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Exploration of the Q factor for a parallel RLC circuit

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An important property of oscillating systems like RLC circuits is the Q factor, which quantifies the strength of damping in the system. The Q factor is inversely proportional to the resistance for a series RLC circuit but increases with the resistance in a parallel RLC circuit. The surprising behavior of the parallel RLC circuit makes building and modeling this circuit an interesting project for a student laboratory. We describe an experiment that has been performed to explore this topic, share an example of the results that can be obtained, and suggest analyses that students might perform. © 2022 Published under an exclusive license by American Association of Physics Teachers.

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I. INTRODUCTION

Resonant circuits that contain combinations of resistors, capacitors, and inductors (known as RLC circuits) are one of the basic building blocks of modern electronics. Their ability to be tuned to a certain frequency makes them very useful in scientific instrumentation as, for example, a first step in signal extraction.¹ A basic understanding of RLC circuits can be gained using concepts of the harmonic oscillator, but a deeper look into such systems reveals more complex behavior. RLC circuits are, therefore, an interesting system for upper-level physics students to explore.^{2,3} For example, a student could use basic concepts of the harmonic oscillator to develop a simple model describing the current in the circuit but would quickly find that the model does not accurately predict how the circuit actually behaves. They would then need to determine where the model fails and revise it accordingly. Such a project provides an excellent opportunity to learn about the scientific process in a laboratory.

An important concept of oscillating systems is the quality factor (also known as the Q factor, or just Q), which quantifies the dissipation in the system. The Q factor is inversely proportional to the damping in the system; therefore, as the strength of the dissipation increases, the Q factor decreases. In an RLC circuit, energy is usually dissipated by electrical resistances in the circuit.

Interesting behavior becomes apparent when comparing the Q factor for series (Fig. 1, top) and parallel (Fig. 1, bottom) RLC circuits. For the series configuration, Q decreases as R increases,⁴⁻⁶ exactly as one might expect for a component that dissipates energy. For the parallel circuit, however, the opposite is found: Q increases as R increases.^{4,6} At first, this behavior appears to be counterintuitive. A resistor is typically a component that dissipates energy, so intuitively one might expect that increasing the value of a resistor should increase the dissipation in the system. Surprisingly, most textbooks that treat the parallel RLC circuit state the result but do not explain this behavior, though one textbook makes a brief mention of it when analyzing leakage current in the capacitor.⁶ Some previous studies of RLC circuits have also shown this effect,³ but do not elaborate on it. How is it possible that increasing the resistance in the parallel RLC circuit can actually reduce the energy loss? Studying the circuit's behavior to come up with an answer to this question is an excellent project for students to explore in a laboratory activity.

In this paper, we present an activity for students in an electronics or advanced laboratory class that explores the

behavior of the parallel RLC circuit. We begin in Sec. II by presenting the theory of resonant circuits, drawing parallels between RLC circuits and harmonic oscillators, and developing a simple model for the dependence of the Q factor on the circuit resistance. In Sec. III, we demonstrate an experiment where we use the decay of transient oscillations to determine the circuit's Q factor and then in Sec. IV, we show that the simple model is not sufficient to accurately predict the behavior of the circuit. We provide a revised model that includes non-ideal factors in the circuit, and we show that this updated model is consistent with the experiment. Finally, we propose an explanation for the difference in behavior between the parallel and series circuits by examining energy transfer between the circuit components and show an example analysis that students can perform to demonstrate the energy losses in the circuit. In the end, it will be shown that the behavior of the parallel RLC circuit makes for an instructive project that includes non-trivial behavior, which can be accurately modeled and explained using concepts accessible to most advanced undergraduate students.

II. BACKGROUND

A. Damped harmonic oscillator

RLC circuits can be understood as a damped harmonic oscillator^{7,8} that consists of a mass, m , attached to a spring with spring constant k , along with a velocity dependent damping characterized by b . As we are only concerned with the transient response of the system, there is no driving force and the system is allowed to evolve freely. The equation of motion for x , the displacement from equilibrium, is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2x = 0, \quad (1)$$

where $\beta = b/2m$ is the damping constant and $\omega_0 = \sqrt{k/m}$ is the resonant (or natural) frequency of the system. The solution to Eq. (1) in the underdamped region is

$$x(t) = A_0e^{-t/\tau} \cos(\omega_1 t - \delta), \quad (2)$$

where A_0 is the initial amplitude of the transient oscillations, δ is a phase offset, $\tau = 1/\beta$ is the time constant for the decay envelope of the oscillations, and $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ is the natural frequency of the damped oscillator.

