Rapidly Rotating Nuclei in the $SU_3$ Model

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Rapid rotational motion is studied in the microscopic $SU_3$ model by using the Dynamical Nuclear Field Theory. Assuming small triaxiality, three-dimensional slow rotation is superposed on the fast rotation about a specified body-fixed axis. We use the quantized cranking model. Two basic operators, quadrupole and angular momentum operators, are represented in effective forms. We point out the importance of the higher-order dynamical coupling potentials induced by the constraints imposed on these operators. Effective Hamiltonians are derived in three ways and compared with the exact result.

§ 1. Introduction

Since the invention of the cranking model by Inglis,\textsuperscript{1} our microscopic understanding of nuclear collective rotation has much progressed. In this approach, a nucleus deformed axially symmetrically is assumed to rotate about an intrinsic axis (called the $x$ axis) perpendicular to the symmetry axis. In the case of uniform rotation, we can find stationary states for the Hamiltonian with Coriolis coupling in the rotating frame of reference. We note that a quantal analogue of the cranking model was proposed by Ripka et al.\textsuperscript{2}

However, as shown by simple calculations using a cranked harmonic oscillator (CHO),\textsuperscript{3} the nucleus loses axial symmetry and deforms triaxially with increasing angular velocity. Then the rotation axis need not coincide with the $x$ axis. In this paper we want to study the cranking model in which a three-dimensional rotation is superimposed on the main rotation about the $x$ axis, assuming small triaxiality.

Previously, Kinouchi, Kishimoto and Kammuri\textsuperscript{4,5} (first two papers are referred to as KKK1 and KKK2, respectively) attempted to extend the cranking model in various ways. In order to get some insight into the structure of the motion, they took a simple $SU_3$ model for which the group theory gives the exact answers.\textsuperscript{7} Namely they assumed that nucleons move in a spherical harmonic oscillator potential with the quadrupole-quadrupole ($QQ$) interaction neglecting the effects of $\Delta \hbar \omega$ admixtures. The quantized cranking theory was developed in KKK2 by using the Dynamical Nuclear Field Theory (DNFT), but was restricted to the case of slow rotation around the $x$ axis.

In this paper we extend these treatments to study a rapid three-dimensional rotation of the $SU_3$ system. The main rotation around the $x$ axis is incorporated in the independent-particle part semiclassically as in KKK1. On the other hand, the slow rotation around an arbitrary axis superposed on the fast rotation is treated quantum-mechanically, extending the method used in KKK2. In developing the DNFT analysis, we want to emphasize the importance of the effects of two constraints on the angular momentum and quadrupole operators. We will see that two kinds of
potentials which couple particles with the rotation and deformation are essentially affected by these conditions.

In § 2, after giving an outline of the DNFT method as applied to the rotational motion, we describe the properties of the lowest-order driving Hamiltonian which takes the form of CHO. In § 3 we construct the coupling potentials between particles and rotation in first and second orders, by taking into account the self-consistency. For this purpose we derive and use effective representations of the angular momentum and quadrupole operators referring to the body-fixed frame. In § 4, the corresponding expressions of these effective operators in the laboratory frame are derived. The effective Hamiltonians are compared in § 5 with the exact $SU_3$ result. Section 6 gives a summary and discussion. Appendix A includes definitions of some basic operators, while in Appendix B we summarize formulae needed in our perturbative expansion.

§ 2. General formulation

2.1. DNFT method

We consider a system of $A$ nucleons interacting via the quadrupole-quadrupole force

$$H = H_{sp} - \frac{1}{2} \chi \sum_{\mu} \hat{Q}_{2\mu} \hat{Q}_{2\mu}, \quad (2.1)$$

where $H_{sp}$ contains the isotropic harmonic oscillator potential

$$H_{sp} = \sum_{n=1}^{A} (h_{sp})_{n}, \quad h_{sp} = \frac{1}{2M} p^2 + \frac{1}{2} M \omega_0^2 r^2. \quad (2.2)$$

The quadrupole operator $\hat{Q}_{2\mu}$ is here assumed to represent its $\Delta N=0$ part only. The index $\mu$ stands for the component referring to the laboratory frame.

As in KKK1, we start from the time-dependent Schrödinger equation in the deformed Hartree-field approximation. By means of the rotation operator ($\hbar=1$)

$$R = \exp(-i\Omega \cdot \vec{L} t), \quad (2.3)$$

we transform the Schrödinger equation to the one in the rotation frame. Here $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ and $\vec{L} = (\vec{L}_x, \vec{L}_y, \vec{L}_z)$ are the angular velocity and particle angular momentum with components along the intrinsic axes. We obtain the stationary equation

$$\tilde{H}\phi = E\phi, \quad (2.4)$$

where the driving Hamiltonian $\tilde{H}$ is given by

$$\tilde{H} = H_{sp} + V_0 + V_c, \quad (2.5a)$$

$$V_0 = -\sum_{\nu=-2}^{2} a_\nu^* \hat{Q}_\nu, \quad V_c = -\sum_{k=x,y,z} \Omega_k \vec{L}_k. \quad (2.5b)$$

Here $\hat{Q}_\nu (\nu=0, \pm 1, \pm 2)$ specifies the spherical component of the quadrupole operator in the body-fixed frame.
In the following we decompose $\hat{H}$ into the intrinsic part $H_0$ and the coupling $V$ between particles and collective fields

$$\hat{H} = H_0 + V.$$  \hfill (2.6)

In the DNFT approach, we first choose a set of eigenstates of $H_0$, adjusting ourselves to the problem of interest. The projection operator onto this set is called $p = \{p_i\}$ and $q$ is given by $q = 1 - p = \{q_i\}$. The effective representation of an arbitrary operator $\hat{O}$ taken between two states $p_1$ and $p_2$ belonging to the $p$ space can be expanded in powers of $V$

$$\hat{O}_{\text{eff}} = \sum_{n=0}^{\infty} \hat{O}_{\text{eff}}[n],$$  \hfill (2.7)

where

\begin{align*}
\hat{p}_1 \hat{O}_{\text{eff}}[0] \hat{p}_2 &= (\hat{p}_1 \hat{O} \hat{p}_2), \\
\hat{p}_1 \hat{O}_{\text{eff}}[1] \hat{p}_2 &= -\sum_i \{ \gamma_{i1} (\hat{p}_1 V q_i \hat{O} \hat{p}_2) + \gamma_{i2} (\hat{p}_1 \hat{O} q_i V \hat{p}_2) \}, \\
\hat{p}_1 \hat{O}_{\text{eff}}[2] \hat{p}_2 &= \sum_i \left[ \gamma_{i1} \gamma_{i1} (\hat{p}_1 V q_i \hat{O} q_i \hat{p}_2) + \gamma_{i2} \gamma_{i2} (\hat{p}_1 \hat{O} q_i V q_i \hat{p}_2) \\
&\quad + \gamma_{i1} \gamma_{i2} (\hat{p}_1 V q_i \hat{O} q_i V \hat{p}_2) \\
&\quad - \gamma_{i1} \gamma_{i1} \left( \frac{1}{2} (\hat{p}_1 V q_i V q_i \hat{O} \hat{p}_2) + (\hat{p}_1 \hat{O} q_i V q_i V \hat{p}_2) \right) \right]. \hfill (2.8c)
\end{align*}

where $\gamma_{ij} = (\varepsilon_i - \varepsilon_j)^{-1}$ and $\varepsilon_i$ is an eigenvalue of $H_0$, and round brackets mean the integration over intrinsic variables only. Since we are concerned with a rotational band built on an intrinsic state $\Phi_0$ or $|0\rangle$ of $H_0$, the $p$ space is composed of just one state.

The deformation parameters $a_\nu$ are determined by the Hartree self-consistency condition

$$a_\nu = \chi \hat{Q}_{\nu,\text{eff}}, \quad (\nu = 0, \pm 1, \pm 2)$$  \hfill (2.9)

where the expectation value of $\hat{Q}_\nu$ in an eigenstate of $\hat{H}$ is definitely defined as the average of the effective $\hat{Q}_\nu$ operator in the corresponding intrinsic eigenstate of $H_0$ specified below, leaving aside the collective variables.

