \cdot \Phi^{\nu}_{\mu\sigma} - \partial_{\sigma}\delta_{XY}(x^{\sigma}) \cdot g^{av}_{\nu\rho} g_{av \rho} - \partial_{\rho}\partial_{\sigma}\delta_{XY}(x^{\nu}) \cdot g^{av}_{\nu\rho} g_{av \rho}. \]

These, of course, conflict with (0).

\(^*\) See the footnote on p. 1348.
In the case of the solution involving the fields other than the 'quantum vielbein' 

In this case, the nontrivial solution is expressible in terms of $b_{\nu}$, $c^\sigma$ and $\xi_{\sigma}$. In § 2, we have fully prepared for the answer. From (0), (i) and (ii), the solution must be expressed in terms of the combination of the $OSp$-tensors $\xi_{\mu\nu}$ and $g^{\mu\nu}$. We have the odd-rank $OSp$-tensor $F_{\mu\nu} = g^{\nu\lambda} \nabla_{\mu} \xi_{\lambda\sigma}$ or $g^{\nu\lambda} \nabla_{\mu} \xi_{\lambda\rho}$, but, of course, these conflict with (iii). Since we cannot use any extra differentiation, there does not exist the nontrivial solution.

Thus we infer the solution to Eq. (3.9) is equal to zero and unique. Hence Eq. (3.7) implies

$$D\nu h^{\nu\sigma} = D\nu h^{\nu\sigma} + \left\{ \nu \right\}_\sigma h^{\sigma\sigma} = 0.$$ (3.10)

We should note that Eq. (3.10), corresponding to the vielbein postulate in the classical theories, was logically obtained in our framework for the first time. Thus the 'connection' 1-form satisfies $\Gamma_{\mu\nu} = (\nu)$, and it should be called $OSp$-connection' or 'quantum Levi-Civita connection' (not a geometrical Leve-Civita connection).

It is now a straightforward calculation to obtain the 'quantum' Einstein action. Since Eq. (3.10) is written as

$$D\nu \gamma^\mu = \nabla_{\nu} \gamma^\mu + \kappa [\omega_{\mu}, \gamma^\mu] = 0, \quad \gamma^a = \tilde{\gamma}_{\sigma} h^{\sigma\sigma},$$ (3.11)

we obtain the following “Ricci formula” for the $OSp$-covariant derivation ($D\nu$) with the help of the Jacobi identity:

$$0 = [D\nu, D\sigma] \gamma^\rho = R^{\rho}_{\sigma\mu\nu} \gamma^\rho + \kappa [\Omega_{\mu\nu}, \gamma^\rho]$$

$$= \tilde{\gamma}_{a} (R^{\rho}_{\sigma\mu\nu} h^{\sigma\sigma} + \kappa \eta_{cb} \Omega^{ab}_{\mu\nu} h^{\sigma\sigma}),$$ (3.12)

where

$$\Omega_{\mu\nu} = 2 \partial [\omega_{\mu}, \omega_{\nu}] + \kappa [\omega_{\mu}, \omega_{\nu}]$$

$$= \frac{1}{2} \tilde{S}_{ab} \left[ (\partial_{\mu} \omega^{ab}_{\nu} + \kappa \omega^{ac}_{\mu} \omega_{c\nu}^{b}) - (\mu \leftrightarrow \nu) \right]$$ (3.13)

and

$$\Omega^{ab}_{\mu\nu} = (\partial_{\mu} \omega^{ab}_{\nu} + \kappa \omega^{ac}_{\mu} \omega_{c\nu}^{b}) - (\mu \leftrightarrow \nu).$$ (3.14)

In Eq. (3.12), putting $R^{\rho}_{\sigma\mu\nu} = R_{\rho\mu\nu} \equiv R_{\mu\sigma}$ and suitably multiplying it by $\gamma^a$ from the right or the left, we obtain

$$\frac{1}{\kappa} R = - h^{\sigma\sigma} \Omega_{\sigma\mu\nu} \chi_{\mu\nu}, \quad R = g^{\sigma\rho} R_{\mu\sigma}.$$ (3.15)