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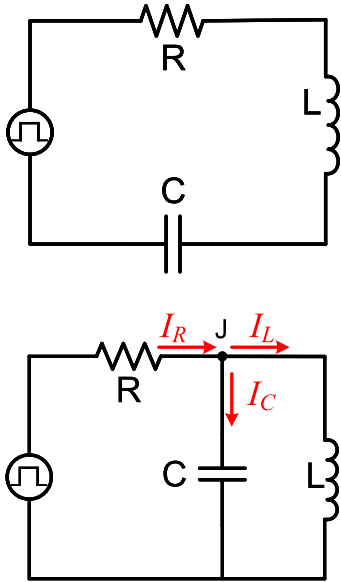


Fig. 1. Idealized RLC circuits showing the series configuration (top) and parallel configuration (bottom). The parallel circuit also shows the currents at junction J when the pulse generator outputs a positive voltage.

The quality factor is proportional to the ratio between the natural frequency of the damped oscillator and the damping constant given by^{7,8}

$$Q \equiv \frac{\omega_1}{2\beta} = \sqrt{\left(\frac{\omega_0}{2\beta}\right)^2 - \frac{1}{4}}. \quad (3)$$

This definition of Q is equivalent to the ratio of the energy stored in the system to the energy lost in one cycle, so Q can also be used as a measure of the amount of energy leaving the system (i.e., damping). Importantly, as the damping coefficient b increases, Q decreases.

Equation (3) also gives us a method for determining Q for a damped harmonic oscillator from the decay of the system's transient oscillations. Since β is related to τ (the time constant for the decay envelope of the transient oscillations), we can compute the oscillator's Q-factor by measuring the oscillation frequency and this time constant.

B. Series RLC circuit

The series RLC circuit, shown in the top of Fig. 1, is mathematically analogous to the damped harmonic oscillator. In this case, Kirchhoff's loop rule yields a second-order differential equation for the current, I ,⁴⁻⁶

$$\ddot{I} + 2\beta\dot{I} + \omega_0^2 I = 0, \quad (4)$$

where the resonant frequency is $\omega_0 = \sqrt{1/LC}$ and the dissipation factor is $\beta = R/2L$. This leads to the same decaying exponential solution as Eq. (2), only with the initial amplitude, A_0 , replaced by I_0 , the initial amplitude of the current oscillations,

$$I(t) = I_0 e^{-\beta t} \cos(\omega_1 t - \delta). \quad (5)$$

With $m \rightarrow L$, $k \rightarrow 1/C$, and $b \rightarrow R$,^{7,9} we find from Eq. (3) that Q varies as $1/R$. As the resistance increases, the

damping constant increases and more energy is lost per cycle resulting in a lower value of Q .

C. Parallel RLC circuit

The parallel RLC circuit is shown in the bottom of Fig. 1. The capacitor and inductor are connected in parallel, and both are in series with a resistor, R , which we call the input resistance. This resistance could be a resistor placed in the circuit, the output resistance of the voltage source in the circuit, or a combination of the two.

For this circuit, there are three branches with currents I_R , I_L , and I_C , as shown in Fig. 1, and the voltage source supplies a voltage $V(t)$. This leads to the following two loop equations (here q is the charge on the capacitor):

$$I_R R + \frac{q}{C} = V(t), \quad (6)$$

$$I_R R + L \frac{dI_L}{dt} = V(t), \quad (7)$$

along with the junction equation at the junction marked as J in Fig. 1,

$$I_R = I_L + I_C. \quad (8)$$

To proceed, we differentiate Eqs. (6) and (7) and combine them while taking advantage of the fact that $dq/dt = I_C$ to get

$$\frac{I_C}{C} = L \frac{d^2 I_L}{dt^2}. \quad (9)$$

We then use Eq. (8) to eliminate I_C , while I_R can be found using Eq. (7). Combining all of this together, and noting that we are only interested in the transient oscillations when $V(t) = 0$, yields the equation of motion for the current through the inductor