Following Ref. 2), we relate the angular velocity $\Omega_k$ about the intrinsic $k$ axis to the total angular momentum $\hat{I}_k$ dependent on Euler’s angles as

$$\mathcal{J}_k \Omega_k = \hat{I}_k,$$  \hfill (2.10)

where $\mathcal{J}_k$ is the moment of inertia. We thereby obtain the quantal analogue of the usual Coriolis coupling

$$V_c = -\sum_k \hat{I}_k \hat{L}_k / \mathcal{J}_k.$$  \hfill (2.11)
Then the value of the particle angular momentum $\vec{L}_k$ in an eigenstate of $\vec{H}$, averaged only over the intrinsic variables, becomes a function of $\vec{I}_k$, for which we require the relation

$$\vec{I}_k = \vec{L}_{k, \text{avg}}.$$  \hfill (2·12)

Using Eq. (2·11) in the calculation of the r.h.s. of Eq. (2·12), we can determine $\vec{J}_k$.

In this paper we consider the system rotating rapidly mainly around the $x$ axis and aim at treating the motion quantum-mechanically in the $SU_3$ model (semiclassical discussion was given in KKK1). We superimpose a slow rotation with the angular velocities $\delta\Omega_k (k=x, y, z)$ on a rapid rotation with $\Omega_0$ about the $x$ axis,

$$\Omega_k = \Omega_0 \delta\Omega_k + \delta\Omega_k \quad (k=x, y, z) \hfill (2·13)$$

Both the effects of rapid rotation and the zeroth-order deformation are incorporated in the following CHO Hamiltonian

$$H_0 = \sum_{n=1}^A (J_0)_n, \hfill (2·14a)$$

$$\delta h_0 = h_{sp} + \nu_0, \quad \nu_0 = \nu_0^{(0)} + \nu_c^{(0)}, \hfill (2·14b)$$

$$\nu_0^{(0)} = - \sum_{\nu=0, \pm 2} a_\nu [0] q_\nu, \quad \nu_c^{(0)} = - \Omega_0 I_x. \hfill (2·14c)$$

In the present perturbation calculations we utilize the moment of inertia $J$ determined by the lowest order self-consistency

$$J_k = J = (3\chi)^{-1}. \hfill (2·15)$$

Due to higher order corrections, we will find the values of $J_k$ to deviate from $J$, the effects of which will be included in the residual coupling. But since the deviations occur in the orders higher than the first order and we do not take into account the feedback effects from higher-order corrections to the lower-order results, we simply multiply $J$ on Eq. (2·13), and write the total angular momentum operators as

$$\vec{I}_k = J_0 \delta\Omega_k + \vec{R}_k, \hfill (2·16a)$$

where

$$J_0 = J \Omega_0, \quad \vec{R}_k = J \delta\Omega_k. \hfill (2·16b)$$

The commutation relations satisfied by $\vec{I}_k$ are taken to be

$$[\vec{I}_x, \vec{I}_y] = -i I_x, \quad \text{etc.} \hfill (2·17)$$

Note that Bohr and Mottelson\textsuperscript{9}) used the approximate relation $[\vec{I}_y, \vec{I}_z] = -i J_0$ in their treatment of the wobbling motion of a triaxial nucleus.

2.2. **CHO Hamiltonian**

In terms of the oscillator-quanta operator $c_k \; (k=x, y, z)$ defined with the phase convention of Bohr and Mottelson,\textsuperscript{9}) $h_{sp}$ is
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\[ h_{sp} = \omega_0 \sum_{k=x,y,z} \left( c_k^\dagger c_k + \frac{1}{2} \right) . \]  
(2.18)

Using Eqs. (A·2) and (A·5) of Appendix A for \( l_x \) and \( q_v \), we have

\[ h_0 = \sum_{k=x,y,z} \omega_k \left( c_k^\dagger c_k + \frac{1}{2} \right) - \Omega_0 (c_x c_y + c_x c_z), \]  
(2.19)

where

\[ \omega_x = \omega_0 + a_0[0] - \sqrt{6} a_2[0], \quad \omega_y = \omega_0 + a_0[0] + \sqrt{6} a_2[0], \quad \omega_z = \omega_0 - 2 a_0[0]. \]  
(2.20)

By means of the transformation

\[ c_x = c_1, \quad c_y = uc_2 + vc_3, \quad c_z = -vc_2 + uc_3 \]  
(2.21)

with

\[ u = [(1+q)/2]^{1/2}, \quad v = [(1-q)/2]^{1/2}, \]
\[ q = (\omega_y - \omega_x)/d, \quad d = [(\omega_y - \omega_x)^2 + 4\Omega_0^2]^{1/2}, \]  
(2.22)

we can diagonalize \( h_0 \) as

\[ h_0 = \sum_{i=1}^3 \omega_i (c_i^\dagger c_i + \frac{1}{2}). \]  
(2.23)

The eigenfrequencies \( \omega_i \) are given by

\[ \omega_1 = \omega_x, \quad \omega_2 = \frac{1}{2} (\omega_x + \omega_z) - \frac{1}{2} d, \quad \omega_3 = \frac{1}{2} (\omega_x + \omega_z) + \frac{1}{2} d. \]  
(2.24)

The intrinsic state we are concerned with is assumed to have the configuration \((\Sigma_1, \Sigma_2, \Sigma_3)\), where, for example, \( \Sigma_1 \) is the total number of oscillator quanta along the \( x \) axis as given by Eq. (A·11).

We determine \( a_v[0] \) and \( \beta \) from the zeroth-order self-consistency conditions. From the relation

\[ a_v[0] = \chi \bar{Q}_{\nu, v}[0] = \chi \langle \bar{Q}_v \rangle, \]  
(2.25)

we obtain

\[ a_0[0] = \chi \langle \bar{S}_5 \rangle = \frac{1}{2} \chi (S_0 + 3q S_1), \quad a_0[0] = 0, \]
\[ a_2[0] = -\sqrt{\frac{3}{2}} \chi \langle \bar{S}_7 \rangle = -\sqrt{\frac{3}{8}} \chi (S_0 - q S_1). \]  
(2.26)

Here the average is taken with respect to the state \(| \Sigma_1, \Sigma_2, \Sigma_3 \rangle \), and the definitions of \( \bar{S}_i \) operators are given in Appendix A. The eigenfrequencies \( \omega_i \) (\( i = 1 \sim 3 \)) turn out to be independent of \( \Omega_0 \)

\[ \omega_1 = \omega_0 + 2\chi(\lambda + 2\mu), \quad \omega_2 = \omega_0 + 2\chi(\lambda - \mu), \quad \omega_3 = \omega_0 - 2\chi(2\lambda + \mu). \]  
(2.27)

The corresponding expressions for \( \omega_k \) (\( k = x, y, z \)) are given by Eq. (3·23) of KKK1.
Next, the equation
\[ g \approx L_{x,\text{err}}[0] = \langle \hat{L}_x \rangle = 2uvS_1 \] (2.28)
gives the moment of inertia about the \( x \) axis and \( J_0 \) as
\[ g_x = \mathcal{I} = 1/3 \chi, \quad J_0 = 2uvS_1. \] (2.29)
Here we have used
\[ q = [1 - (\Omega_0/\Omega_c)^2]^{1/2}, \quad \Omega_c = 3\chi\lambda. \] (2.30)

§ 3. Particle-rotation couplings and effective intrinsic operators

3.1. First-order couplings

The first-order coupling \( v_1 \) is composed of two parts
\[ v_1 = v_q^{(1)} + v_c^{(1)}, \] (3.1a)
\[ v_q^{(1)} = - \sum_{\nu=2} q_v[1] q_v, \quad v_c^{(1)} = - \sum_{k=x,y,z} \delta \omega_k l_k, \] (3.1b)
where the first order deformation parameters \( a_v[1] \) are related with \( \tilde{Q}_{v,\text{err}} \) resulting from the operation of \( V_1 \) once
\[ a_v[1] = \chi \tilde{Q}_{v,\text{err}}[1, V_1]. \] (3.2)
Using Eq. (2.8b), we rewrite the r.h.s. as
\[ a_v[1] = - \chi \sum_{\nu=2} a_v[1] Q_v[Q_v] - \chi \sum_{k=x,y,z} \delta \omega_k Q_v[L_k], \] (3.3)
where \( Q_v[Q_v] \) and \( Q_v[L_k] \) are given in Appendix B.