In Eq. (3.12), $\Omega$ is a curvature 2-form, $\Omega_{\nu} \in \Gamma((S \times_{Ad} S_0(1, D-1)) \otimes T^*(Ad|U))$,

$$\Omega(x) = \frac{1}{2} \Omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu.$$ (3.16)

Now, we define a bilinear functional as follows:
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\[ \varphi: \mathfrak{so}(1, D-1) \longrightarrow \text{Hom}(\mathfrak{c} \times \mathfrak{c}, \mathbb{R}) \]  \hspace{1cm} (3.17)

\[ \frac{1}{2} \mathcal{S}_{ab} \quad \longrightarrow \quad \frac{1}{2} \mathcal{Q}_{ab} \]

with

\[ \frac{1}{2} \mathcal{Q}_{ab} (\tilde{\gamma}_c, \tilde{\gamma}_d) = \frac{1}{D} \left( \frac{1}{2} \mathcal{S}_{ab} \tilde{\gamma}_c \tilde{\gamma}_d \right) = \frac{1}{D} \text{Tr} \left( \frac{1}{2} \mathcal{S}_{ab} \tilde{\gamma}_c \tilde{\gamma}_d \right), \]  \hspace{1cm} (3.18)

and then we have

\[ \varphi(O)(x) = \frac{1}{2} \mathcal{Q}_{ab} \otimes O_x^{ab}, \]  \hspace{1cm} (3.19)

where

\[ O_x^{ab} = \frac{1}{2} O^{ab}_{\mu}(x) dx^\mu \wedge dx^\nu. \]  \hspace{1cm} (3.20)

Thus we get the following bilinear form on \( \mathcal{E}_x \):

\[
\varphi(O)(h, h)(x) = h^a h^{ab} \frac{1}{D} \left\langle \frac{1}{2} \mathcal{S}_{ab} \tilde{\gamma}_c \tilde{\gamma}_d \right\rangle O_x^{ab}(\partial_\mu, \partial_\nu) \\
= -\frac{1}{2} \left( h_{a}^{a'} h^{b'} O^{ab}_{\mu}(x) \right) = \frac{1}{2\kappa} R(x) \]  \hspace{1cm} (3.21)

from (3.15). This (3.21) is the 'quantum' Einstein Lagrangian.

Summing up (2.19) and (3.21), we consequently get the following bilinear forms on \( \Gamma'(\mathcal{E})^* \):

\[
(\varphi(O) + \bar{\xi})(h, h) = \int_{\mathbb{R}^{d}} d^D x \text{det}[h_{\mu\nu}] \frac{1}{2D} h^{a_{\nu}} h^{a_{\mu}} \left[ \left\langle \mathcal{S}_{cd} \tilde{\gamma}_a \tilde{\gamma}_b \right\rangle O_x^{cd}(\partial_\mu, \partial_\nu) \right. \\
+ \left. \left\langle \tilde{\gamma}_a, \tilde{\gamma}_b \right\rangle \bar{\xi}_x(\partial_\mu, \partial_\nu) \right] \\
= \int_{\mathbb{R}^{d}} d^D x \left[ \mathcal{L}_g + \mathcal{L}_{\text{CP}} + \mathcal{L}_{\text{FP}} \right]. \]  \hspace{1cm} (3.22)

§ 4. Concluding remarks

As understood from this formulation, the 'connection' \( (\Gamma'^{\mu}_{\nu}) \) is not explained within the usual connection theory. In this sense, the \( \text{OSp} \)-'connection' is not Riemann-geometrical. This is caused by the fact that the \( \text{OSp} \)-symmetry is a 'quantum symmetry'. As stated in Ref. 4) the 'vielbein' based on the \( \text{OSp} \)-symmetry is not a geometrical vielbein, but a 'quantum' field at Lagrangian level. In general, a vielbein is nothing but a moving frame on a smooth manifold. If we dare to refer to it in our theory, the \( \bar{\xi} \)-field \( ([\partial_\nu U] = \mathbb{E}) \) may have correspondence to it (of course, not

\[^{*)} \text{R}^2\text{-type Lagrangian, which causes unitarity-trouble, is not formulated as a bilinear form on } \Gamma'(\mathcal{E}). \]  