$$\ddot{I}_L + \frac{1}{RC} \dot{I}_L + \frac{1}{LC} I_L = 0. \quad (10)$$

Equation (10) has the same form as Eq. (4), and the solution is again the decaying oscillations described in Eq. (5). Comparing Eqs. (10) and (4), it is found that $\omega_0^2 = 1/LC$ exactly as before, but now $\beta = 1/(2RC)$. Thus, the damping constant in this circuit is *inversely* proportional to R , meaning that larger resistances lead to less damping and a higher value for Q . This is the opposite of the series RLC circuit and the harmonic oscillator where increasing the value of the dissipative element decreases the value of Q . The Q factor for the parallel RLC circuit is found using Eq. (3)

$$Q = \sqrt{\frac{R^2 C}{L} - \frac{1}{4}}, \quad (11)$$

showing the non-intuitive behavior that Q increases as R is increased. For $R \gg \sqrt{L/(4C)}$, $Q \approx R\sqrt{C/L}$, and Q is proportional to R .

III. EXPERIMENT

To test the dependence of the Q factor on the resistance, we induced oscillations in a parallel RLC circuit by applying

a 0 to 5 V square wave of frequency 100 Hz and then observed the system's transient oscillations starting on the negative edge of the square wave (when the voltage changes from 5 to 0 V) for various values of R . We measured the voltage across the inductor/capacitor parallel combination, V_L , which is related to the current in the inductor by a derivative, $V_L = Ldi/dt$. Using Eq. (5) for the inductor current then gives

$$V_L(t) = -LI_0 e^{-\beta t} [\beta \cos(\omega_1 t - \delta) + \omega \sin(\omega_1 t - \delta)] = V_0 e^{-\beta t} \sin(\omega_1 t - \phi), \quad (12)$$

where ϕ is a phase offset which can be related to ω_1 , β , and δ .¹⁰ We fit the measured voltage vs. time data using Eq. (12) with V_0 , β , ω_1 , and ϕ as fitting parameters. We performed this experiment for twelve different values of R ranging from 47 Ω to 67 k Ω , while L and C were held fixed and measured separately to be 10.54 mH and 0.5639 μ F, respectively.

The results of these measurements are summarized in Fig. 2, which shows V_L measured over time for two values of R (for the full set of data see the supplementary material¹²). The underdamped nature of the oscillations is apparent, and it is easy to see that there is a longer decay time constant for the larger value of R . As expected for a circuit consisting of two impedances in series (the input resistor, R , is in series with the parallel inductor/capacitor parallel combination), the size of the initial peak depends strongly on the value of R . In all, Eq. (12) fits the data (with appropriately chosen values of ω_1 , β , V_0 , and ϕ), and hence, it is clear this equation accurately captures the time dependence of V_L for this circuit.

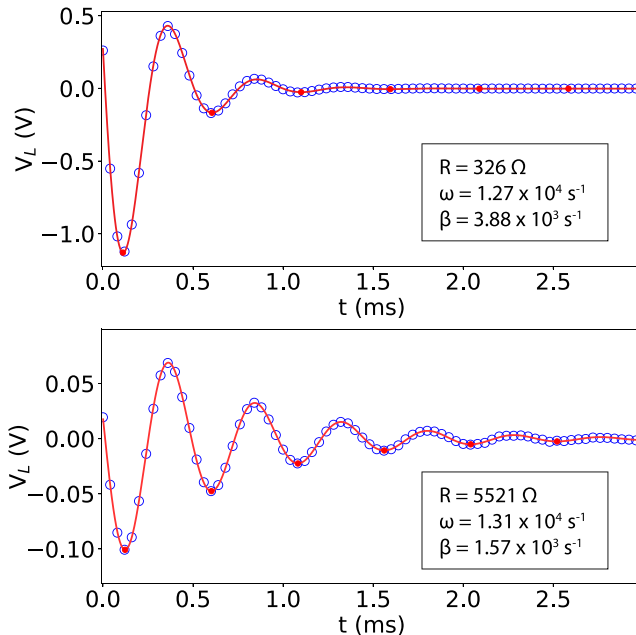


Fig. 2. Voltage measured across the L/C parallel combination after the input voltage was changed from 5 to 0 V (blue open circles) for $R = 326 \Omega$ (top) and $R = 5.52 \text{ k}\Omega$ (bottom). In both cases, $C = 0.5639 \mu\text{F}$ and $L = 10.54 \text{ mH}$. The red solid line shows a fit to the data using Eq. (12), and the red dots show locations of local minima in the fit.