Writing down Eq. (3.3) explicitly, we have the following equations
\[ a_0[1] = \chi \mathcal{I} (3uv)^2 \left[ a_v[1] + \sqrt{\frac{1}{6}} (a_x[1] + a_x[2]) \right] - 3\chi \mathcal{I} quv\delta \Omega_x, \] (3.4a)
\[ a_1[1] = - 3\chi \mathcal{I} a_v[1], \] (3.4b)
\[ a_2[1] = \frac{1}{\sqrt{6}} \chi \mathcal{I} (3uv)^2 \left[ a_v[1] + \sqrt{\frac{1}{6}} (a_x[1] + a_x[2]) \right] - \frac{3}{2} \chi \mathcal{I} (a_v[1] - a_v[2]) \]
\[ - \frac{3}{\sqrt{6}} \chi \mathcal{I} quv\delta \Omega_x. \] (3.4c)
Equation (3.4b) allows us to take \( a_1[1] = 0 \), because of \( a_v^* = (-)^y a_{-y} \). The real part of \( a_2[1] \) satisfies
\[ \text{Re} a_2[1] = \frac{9}{\sqrt{6}} (uv)^2 \chi \mathcal{I} \left[ a_v[1] + \sqrt{\frac{2}{3}} \text{Re} a_2[1] \right] - \frac{3}{\sqrt{6}} \chi \mathcal{I} quv\delta \Omega_x. \] (3.4d)
Comparing this with Eq. (3.4a), we find \( \text{Re} a_2[1] = a_0[1]/\sqrt{6} \). Using \( 3\chi \mathcal{I} = 1 \) and
Eq. (3·4a), we obtain
\[ a_0[1] = -\frac{u^2}{q} \delta \Omega_x, \quad \text{Re} a_2[1] = \frac{a_0[1]}{\sqrt{6}}, \quad \text{Im} a_2[1] = 0, \quad a_1[1] = 0. \quad (3·5) \]

Thus the particle-deformation coupling is expressed as
\[ v_0^{(1)} = \frac{u^2}{q} \delta \Omega_x \left[ q_0 + \frac{1}{\sqrt{6}} (q_x + q_y) \right] = \delta \Omega_x \left[ 2uv \tilde{s}_1 - \frac{(2uv)^2}{q} \tilde{l}_1 \right], \quad (3·6) \]
where we have used Eq. (A·17) for \( q_v \). Summing this with \( v_c^{(1)} \), we obtain a renormalized Coriolis coupling
\[ v_1 = -\left\{ \frac{\delta \Omega_x}{q} l_1 + \delta \Omega_x l_y + \delta \Omega_x l_z \right\}, \quad (3·7) \]
of which the first term agrees with Eq. (5·4) of KKK1. Using Eq. (A·15), its quantal analogue is
\[ v_1 = -\sum_{k,l} \tilde{R}_l / \mathcal{J}_k^{(0)} - \sum_{k,m} \varphi^*_k \tilde{R}_m / \mathcal{J}_k^{(0)}. \quad (3·8) \]

Here the summation is taken over the combinations
\[ (k, l) = (1, x), (2, y), (3, z); \quad (k, m) = (2, z), (3, y) \quad (3·9a) \]
and we have defined
\[ l_k = (q_l, u_l, u_l), \quad \varphi_k = (\varphi_1, i\varphi_2, i\varphi_3), \quad (3·9b) \]
\[ \mathcal{J}_k^{(0)} = (q^2, \mathcal{J}, \mathcal{J}). \quad (k = 1 \sim 3) \quad (3·9c) \]

Using this coupling potential and the formulae of Appendix B, we can check that the relation
\[ \tilde{L}_{k,\text{eff}}[1, V_1] = \tilde{R}_k \quad (k = x \sim z) \quad (3·10) \]
holds. That is, the moment of inertia is given by Eq. (2·15) up to first order.

We add here one remark. The cranking formula for the moment of inertia is
\[ \mathcal{J}_k = 2 \sum_{l=0}^{\infty} \left| \langle \tilde{L}_l | 0 \rangle \right|^2 / \varepsilon_l - \varepsilon_0, \quad (3·11a) \]
where the pair of indices \((k, l)\) is the same as in Eq. (3·9a). Its value in the present model is equal to
\[ \mathcal{J}_k = 2 \tilde{L}_q \tilde{L}_l \rangle / E_k = 2 S \tilde{L} \times (q^2, 1, 1), \quad (3·11b) \]
where
\[ E_1 = \omega_2 - \omega_3, \quad E_2 = \omega_1 - \omega_3, \quad E_3 = \omega_1 - \omega_2. \]

Using the relation \( S \tilde{L} / E_k = 1/6 \chi \mathcal{J} / 2 \), we find it equal to \( \mathcal{J}_k^{(0)} \) (3·9c). The difference between \( \mathcal{J} \) and \( \mathcal{J}_k^{(0)} \) tells us the importance of the self-consistency conditions.

We note that the first-order \( Q_{\nu,\text{eff}} \) can be written as
\begin{equation}
\tilde{Q}_{0,\text{eff}}[1] = -\frac{3\mu v}{q} R_x, \quad \tilde{Q}_{\pm 1,\text{eff}}[1] = 0, \quad \tilde{Q}_{\pm 2,\text{eff}}[1] = -\sqrt{\frac{3}{2}} \frac{\mu v}{q} R_x. \tag{3.12}
\end{equation}

3.2. Second-order couplings

Of the second-order coupling \( v_2 \) given by
\begin{equation}
v_2 = v_q^{(2)} + v_c^{(2)}, \quad v_q^{(2)} = -\sum \alpha^* v [2] q_v, \tag{3.13}
\end{equation}

we first consider the additional coupling \( v_q^{(2)} \) between particles and deformations. The self-consistency condition between \( \alpha_v[2] \) and \( \tilde{Q}_{v,\text{eff}} \) obtained by acting \( V_i \) twice and \( V_2 \) once is
\begin{equation}
\alpha_v[2] = \chi \tilde{Q}_{v,\text{eff}}[2, V_1 + V_2] = \chi \tilde{Q}_{v,\text{eff}}[1, V_2] + \chi \tilde{Q}_{v,\text{eff}}[2, V_1]. \tag{3.14}
\end{equation}

In writing down Eq. (3.14), we have neglected the contributions of \( V_c^{(2)} \), which can be justified at the end of § 3.2. We then obtain equations similar to Eqs. (3.4a~c) except for the replacement of \( \chi \tilde{Q}_{v,\text{eff}}[1, V_c^{(1)}] \) by \( \chi \tilde{Q}_{v,\text{eff}}[2, V_1] \) in each r.h.s. Using formulae given in Appendix B, \( \tilde{Q}_{v,\text{eff}}[2, V_1] \) can be arranged as
\begin{equation}
\tilde{Q}_{v,\text{eff}}[2, V_1] = \sum_{m=-2}^2 (A_{\nu m}[2] \tilde{R}_m^2 + B_{\nu m}[2] \tilde{I}_m), \tag{3.15}
\end{equation}

whose non-zero coefficients are tabulated in the second column of Table I. In the