Since we should lay more emphasis upon the unitarity than the renormalizability, this bilinear formulation is important.
a true vielbein). Thus the conventional $g_{\mu\nu}$ composed of the 'quantum vielbein' is not a metric. And so we should not interpret $\nabla_{\mu}g^{\rho\sigma}=0$ (reciprocally $\nabla_{\nu}g^{\rho\sigma}=0$ (A·13)) as a metric compatibility as in a Riemannian geometry:

$$0=(\nabla_xg)(Y, Z)=\nabla_x(g(Y, Z))-g(\nabla_xY, Z)-g(Y, \nabla_xZ),$$

for $Y, Z \in \Gamma(M, TM)$. After all, the $OSp$-connection $\Gamma^{\mu}_{\nu\rho}$ represented by the $g^{\mu\nu}$'s and the $g_{\mu\nu}$'s is not a geometrical Levi-Civita connection, but a 'quantum'-composite field.

We have clarified the "geometrical structure" of QEG based on the 'quantum symmetry', intrinsic $OSp(2D; 2D)$, in Ref. 4) and this paper.*l

From the viewpoint of the canonical quantization of the 'quantum vielbein' formalism, this reconstruction of QEG is not yet complete, because the freedom of internal local $SO(1, D-1)$-symmetry remains. It is the next work how to formulate this in the application-scope of our framework.

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**Appendix A**

First, we collect conventions and some useful formulae in this paper:

$$\{ \hat{\gamma}_a, \hat{\gamma}_b \} \equiv \hat{\gamma}_a \hat{\gamma}_b + \hat{\gamma}_b \hat{\gamma}_a = 2\eta_{ab}1_D, \quad \eta_{ab} = \text{diag}(+1, -1, \cdots, -1) \quad (D-1)\text{times}$$

$$\hat{S}_{ab} \equiv \frac{1}{2} \hat{\gamma}_a \hat{\gamma}_b \equiv \frac{1}{4} \left[ \hat{\gamma}_a, \hat{\gamma}_b \right] \equiv \frac{1}{4} \left( \hat{\gamma}_a \hat{\gamma}_b - \hat{\gamma}_b \hat{\gamma}_a \right),$$

$$\left[ \frac{1}{2} \hat{S}_{ab}, \frac{1}{2} \hat{S}_{cd} \right] = f_{[ab][cd]}^{[ef]} \frac{1}{2} \hat{S}_{ef}, \quad f_{[ab][cd]}^{[ef]} = \frac{1}{4} \left[ \left( \eta_{ac}\delta^e_d \delta^f_d - \eta_{bc}\delta^e_d \delta^f_d - \eta_{ad}\delta^e_d \delta^f_c + \eta_{bd}\delta^e_d \delta^f_c \right) = \delta_{ef} \right],$$

$$\left[ \hat{S}_{ab}, \hat{\gamma}_c \right] = \eta_{cb} \hat{\gamma}_a - \eta_{ca} \hat{\gamma}_b,$$

$$\langle \hat{\gamma}_a, \hat{\gamma}_b \rangle \equiv \text{Tr}(\hat{\gamma}_a \hat{\gamma}_b) = D \eta_{ab},$$

$$\langle \hat{S}_{ab} \hat{\gamma}_c, \hat{\gamma}_d \rangle \equiv \text{Tr}(\hat{S}_{ab} \hat{\gamma}_c \hat{\gamma}_d) = \frac{D}{2} \left( \eta_{da} \eta_{bc} - \eta_{db} \eta_{ac} \right),$$

$$\gamma^{\mu} \equiv \hat{\gamma}_a h^{\mu a}, \quad g^{\mu\nu} \equiv h^{\mu a} h^{\nu a},$$

$$\{ \gamma^{\mu}, \gamma^{\nu} \} = 2g^{\mu\nu}1_D .$$

Next, we give some corresponding formulae for the 'quantum vielbein' to the

*l It is instructive that the quantum theory of gravity is constructed from a 'quantum' principle.
classical theories:

Let \([h\alpha\nu]\) be the inverse of \([h^\alpha\nu]\), i.e.,

\[
h_{\alpha\nu}h_{\beta\mu} = \delta_\mu^\nu, \quad h^{\alpha\nu}h_{\beta\nu} = \delta_\beta^\alpha,
\]

then

\[
h^{\alpha\nu}h^{\beta\mu} = h^{\alpha\nu}h_{\epsilon\mu}\eta^{\beta\epsilon} = \eta^{\alpha\beta},
\]

and similarly

\[
h_{\alpha\nu}h_{\beta\mu} = \eta_{\alpha\beta}.
\]

Let \([g_{\mu\nu}]\) be the inverse of \([g^{\mu\nu}]\), then from (A·7) and (A·8)

\[
g_{\mu\sigma} = h^\alpha_{\mu\nu}h_{\alpha\sigma}.
\]

Now, the following formulae are the subsequent ones to Eq. (3·10) in § 3. Combining (3·10) with (A·8), we have

\[
D_{\mu}h_{\alpha\nu} = \left\{ \begin{array}{l}
\sigma \\
\mu
\end{array} \right\} h_{\alpha\sigma}.
\]

From (3·10) and (A·7), we have

\[
D_{\mu}g_{\rho\sigma} = - \left\{ \begin{array}{l}
\rho \\
\mu
\end{array} \right\} g_{\sigma\epsilon} - \left\{ \begin{array}{l}
\sigma \\
\mu
\end{array} \right\} g_{\epsilon\rho} = \partial_{\mu}g_{\rho\sigma} \quad \text{or} \quad \nabla_{\mu}g_{\rho\sigma} = 0.
\]

This result, of course, guarantees the existence of the 'quantum Levi-Civita connection' \((\xi_{\mu\nu})\).

**Appendix B**

--- Check on (2·20) ---

Since the \(\xi_{\mu\nu}\) is given by (2·18) together with (2·2), due to the \(\xi\)-field realization (2·9), we have

\[
\delta_{XY}(\xi_{\mu\nu}) = 2\sum_{\mu,\nu} \bar{\eta}_{\mu\nu}\partial_{\mu}\delta_{XY}(U) \cdot \partial_{\nu} V - \partial_{\mu}\delta_{XY}(x^\sigma) \cdot \xi_{\sigma\nu} - \partial_{\nu}\delta_{XY}(x^\sigma) \cdot \xi_{\mu\sigma}.
\]

From (2·3)~(2·7), the first term of the right-hand side in (B·1) becomes

\[
i \sum_{\nu,\omega} \sqrt{\epsilon(V)} \sqrt{-\epsilon(XY, V)} [\delta(Y, V) \delta(X, W) - \epsilon(X, Y) \delta(X, V) \delta(Y, W)] \partial_{\nu} W \cdot \partial_{\omega} V
\]

\[
= i \left[ \frac{\sqrt{\epsilon(Y)} \sqrt{-\epsilon(XY, Y)} \partial_{\nu} X \cdot \partial_{\nu} Y - \sqrt{\epsilon(X)} \sqrt{-\epsilon(XY, X)} \epsilon(X, Y) \partial_{\nu} Y \cdot \partial_{\nu} X}{2} + \sqrt{\epsilon(Y)} \sqrt{-\epsilon(XY, Y)} \partial_{\nu} X \cdot \partial_{\mu} Y - \sqrt{\epsilon(X)} \sqrt{-\epsilon(XY, X)} \epsilon(X, Y) \partial_{\nu} Y \cdot \partial_{\mu} X} \right],
\]

where the property \(\xi_{\mu\nu} = \xi_{\nu\mu}\) was used. Since \(\epsilon(X, Y) = \epsilon(Y, X)\) and (2·6), the following identity holds:
\[ \sqrt{\epsilon(Y)} \sqrt{-\epsilon(XY, Y)} = \sqrt{\epsilon(X)} \sqrt{-\epsilon(XY, X)}, \]

(B·3)

and hence (B·2) vanishes. Therefore (B·1) becomes (2·20).

References