IV. ANALYSIS

A. Q factor

In order to judge the validity of the model, students need to extract ω_1 and β from the fits and use them to compute Q . An example of this is shown in Fig. 3—a plot of the measured Q factor of the circuit for the different values of R used in the experiment. At lower values of R , Q is seen to increase as R increases, while at higher values ($R \gtrsim 10 \text{ k}\Omega$), the curve flattens indicating that Q only depends weakly on R in this region. Students should quickly realize that the behavior at large R is not consistent with Eq. (11), which predicts that Q should continue to increase, approaching a linear relationship. Relatedly, students might realize that there is a discrepancy between the values of β resulting from the fits and those found from $\beta = 1/(2RC)$ (using the measured values of R and C). This presents students with their first challenge: To determine why the model fails at higher resistances and revise it accordingly.

In this case, the revision involves relaxing the assumption of an ideal RLC circuit, where the resistance is only due to the resistor R . Real circuits have other sources of resistance, often referred to as parasitic resistances, that need to be incorporated into the model. In this circuit, the main source of the parasitic resistance is the inductor, which can be modeled as an ideal inductor, L , in series with a resistor, r , as shown in Fig. 4. In this updated circuit, we have replaced the square-wave function generator with a switch that can toggle between a DC voltage of V_0 and 0 V. We have also included the output impedance of the voltage source, R_S , which in our experiment was 50 Ω .

Following the same process described in Sec. II C to find the equation of motion for the current in the circuit, we still find a differential equation in the form of Eq. (4); however, now the natural frequency is

$$\omega_0^2 = \frac{1}{LC} \left(\frac{r}{R + R_S} + 1 \right), \quad (13)$$

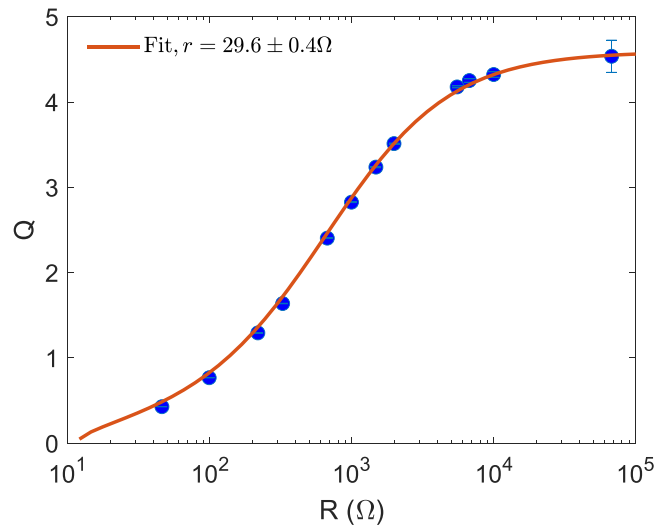


Fig. 3. Computed Q factors of the experimental data as the value of R is changed. The fitting algorithm's 95% confidence level bounds on the parameters β and ω_1 were used as their uncertainties. The uncertainty in Q was determined using error propagation in Eq. (3), and these uncertainties were added as error bars. For all except the last data point, they are smaller than the size of the symbols. The solid line is a fit to the data using Eq. (15) with r used as a fitting parameter.

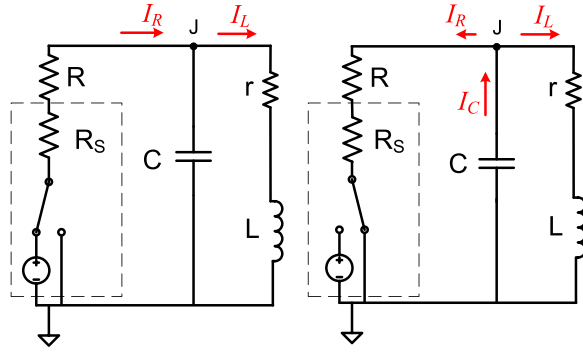


Fig. 4. Model for a real RLC circuit. The parasitic resistance, r , is mainly due to the resistance of the inductor. The dashed box shows the components that make up the function generator including a switch which sets the input at either 5 or 0 V, and the output impedance, $R_S = 50 \Omega$. The arrows show the directions of the currents at junction J when the switch is set to the voltage source and the circuit is in a steady state (left), and shortly after the switch is moved to the ground position (right).

while the damping constant becomes

$$\beta = \frac{1}{2} \left(\frac{1}{(R + R_S)C} + \frac{r}{L} \right). \quad (14)$$