<table>
<thead>
<tr>
<th>( \nu = 0 )</th>
<th>( V_1 )</th>
<th>( V_2 )</th>
<th>( V_1 + V_2 )</th>
</tr>
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<tbody>
<tr>
<td>( A_{xx} )</td>
<td>( -\frac{12}{q S_1} )</td>
<td>( -\frac{12(1-q^2)}{q^2 S_1} )</td>
<td>( -\frac{12}{q^2 S_1} )</td>
</tr>
<tr>
<td>( A_{xy} )</td>
<td>( -3[2qS_+-S_+qS_+]+3\frac{1-q^2}{q}(S_+-qS_+) )</td>
<td>( -3\frac{1-q^2}{q}(S_-+qS_-) - \frac{3}{q}(S_++qS_+) )</td>
<td></td>
</tr>
<tr>
<td>( A_{xz} )</td>
<td>( -3(1-q^2)S_+ )</td>
<td>( -\frac{3}{q}(1-q^2)S_- )</td>
<td></td>
</tr>
<tr>
<td>( B_{xz} )</td>
<td>( -6uv(S_+qS_+) )</td>
<td>( -6uv\frac{1-q^2}{q}S_+ )</td>
<td>( -6uv\left(S_++\frac{1}{q}S_+\right) )</td>
</tr>
<tr>
<td>( \nu = 1 )</td>
<td>( B_{xx} )</td>
<td>( -\frac{8\mu v}{q S_1} )</td>
<td>0</td>
</tr>
<tr>
<td>( \nu = 2 )</td>
<td>( A_{xx} )</td>
<td>( -\frac{4}{q S_1} )</td>
<td>( -\frac{4(1-q^2)}{q^2 S_1} )</td>
</tr>
<tr>
<td>( A_{xy} )</td>
<td>( 2qS_+\left(3q^2\right)S_+ )</td>
<td>( -\frac{4}{q}\left(3q^2\right)(S_+qS_+) )</td>
<td>( -\frac{1-3q^2}{q}\left(3q^2\right)S_+ + 5S_+ )</td>
</tr>
<tr>
<td>( A_{xz} )</td>
<td>( -4qS_+\left(3q^2\right)S_+ )</td>
<td>( -\frac{4}{q}\left(3q^2\right)(S_-qS_-) )</td>
<td>( -\frac{1+3q^2}{q}\left(3q^2\right)S_- + 5S_- )</td>
</tr>
<tr>
<td>( B_{xz} )</td>
<td>( 2uv(3q^2S_+qS_+) )</td>
<td>( -2uv\left(1-q^2\right)S_+ )</td>
<td>( 2uv\left(3q^2\right)S_+ )</td>
</tr>
<tr>
<td>( B_{vy} )</td>
<td>( \frac{4\mu iv}{q S_2} )</td>
<td>0</td>
<td>( \frac{4\mu iv}{q S_2} )</td>
</tr>
</tbody>
</table>

All the coefficients should be multiplied by the factor of 1/16 for \( \nu = 0, \sqrt{3}/16\sqrt{2} \) for \( \nu = 1 \) and 2, respectively.
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expansion (3·15), the terms linear in \( \tilde{I}_m \) are due to the non-commutativity between \( \tilde{I}_m \)'s (2·17). We solve the equations for \( a_m[2] \), ignoring these quantal effects. We substitute these \( a_m[2] \) into Eq. (3·13) and omit the diagonal parts involving \( \tilde{s}_s \) and \( \tilde{s}_r \) because they do not contribute in the following calculations.

We then obtain the coupling potential

\[
v_q^{(2)} = -l_i \sum_{m=x-z} W_m \tilde{R}_m^2,
\]

where the coefficients \( (W_x, W_y, W_z) \) are given as

\[
\frac{uv}{4q \mathcal{A}} \left( \frac{4}{q^2 S_1}, S_+ q S_+, S_- q S_+ \right).
\]

We note that \( \tilde{R}_x^2 \) dependent part of \( v_q^{(2)} \) is the same coupling potential as obtained in Eq. (5·14) of KKKK1. Contributions of this \( v_q^{(2)} \) to \( \hat{Q}_{v,\text{eff}}[2] \) are given in the third column of Table I, except for \( B_{0x} \) and \( B_{2x} \).

The effective representation of the \( k \)th intrinsic component of the particle angular momentum due to the operation of \( V_1 \) can be summarized as

\[
\hat{L}_{k,\text{eff}}[2, V_1] = \sum_{m=x-z} A_{km}[2] \tilde{R}_m^2 + B_k[2] \tilde{I}_k. \quad (k=x-z)
\]

The following coefficients are not zero,

\[
A_{km}[2] = -\frac{uv}{4} \left( \frac{4}{q^2 S_1}, S_+ q S_+, S_- q S_+ \right), \quad (m=x-z)
\]

\[
B_k[2] = \left( -\frac{1}{8} (1 - q^2) S_+, \frac{u^2}{2q S_2}, \frac{v^2}{2q S_2} \right). \quad (k=x-z)
\]

As can be inferred from the proportionality of \( W_m \) to \( A_{xm}[2] \), \( v_q^{(2)} \) has the effect to cancel out the \( \tilde{R}_m^2 \) dependent terms, so that we retain only

\[
\hat{L}_{k,\text{eff}}[2, V_1 + V_q^{(2)}] = B_k[2] \tilde{I}_k. \quad (3·20)
\]

We now consider the second-order Coriolis coupling \( v_c^{(2)} \) arising from the change of \( \mathcal{J}_k \) from \( \mathcal{J} \) and its effects. Writing \( \mathcal{J}_k = \mathcal{J} + \delta \mathcal{J}_k[2] \), the coupling can be written as

\[
v_c^{(2)} = \frac{1}{\mathcal{J}_k^2} \sum_k \delta \mathcal{J}_k[2] l_k \tilde{I}_k.
\]

Since the condition (2·12) is fulfilled by taking only the lowest two terms of \( \hat{L}_{k,\text{eff}} \) given by Eqs. (2·28) and (3·10), we must have

\[
\hat{L}_{k,\text{eff}}[2, V_1 + V_2] = 0 \quad (3·22)
\]

in second order. Substituting Eq. (3·21) and using Eqs. (3·20) and (B·4), we have

\[
\delta \mathcal{J}_k[2] = -\mathcal{J} B_k[2] [L_k][L_k]. \quad (3·23)
\]

We thus find the second-order Coriolis coupling as
\[ v_c^{(2)} = \sum_k \frac{f_k}{g} J_k \hat{I}_k, \quad (3.24) \]

where

\[ f_k = -\frac{1-q^2}{8q^2} S_+^k, \quad \frac{u^2}{2qS_2}, \quad \frac{v^2}{2qS_2} \quad (3.25) \]

for \( k = x, y, z \) respectively.

The third column of Table I includes the contributions of \( v_c^{(2)} \) to \( \tilde{Q}_{\nu,\text{eff}} \). Only the \( \hat{I}_x \)-dependent terms \( (B_{0z}, B_{2z}) \) are affected by \( v_c^{(2)} \). Total second order corrections to \( \tilde{Q}_{\nu,\text{eff}} \) can also be expressed as Eq. (3.15), whose coefficients are now listed in the last column.

\section*{§ 4. Effective operators referring to the laboratory frame}

A spherical tensor \( \tilde{T}_{\lambda \nu}(x) \) referring to the laboratory frame is related with the corresponding intrinsic operator \( \tilde{T}_{\lambda \nu}(x') \) by the transformation

\[ \tilde{T}_{\lambda \nu}(x) = \sum_D \lambda \nu \omega \tilde{T}_{\lambda \nu}(x'). \quad (4.1) \]

However this relation is invalid for the effective operators because of the rotational disturbances. We thereby derive here the effective forms of \( \tilde{Q}_{2 \mu} \) and \( \tilde{L}_{1 \mu} \) operators referring to the laboratory frame, using Eqs. (2.8a–c) anew.

\subsection*{4.1. Quadrupole operators}

As in KKK2, we express \( \tilde{Q}_{2 \mu} \) as

\[ \tilde{Q}_{2 \mu} = D_{l l}^\mu \tilde{Q}_l - D_l^\mu \tilde{Q}_l + \sum D_{l l}^\mu \tilde{Q}_l, \quad (4.2) \]

where the sum \((i, j)\) is of the same type as \((k, l)\) in Eq. (3.9a), and \( \tilde{\Phi}_l, \tilde{Q}_l, \tilde{Q}_l \) are defined in Eqs. (A.4) and (A.6) of Appendix A. Linear combinations of \( D_{2 l}^\mu \) and \( D_{l 2}^\mu \) are denoted by \( D_{l l}^\mu \),

\[ D_{l l}^\mu = D_{l 0}^\mu, \quad D_l^\mu = \sqrt{\frac{3}{2}} (D_{l l}^2 + D_{l -2}^2), \quad D_l^\mu = \sqrt{\frac{3}{2}} (D_{l l}^2 - D_{l -2}^2), \]

\[ D_{l l}^\mu = i \sqrt{\frac{3}{2}} (D_{l l}^1 + D_{l 1}^2), \quad D_{l l}^\mu = -\sqrt{\frac{3}{2}} (D_{l l}^1 - D_{l 1}^2), \quad (4.3a) \]

\[ D_{l l}^\mu = 3D_{l l}^\mu + D_l^\mu. \quad (4.3b) \]

The commutation relations \([ \hat{I}_k, D_l^\mu ]\) are given in Table II(a) of KKK2.