If we use these two relationships in Eq. (3), then we find that the Q factor of the real RLC circuit is

$$Q = \sqrt{\frac{L}{r^2 C} \frac{\frac{r}{R + R_S} + 1}{\left(\frac{L}{r(R + R_S)C} + 1 \right)^2} - \frac{1}{4}}. \quad (15)$$

In our experiment, the inductor was measured to have a resistance of about 28Ω . Two other sources that may also contribute to the parasitic resistance are the equivalent series resistance (ESR) of the capacitor,⁵ which for the film capacitors used in the experiment has a maximum value of $\sim 1.5 \Omega$, and resistance from the breadboard connections, which was found to be up to $\sim 0.5 \Omega$, though there could be considerable variations on both of these values. In total, we estimate the total parasitic resistance of the circuit to be $r \approx 30 \pm 2 \Omega$.

To test this revised model, we fit Eq. (15) to the Q vs R data with the parasitic resistance, r , used as a fitting parameter. The result of this fit is shown in Fig. 3, where it is found to have excellent agreement with the collected data. The value of the parasitic resistance extracted from the fit was found to be $r = 29.6 \pm 0.4 \Omega$, which, again, is in excellent agreement with the measured values of the parasitic resistance described previously.

B. Proposed explanation

We have developed a model that predicts Q increasing with R and have shown that the model's predictions are consistent with experiment. A simple explanation for this behavior is that the power delivered to R is simply V_R^2/R , so as R increases the amount of energy lost by R will decrease, thus increasing Q . However, a deeper understanding of the Q factor in this circuit can be gained by exploring the energy transfer between various components in the circuit.

We start by noting that the oscillations in the circuit depend on energy transfer between the capacitor and

inductor; and if less energy is transferred between the two components, the oscillations will die out sooner resulting in a smaller value of Q . The power delivered to the input resistor and inductor (i.e., the energy flow) are $P_R = I_R V_R$ and $P_L = I_L V_L$, respectively.¹¹ Since these two components are connected in parallel when the switch is set to ground, $V_L = V_R$, and the energy flowing through each branch of the circuit is then proportional to the current in that branch.

The dynamics of the currents in the circuit are complicated, with the three branch currents I_C , I_L , and I_R all oscillating with different phases and amplitudes. However, we can gain insight into the circuit behavior by considering the current flow at junction J (shown in Fig. 4) in the moments after the switch is toggled. We start with the switch set to 5 V and with the circuit in a steady state (Fig. 4, left). At this time, the charge on the capacitor is at its maximum, meaning that the energy stored in the capacitor is also at a maximum, and there is a steady current flowing through both the input resistor and the inductor. When the switch is moved to ground, as shown in Fig. 4 (right), the capacitor starts to discharge, and current flows out of the capacitor. At junction J, this current splits between the inductor and the resistor branches, each carrying a fraction of the energy originally stored in the capacitor. The current going through the resistor branch, which is V_L/R , does not transfer any energy to the inductor and thus can be dissipated across the resistor R . Since r and L are held constant, then as R is increased, a smaller fraction of the current goes through R , and a smaller fraction of the stored energy is dissipated across it, thus increasing the Q factor.

As R is further increased, a very small fraction of the circuit's energy will be dissipated by the input resistor, and most of the energy losses will instead occur through the parasitic resistance, r . This is consistent with Eq. (15), which reduces to $Q = \sqrt{(L/r^2 C)} - 1/4$ in the limit of large values of R , which is the same result for the Q factor for the series RLC circuit.⁴⁻⁶ Using the values of L , C , and r given above, we find that in this limit, $Q \rightarrow 4.6$, which is consistent with the data shown in Fig. 3.

C. Energy

To demonstrate the role that the input resistor plays in the energy losses in the circuit, students can use their collected data to compute the fraction of the energy lost in different portions of the circuit. According to our explanation, there are two ways that energy can leave the system: It can either be dissipated by the parasitic resistance or it can leak out through the input resistor. Thus, the total energy loss from the circuit, ΔE_T , is the sum of the energy loss through the parasitic resistance, ΔE_r , and the energy that leaks out through the input resistor, ΔE_R ,

$$\Delta E_T = \Delta E_r + \Delta E_R. \quad (16)$$

Here, we will show how these energy losses can be computed with the data that has been collected.