Using Eqs. (A.16) and (A.17), we rewrite Eq. (4.2) in terms of \( c_l \) \((i=1 \sim 3)\) operators

\[ \tilde{Q}_{2 \mu} = D_{l l}^\mu \tilde{S}_l - D_l^\mu \tilde{S}_l + \sum_{k=1}^3 D_{k l}^\mu \tilde{\Phi}_k + \sum_{k, m} D_{k m}^\mu \tilde{L}_k, \quad (4.4) \]

in which \((k, m)\) = (1, 3), (2, 3), (3, 2) and...
Rapidly Rotating Nuclei in the SU$_3$ Model

\[
\tilde{L}_h^\nu = (-uv\tilde{L}_1, iv\tilde{L}_2, iv\tilde{L}_3), \quad \Phi_h = (\Phi_1, u\Phi_2, u\Phi_3).
\] (4.5)

The effective $\tilde{Q}_{2\mu}$ in zeroth order is given by

\[
\tilde{Q}_{2\mu,\text{eff}}[0] = D\tilde{\nu}^\alpha S_\alpha - D\tilde{\nu}^\alpha S_\gamma,
\] (4.6a)

where

\[
S_\alpha = \frac{1}{2} (\lambda + 2\mu + 3q\lambda) \quad \text{and} \quad S_\gamma = \frac{1}{2} (\lambda + 2\mu - q\lambda).
\] (4.6b)

The first order term can be expressed as

\[
\tilde{Q}_{2\mu,\text{eff}}[1] = \sum_{k=\beta,\tau,1} D_k^\alpha (B_k[1]\tilde{R}_x + C_k[1]),
\] (4.7a)

where nonzero coefficients are

\[
B_{k[1]} = -\frac{3uv}{q} = 3B_{[1]},
\]

\[
C_{k[1]} = \frac{3}{2} \left(1 + \frac{1}{q}\right), \quad C_{\gamma[1]} = -\frac{1}{2} \left(3 - \frac{1}{q}\right), \quad C_{\lambda[1]} = -\frac{2uv}{q}.
\] (4.7b)

We calculate the second order part by using formulae given in Appendix B and commutation relations of KKK2 and rearrange it as

\[
\tilde{Q}_{2\mu,\text{eff}}[2] = \sum_{k=\beta,\tau,1} D_k^\alpha \left(\sum_{m=x-z} (A_{km}[2]\tilde{R}_m^2 + B_{km}[2]\tilde{I}_m) + C_k[2]\right).
\] (4.8a)

<table>
<thead>
<tr>
<th>$A_{km}$</th>
<th>$A_{lm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{2/3}A_{km}$ (m=x-z)</td>
<td>$\sqrt{2/3}A_{km}$ (m=x-z)</td>
</tr>
<tr>
<td>$B_{xx}$</td>
<td>-3uv \left(\frac{1}{8} S_+ + \frac{3uv}{q^2 S_1}\right)</td>
</tr>
<tr>
<td>$B_{xz}$</td>
<td>\frac{1}{8} uv(3S_- - \frac{1}{q} S_+) + \frac{uv}{q^2 S_1}</td>
</tr>
<tr>
<td>$B_{xz}$</td>
<td>-\frac{1}{q} S_+</td>
</tr>
<tr>
<td>$B_{xy}$</td>
<td>\frac{1}{4} uv[(S_x + qS_1 - 3S_)- qS_] - \frac{1}{8q} (S_+ + qS_+)</td>
</tr>
<tr>
<td>$B_{yy}$</td>
<td>\frac{1}{4} uv[(S_x + qS_1 - 3S_)+ qS_] + \frac{uv(2+q)}{4q} (S_+ + qS_+)</td>
</tr>
<tr>
<td>$B_{yz}$</td>
<td>\frac{1}{8} \left[4qS_- - (3 + q^2) S_+\right] + \frac{-q^2}{8q} (S_+ - qS_+)</td>
</tr>
<tr>
<td>$C_\nu$</td>
<td>\frac{3}{4q} \left[\frac{1}{16S_2 S_3} (S_3^2 - q^2S_1^2) + \frac{3}{8q S_2} (1 + 3q + q^2 - q^3)\right]</td>
</tr>
<tr>
<td>$C_\nu$</td>
<td>\frac{3}{4q} \left[\frac{1}{16S_2 S_3} (S_3^2 - q^2S_1^2) + \frac{3}{8q S_2} (1 + 3q + q^2 - q^3)\right]</td>
</tr>
<tr>
<td>$C_\gamma$</td>
<td>\frac{1}{4q} \left[\frac{3}{16S_2 S_3} (S_3^2 - q^2S_1^2) + \frac{1}{8q S_2} (1 + 9q + q^2 + 3q^3)\right]</td>
</tr>
<tr>
<td>$C_\gamma$</td>
<td>\frac{1}{4q} \left[\frac{3}{16S_2 S_3} (S_3^2 - q^2S_1^2) + \frac{1}{8q S_2} (1 + 9q + q^2 + 3q^3)\right]</td>
</tr>
<tr>
<td>$C_{\lambda}[1]$</td>
<td>-\frac{uv}{2S_2} \left(\frac{1}{S_1} + \frac{q^2}{8 S_1}\right) + \frac{1}{q^2 S_1}</td>
</tr>
</tbody>
</table>

Table II. $\tilde{Q}_{2\mu,\text{eff}}[2]$, the effective representation of the laboratory-frame quadrupole operator in second order.
Calculated coefficients are listed in Table II. As compared with those of \( Q_{v,\text{eff}}[2] \), we find

\[
A_{3m}[2] = A_{v=0,m}[2], \quad A_{3m}[2] = \sqrt{\frac{2}{3}} A_{v=2,m}[2] \quad (m = x \sim z) \tag{4.8b}
\]

for each operation of \( V_i^2 \) or \( V_2 \). On the other hand, similar equalities of \( B_{px}[2] \) with \( B_{ox}[2] \) and of \( B_{rz}[2] \) with \( B_{zr}[2] \), respectively, hold only for the case of \( V_i^2 \), but not for the total \( V_1 + V_2 \) operation. The last two terms on the r.h.s. of Eq. (4.4) contribute only to the second terms of \( B_{2x}, B_{2z}, C_1 \) and to the fourth terms of \( C_h, C_r \). Expressions for \( B \) and \( C \) coefficients are lengthy because of the non-commutativity of \( \vec{I}_j \) and \( \vec{D}_k \).

### 4.2. Angular momentum operators

The spherical components \( \hat{L}_{1\mu} \) of the particle angular momentum referring to the laboratory frame are expressed in terms of \( \hat{L}_k \), the cartesian components along the body-fixed axes

\[
\hat{L}_{1\mu} = \sum_{k,l} D_k^\mu \hat{L}_l
\tag{4.9}
\]

with the sum \((k, l)\) given by Eq. (3.9a). The \( D_k^\mu \) functions in the present case are of rank 1

\[
D_1^\mu = \frac{1}{\sqrt{2}} (D_{\mu - 1}^1 - D_{\mu 1}^1), \quad D_2^\mu = \frac{-i}{\sqrt{2}} (D_{\mu - 1}^1 + D_{\mu 1}^1), \quad D_3^\mu = D_3^0.
\tag{4.10}
\]

Commutation relations between these \( D_k^\mu \) with \( \vec{I}_l \) can be seen in Table II(b) of KKK2. In terms of \( c_i \) \((i = 1 \sim 3)\) operators, Eq. (4.9) can be rewritten as

\[
\hat{L}_{1\mu} = 2uvS_1D_1^\mu + \sum_{k=1}^3 D_k^\mu \hat{L}_k + \sum_{k,m} D_{km}^\mu \hat{\phi}_k^\mu,
\tag{4.11}
\]

where \((k, m) = (2, 3), (3, 2), \) and \( \hat{L}_k, \hat{\phi}_k^\mu \) are defined by Eq. (3.9b).

The lowest two terms of \( \hat{L}_{1\mu, \text{eff}} \) are given by

\[
\hat{L}_{1\mu, \text{eff}}[0] = J_0D_1^\mu, \quad \hat{L}_{1\mu, \text{eff}}[1] = \sum_{k,l} D_{kl}^\mu \vec{R}_l
\tag{4.12}
\]

with Eq. (3.9a) for \((k, l)\) and \( J_0 = 2uvS_1 \). We rearrange the second-order term in the same way as in the case of \( \hat{Q}_{2\mu, \text{eff}}[2] \) (4.8a)

\[
\hat{L}_{1\mu, \text{eff}}[2] = \sum_{k=1}^3 D_k^\mu \left( \sum_{m=x \sim z} (A_{km}[2] \vec{R}_m^2 + B_{km}[2] \vec{I}_m) + C_k[2] \right).
\tag{4.13}
\]

Under the separate operation of \( V_1 \) and \( V_2 \), \( A_{km}[2] \) \((k = 2, 3)\) turns out to be zero, while \( A_{1m}[2] \) is equal to the corresponding \( A_{1m}[2] \) coefficient of \( \hat{L}_{x,\text{eff}}[2] \). As in the \( \hat{L}_{x,\text{eff}} \) case, these two potentials have the opposite effects, so that the composite \( A_{1m}[2] \) becomes zero. Concerning the \( B \) coefficients, we find that \( B_{1x}[2, V_1] \) is equal to \( B_{2x}[2, V_1] \) given by Eq. (3.19b) and that \( B_{1x}[2, V_1 + V_2] \) vanishes. Similarly, the \( V_c^{(2)} \) contributions to \( B_{2x} \) and \( B_{3y} \) cancel away the \( V_1 \) contributions. However, in spite of these cancellations, we still have the following nonzero coefficients.
$B_{2x}[2] = -\frac{1}{2} (uv)^{(S_x - S_y) + \left( \frac{q}{2} - 1 \right) \frac{u^2}{S_z}}$, \\
$B_{3z}[2] = -\frac{1}{2} (uv)^{(S_x - S_y) - \left( \frac{q}{2} - 1 \right) \frac{v^2}{S_z}}$, \\
$C[2] = \frac{1}{4} u v (S_z + 1)$, 

(4.14)

in contrast with the case of $L_{k,\text{eff}}[2]$ (3.22).

§ 5. Effective Hamiltonian

Group-theoretically, the $SU_3$ Hamiltonian (2.1) is known to have the eigenvalue

$$E(\lambda, \mu, L) = \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3) - 2 \chi (\lambda^2 + \lambda \mu + \mu^2 + 3 \lambda + 3 \mu) + \frac{3}{2} \chi L (L + 1).$$

(5.1)

In the following we evaluate effective forms of the Hamiltonian up to second order in three ways and compare these results with Eq. (5.1).

1° As the first method, we make use of the zeroth-order driving Hamiltonian $H_0$ (2.14), whose effective form is composed of

$$H_{0,\text{eff}}[0] = \omega_1 \Sigma_1 + \omega_2 \Sigma_2 + \omega_3 \Sigma_3 = \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3) - 4 \chi (\lambda^2 + \lambda \mu + \mu^2),$$

(5.2a)

$$H_{0,\text{eff}}[1] = 0, \quad H_{0,\text{eff}}[2] = \sum_{k=x-z} \frac{\tilde{R}_k^2}{2 \tilde{g}_{k,00}^2}. $$

(5.2b)

The original Hamiltonian can be expressed as

$$H = H_0 + \left( \sum_\nu q_\nu [0] \tilde{Q}_\nu - \frac{1}{2} \chi \sum_\nu \tilde{Q}_\nu^+ \tilde{Q}_\nu \right) + Q_0 \bar{L}_x,$$

(5.3)

so that its effective form is given by

$$H_{\text{eff}} = H_{0,\text{eff}} + \Delta E_{00} + Q_0 \bar{L}_x, $$

(5.4)

Here the second term is

$$\Delta E_{00} = \sum_\nu q_\nu [0] \tilde{Q}_{\nu,\text{eff}} - \frac{1}{2} \chi \sum_\nu |\tilde{Q}_{\nu,\text{eff}}|^2, $$

(5.5a)

which can be rewritten up to second order as

$$\Delta E_{00} = \Delta E_{00}[0] - \Delta E_{00}[1], \quad \Delta E_{00}[n] = \frac{1}{2} \chi \sum_\nu |\tilde{Q}_{\nu,\text{eff}}[n]|^2. $$

(5.5b)

Using Eqs. (2.26) and (2.29) for $n=0$ and Eq. (3.12) for $n=1$, we obtain

$$\Delta E_{00}[0] = 2 \chi (\lambda^2 + \lambda \mu + \mu^2) - \frac{3}{2} \chi J_0^2, $$

(5.5c)

$$\Delta E_{00}[1] = \frac{1}{2} \tilde{g} \left( 1 - \frac{1}{q^2} \right) \tilde{R}_x^2. $$

(5.5d)
As $\tilde{L}_{x,\text{eff}}$ is already calculated as Eqs. (2·28), (3·10) and (3·22), it follows that

$$H_{\text{eff}[0]} = \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3) - 2\chi (\lambda^2 + \lambda \mu + \mu^2) + \frac{1}{2\tilde{g}} J_0^2, \quad (5·6a)$$

$$H_{\text{eff}[1]} = \frac{1}{\tilde{g}} J_0 \tilde{R}_x, \quad (5·6b)$$

$$H_{\text{eff}[2]} = \frac{1}{2\tilde{g}} \sum_{k=x-z} \tilde{R}_k^2. \quad (5·6c)$$

Therefore we can arrange $H_{\text{eff}}$ as

$$H_{\text{eff}[n \leq 2]} = \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3) - 2\chi (\lambda^2 + \lambda \mu + \mu^2) + \frac{3}{2} \chi \sum_{k=x-z} \tilde{1}_k^2. \quad (5·7)$$

We here meet two kinds of cancellation: First, the second term of $H_{\text{eff}} (5·7)$ is the result of $H_{0,\text{eff}[0]} + \Delta E_{\text{qq}[0]}$; second, the rotational energy is obtained by adding $Q_\alpha L_{x,\text{eff}[0]}$ and $\Delta E_{\text{qq}[0]}$ in zeroth order, and $H_{0,\text{eff}[2]}$ and $\Delta E_{\text{qq}[1]}$ in second order. Our result (5·7) differs from the exact result (5·1) by $\Delta E = 6\chi (\lambda + \mu)$. In KKK1 we argued the necessity to include the fluctuation effect to the Hartree approximation in order to dissolve this discrepancy.

2° Next, we start from Eq. (2·1) expressed with operators in the body-fixed frame. The effective single particle Hamiltonian $H_{\text{sp,eff}}$ vanishes except for the zeroth order

$$H_{\text{sp,eff}[0]} = \omega_0 (\Sigma_1 + \Sigma_2 + \Sigma_3). \quad (5·8)$$

We decompose the $\text{QQ}$ interaction as

$$H_{\text{qq,eff}} = -\frac{1}{2} \chi \sum_{p} \tilde{Q}_{r,\text{eff}}^2$$

$$= [0] \times [0] + [0] \times [1] + [1] \times [1] + [0] \times [2]. \quad (5·9)$$

The zeroth order term is equal to Eq. (5·5c) but with the opposite sign, and the first order term is

$$H_{\text{qq,eff}[1]} = \frac{1}{\tilde{g}} J_0 \tilde{R}_x. \quad (5·10)$$

One of the second-order terms, $[1] \times [1]$, agrees with $\Delta E_{\text{qq}[1]} (5·5d)$, while the rest $[0] \times [2]$ turns out to be equal to $H_{0,\text{eff}[2]} (5·2b)$. We owe this simplicity to the higher-order self-consistency. Hence we can conclude that this method gives the same result as 1°.