First, we compute ΔE_T , the total energy lost in one cycle, by determining the difference in the circuit energy at successive peaks or troughs in Fig. 2. At these points, the total energy in the system is the energy stored in the capacitor plus the energy stored in the inductor. The energy stored in the capacitor at peak i is $(1/2)CV_i^2$, where V_i is the voltage at peak i . The energy stored in the inductor, $(1/2)LI_i^2$, can be

determined by noting that when the voltage is at a local extrema, the current through the capacitor, $I_C = CdV/dt = 0$ and $I_L = I_R = V_i/(R + R_S)$. Thus, the total energy loss between the first two peaks is

$$\Delta E_T = \frac{1}{2} \left(C + \frac{L}{(R + R_S)^2} \right) (V_i^2 - V_{i+1}^2). \quad (17)$$

The extrema, which are indicated by the red dots in Fig. 2, were determined using a peak finding algorithm in Python. We then used the voltages at the first two minima to determine ΔE_T .

The energy lost in the left-hand branch of the circuit includes the energy dissipated by both the input resistor, R , and the output impedance of the voltage source, R_S . The energy loss through both of these resistors together can be computed by

$$\Delta E_R = \int_T \frac{V_R(t)^2}{R + R_S} dt. \quad (18)$$

Since the input resistor is in parallel with the other two components, we can compute ΔE_R by using the measured voltage vs time data integrated over the same two peaks as was used to compute ΔE_T .

Finally, once ΔE_T and ΔE_R are computed, the energy lost through the parasitic resistance, ΔE_r can be computed using Eq. (16). In order to compare energy losses from the different data sets, we scaled each value of ΔE by the maximum energy in the first peak of each dataset, E_{\max} . We performed these computations for the 12 data sets shown in the supplementary material,¹² and the results are plotted in Fig. 5. At lower values of R , it is seen that most of the energy from the circuit is leaking out through the input resistor within the first oscillation. As R increases, the amount of energy leaking out through the input resistor decreases, and as a consequence, an increasing percentage of the energy is dissipated by the parasitic resistance. In this region, Q is increasing as evidenced by the decreasing ΔE_T curve, which indicates that less of the oscillator's energy is lost in the first period of oscillation. When $R \geq 10 \text{ k}\Omega$ very little energy leaks out through the input resistance; almost all the energy is lost

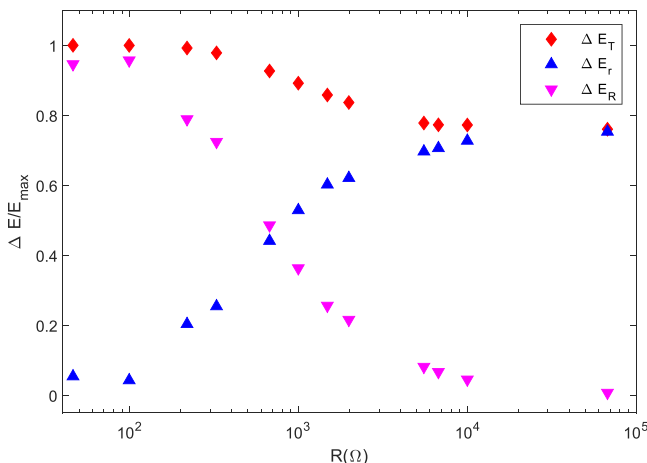


Fig. 5. Computed energy losses through the parasitic resistance and input resistor along with the total energy lost from the circuit in one cycle.

through the parasitic resistance, and Q is independent of R . These observations are entirely consistent with energy loss analysis presented above.

V. CONCLUSION

The parallel RLC circuit presents students with a mystery: If resistance is the source of damping, then why does increasing the resistance decrease the damping? As we have shown, the circuit can be understood through a simple model, and students can also gain an intuitive understanding of its principles, which can then be verified with the data they have collected in the laboratory. Thus, students can experience the satisfying process of discovering a mystery and then explaining it.

Though we have spent some time in this paper drawing comparisons between the series RLC circuits and the simple harmonic oscillator, it is not clear that such a mechanical analog exists for the parallel RLC circuit. One would have to find a mechanism by which increasing the value of a dissipative force would decrease the amount of energy leaving the system, while not affecting the primary route that energy leaves (in the case of the harmonic oscillator that would be retarding forces, while in the RLC circuit it would be the parasitic resistance). We hope that our readers will be inspired to search for such a mechanical system.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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¹¹The subscript L includes both the inductor and its parasitic resistance, r .

¹²See supplementary material at <https://www.scitation.org/doi/suppl/10.1119/5.0074843> for the full set of data collected in the experiment.