3° Lastly, we evaluate the effective form of Eq. (2·1) written in terms of the quadrupole operators in the laboratory frame. We divide $H_{\text{qq,eff}} = -(1/2) \chi \sum_{\mu} \tilde{Q}_{2\mu,\text{eff}}^2$ in the same way as Eq. (5·9). The $D$ functions involved in $\tilde{Q}_{2\mu,\text{eff}}$ can be eliminated by using the orthonormality relations

$$\sum_{\mu} D_{\mu}^{\alpha\beta} D_{\nu}^{\gamma} = \delta_{\nu}(3 - 2\delta_{ij}). \quad (i, j = \beta, \gamma, 1, 2, 3) \quad (5·11)$$
The zeroth order term is equal to \(-\Delta E_{\alpha\alpha}[0]\) as above, while the first order term is given by

\[
H_{\alpha\alpha,\text{eff}[1]} = -\frac{3}{2} \chi \left( S_x \left[ 1 + \frac{1}{q} (1 - 2uv\tilde{R}_x) \right] - S_y \left[ -3 + \frac{1}{q} (1 - 2uv\tilde{R}_x) \right] \right)
\]

\[
= -6\chi(\lambda + \mu) + \frac{1}{\eta} J_0 \tilde{R}_x .
\]

(5.12)

Although the first term of this equation recovers \(\Delta E\), this is merely fortuitous as will be seen below.

One of the second-order terms, \([1] \times [1]\), is

\[
H_{\alpha\alpha,\text{eff}[2a]} = -\frac{1}{2} \chi \left[ \frac{9}{4} \left( 1 + \frac{1}{q} \right)^2 \tilde{R}_x^2 + 3 \left( -3 + \frac{1}{q} \right)^2 \tilde{R}_x^2 \right] 
\]

\[
= -\frac{3}{2} \chi \left[ \frac{1}{2} + \frac{2}{q^2} + 4uv\tilde{R}_x + \left( \frac{2uv}{q} \right)^2 \tilde{R}_x^2 \right] .
\]

(5.13)

The other \([0] \times [2]\) part can be written as

\[
H_{\alpha\alpha,\text{eff}[2b]} = -\chi \left[ \sum_{m=x-z} \tilde{R}_m^2 (S_y A_m[2] - 3S_x A_m[2]) + \tilde{I}_x (S_y B_{mx}[2] - 3S_x B_{mx}[2]) \right] 
\]

\[
+ (S_y C_m[2] - 3S_x C_m[2]) ,
\]

(5.14)

where \(A\), \(B\) and \(C\) coefficients are defined by Eq. (4.8a) and given in the last column of Table II. This reduces to

\[
H_{\alpha\alpha,\text{eff}[2b]} = \frac{3}{2} \chi \left( \frac{\tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{R}_z^2}{q^2} \right) - \frac{6\chi uv}{q^2} \tilde{I}_x - \frac{3}{2} \chi C .
\]

(5.15)

Here \(C\) is

\[
C = -2(\lambda + \mu) + 3 - q - \frac{1}{4} q^2 (1 - q^2) \lambda^2 \left( \frac{1}{\mu} + \frac{1}{\lambda + \mu} \right) + \frac{1}{4} (1 - q^2) \lambda^2 \left( \frac{1}{\mu} - \frac{3}{\lambda + \mu} \right) .
\]

(5.16)

Summing up these together, we find \(H_{\alpha\alpha}[n \leq 2]\) to differ from Eq. (5.7) by a constant term

\[
H_{\alpha\alpha}[n \leq 2] = \text{Eq. (5.7)} - \frac{3}{2} \chi \left[ 4(\lambda + \mu) + \frac{1}{2} + \frac{2}{q^2} + C \right] .
\]

(5.17)

The difference proportional to \(\lambda\) may be related to imposing the constraint (2.12) in the body-fixed frame. There arises similar difference between two expressions of the magnitude of the angular momentum

\[
\sum_{k=x-z} \tilde{L}_k^2, \quad \sum_{\mu=0,\pm 1} \tilde{L}_{1\mu,\text{eff}}^2.
\]

(5.18)
Summary and discussion

We have studied a rapidly rotating SU3 system by extending the DNFT method developed in Refs. 4)~6). The present model is a kind of compromise: Rotation is assumed to be composed of two parts, the main fast rotation around the $x$ axis and the three-dimensional slow rotation. Incorporating the former in the single-particle Hamiltonian by means of the usual CHO diagonalization procedure, we concentrate to treat the latter quantum-mechanically. The effective representations of operators are expanded in powers of the coupling potential between particles and the rotation. The field coupling includes dynamical ones in addition to the simple Coriolis term. These are produced as a result of keeping in higher orders two conditions, the self-consistency between the average quadrupole moment and the nuclear deformation, and the equality of the effective particle angular momentum with the total angular momentum. In higher orders we have corrections to the zeroth-order deformations and moment of inertia, which turn out to induce the dynamical couplings. We note that the higher-order self-consistency conditions played a similar role in the problem of pair excitation modes.10)

These two conditions are imposed in the body-fixed frame. Due to this, the effective forms of two basic intrinsic operators $\tilde{L}_a$ and $\tilde{Q}_v$ give satisfactory results for the effective Hamiltonian. In contrast, $H_{\text{eff}}$ calculated through the use of their laboratory components contains unnecessary terms. This situation cannot be remedied ever if we take into account the residual part of the $QQ$ interaction. We have studied the effects of the following interactions (see also KKK1)

$$V_a = \frac{1}{2} \sum_{n=1}^{3} (\tilde{Q}_n^2 - \tilde{L}_n^2), \quad V_b = \frac{1}{2} \lambda (\tilde{Q}_\rho - \langle \tilde{Q}_\rho \rangle)^2 - \frac{3}{2} \lambda (\tilde{Q}_r - \langle \tilde{Q}_r \rangle)^2. \quad (6\cdot1)$$

However we could not cancel out extra terms appearing in $H_{\text{eff}}$. The discrepancy is smaller than the main term by one order of magnitude with respect to the order parameters ($\lambda, \mu$) which can be regarded as the expansion parameters of another kind from those of our perturbative expansion. This may be related to the result that the part of $a_2[2]$ dependent linearly on $\tilde{I}_m$ does not satisfy the self-consistency condition. In fact, we could not solve the condition (2·9) in second order by expressing it as Eq. (3·4) and using the whole $\tilde{Q}_{\nu, \text{eff}}[2, V_1]$ (3·15). We cannot see whether this is due to our inconsistent way to treat the three-dimensional rotation or some fundamental defects of the DNFT method.

We have here assumed the static deformation and uniform rotation. By relaxing these assumptions we can study the rotation-vibration coupling mode or the wobbling motion. We again note that the former effect was considered in Ref. 10) concerning the pair excitation mode.

Acknowledgements

The author expresses his sincere thanks to late Dr. T. Kishimoto and Dr. S.-I. Kinouchi. This work started while Dr. Kishimoto was still alive and very much
active. Without their cooperation and discussion, this paper could not have been written. He also thanks Miss Y. Miura for carefully preparing the manuscript.

**Appendix A**

--- Some Basic Operators ---

We here summarize definitions of various operators used in this paper. For each single particle operator \(T\), we associate the contribution from the \(n\)th particle denoted as \((t)_n\) by using the lower-case letter

\[
T = \sum_{n=1}^{N}(t)_n.
\]  

(A·1)

We first give operator referring to the body-fixed frame expressed in terms of the original oscillator-quanta operators \(c_k\) \((k=x, y, z)\). The \(\Delta N=0\) parts of the angular momentum operators are given by

\[
\vec{L}_x = E_{yz} + E_{zy}, \quad \vec{L}_y = i(E_{xz} - E_{zx}), \quad \vec{L}_z = -(E_{xy} + E_{yx})
\]  

(A·2)

in terms of the set of operators \((k, l=x \sim z)\)

\[
E_{kl} = \sum_{n=1}^{N} \left( c_k^* c_l + \frac{1}{2} \delta_{kl} \right)_n = \sum_{n=1}^{N} (e_{kl})_n.
\]  

(A·3)

The \(\vec{\Phi}_k\) operators proportional to the angle variables conjugate to \(\vec{L}_k\) in an average sense are defined by

\[
\vec{\Phi}_x = E_{yz} - E_{zy}, \quad \vec{\Phi}_y = i(E_{xz} + E_{zx}), \quad \vec{\Phi}_z = -(E_{xy} - E_{yx}).
\]  

(A·4)

The \(\Delta N=0\) parts of the quadrupole operators \(\vec{Q}_k\) in the rotating frame are then given by

\[
\vec{Q}_0 = \vec{Q}_z, \quad \vec{Q}_{z1} = \sqrt{\frac{3}{2}} \left( \vec{Q}_x \pm i \vec{Q}_y \right), \quad \vec{Q}_{z2} = -\sqrt{\frac{3}{2}} \left( \vec{Q}_x \mp i \vec{Q}_y \right),
\]  

(A·5)

\[
\vec{Q}_x = \vec{S}_x + \vec{S}_y, \quad \vec{Q}_y = \vec{S}_y, \quad \vec{Q}_z = \vec{S}_z,
\]  

(A·6)

with

\[
\vec{S}_x = E_{xz} - E_{zy}, \quad \vec{S}_y = E_{yz} + E_{zx}, \quad \vec{S}_z = E_{xy} - E_{yx}.
\]  

(A·7)

The unperturbed cranking Hamiltonian \(H_0\) \((2·19)\) can be diagonalized by the transformation \((2·21)\). The corresponding operators in the \(c_i\) \((i=1 \sim 3)\) basis are

\[
\vec{L}_1 = E_{23} + E_{32}, \quad \vec{L}_2 = i(E_{13} - E_{31}), \quad \vec{L}_3 = -(E_{12} + E_{21}),
\]  

(A·8)

\[
\vec{\Phi}_1 = E_{23} - E_{32}, \quad \vec{\Phi}_2 = i(E_{13} + E_{31}), \quad \vec{\Phi}_3 = -(E_{12} - E_{21}),
\]  

(A·9)

\[
\vec{S}_1 = E_{33} - E_{22}, \quad \vec{S}_2 = E_{33} - E_{11}, \quad \vec{S}_3 = E_{22} - E_{11},
\]  

(A·10)

in which \(E_{23}\) is obtained by putting \((k, l)\) as \((2, 3)\) in \((A·3)\), for example. We also utilize \(\vec{S}_0 = \vec{S}_2 + \vec{S}_3\).

The average of \(E_{ii}\) in an eigenstate of \(H_0\) is denoted as \(\Sigma_i\)

\[
\Sigma_i = \langle E_{ii} \rangle. \quad (i=1 \sim 3)
\]  

(A·11)
Conversely, this state can be specified as $|\Sigma_1, \Sigma_2, \Sigma_3\rangle$. We assume the configuration to satisfy the inequality $\Sigma_1\leq \Sigma_2 \leq \Sigma_3$, so that

$$E_{32}|=E_{31}|=E_{21}|=0.$$ \hfill (A·12)

The expectation values of $\bar{S}_i$ are denoted by the same letter,

$$S_1 = \Sigma_3 - \Sigma_2 = \lambda, \quad S_2 = \Sigma_3 - \Sigma_1 = \lambda + \mu, \quad S_3 = \Sigma_2 - \Sigma_1 = \mu.$$ \hfill (A·13)

Here $(\lambda, \mu)$ is the same as the label of an irreducible representation of the $SU_3$ group.

We will use $S_\pm$ given by

$$S_\pm = \frac{1}{2} S_1 \mp \frac{1}{2} S_3, \quad S_\mp = \frac{1}{2} S_2 \pm \frac{1}{2} S_3.$$ \hfill (A·14)

Operators $\tilde{L}_i$ (A·2), $\tilde{\Phi}_i$ (A·4) and $\tilde{Q}_\nu$ (A·5) can be expressed by using Eqs. (A·8~10)

$$\tilde{L}_2 = 2uv\tilde{S}_1 + q\tilde{L}_1, \quad \tilde{L}_2 = u\tilde{L}_2 + iv\tilde{\Phi}_2, \quad L_2 = uL_2 + iv\Phi_2,$$ \hfill (A·15)

$$\tilde{\Phi}_2 = \Phi_1, \quad \tilde{\Phi}_2 = u\tilde{\Phi}_2 + iv\tilde{\Phi}_3, \quad \Phi_2 = u\Phi_2 + iv\Phi_2,$$ \hfill (A·16)

$$\tilde{Q}_0 = \tilde{Q}_\pm = \tilde{S}_6 - 3uv\tilde{L}_1, \quad Q_{\pm 1} = \sqrt{\frac{3}{2}} (\Phi_1 \pm iu\tilde{\Phi}_2 \mp v\tilde{L}_3),$$ \hfill (A·17)

$$Q_{\pm 2} = \sqrt{\frac{3}{2}} (\mp Q_{\mp 1} \Phi_3 \mp iv\tilde{L}_2), \quad Q_r = \tilde{S}_r + uv\tilde{L}_1,$$ \hfill (A·18)

where

$$\tilde{S}_6 = \frac{1}{2} (\tilde{S}_6 + 3q\tilde{S}_1), \quad \tilde{S}_r = \frac{1}{2} (\tilde{S}_6 - q\tilde{S}_1).$$ \hfill (A·19)

**Appendix B**

--- Some of Formulae Used in Perturbative Expansions ---

**B.1. Basic first-order relations**

In the case that the $p$ space consists of just one state, we can define

$$A[B] = -\sum_i \gamma_i (p\tilde{A}_i \tilde{B}_p) + (p\tilde{B}_q \tilde{A}_q) = B[A]$$ \hfill (B·1)

for any two operators $\tilde{A}$ and $\tilde{B}$.

For three basic operators $\tilde{L}_i$, $\tilde{\Phi}_i$ and $\tilde{S}_i (i=1\sim3)$ given by Eqs. (A·8~10), the only non-vanishing $A[B]$ are

$$L_i[L_i] = -\Phi_i[\Phi_i] = -\mathcal{A}.$$ \hfill (B·2)

We have used the intrinsic matrix elements calculated in Appendix A of KKKK1. The products of $\tilde{Q}_\nu$ (A·17) with $\tilde{L}_i$ and $\tilde{\Phi}_i$ are

$$Q_0[L_i] = 3uv\mathcal{A}\delta_{i1}, \quad Q_{z1}[L_i] = \pm \sqrt{\frac{3}{2}}v\mathcal{A}\delta_{i3},$$

$$Q_{z2}[L_i] = \sqrt{\frac{3}{2}}\mathcal{A}(uv\delta_{i1} \pm iv\delta_{i2}).$$
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\[ Q_0[\Phi_i] = 0, \quad Q_{\pm 1}[\Phi_i] = \sqrt{\frac{3}{2}} \mathcal{J} (\delta_{i1} \pm i\delta_{i2}), \]
\[ Q_{\pm 2}[\Phi_i] = \mp \sqrt{\frac{3}{2}} \mathcal{J} u \delta_{i3}. \quad (\text{B.3}) \]

From these relations we have
\[ Q_0[Q_v] = -(3uv)^2 \mathcal{J} \left[ \delta_{v0} + \frac{1}{\sqrt{6}} (\delta_{v2} + \delta_{v,-2}) \right], \]
\[ Q_1[Q_v] = 3 \mathcal{J} v_{-1}, \]
\[ Q_2[Q_v] = -\sqrt{\frac{1}{6}} (3uv)^2 \mathcal{J} \delta_{v0} - \frac{3}{2} [(uv)^2 - 1] \mathcal{J} \delta_{v2} - \frac{3}{2} [(uv)^2 + 1] \mathcal{J} \delta_{v,-2}, \]
\[ Q_v[L_x] = 3uvq \mathcal{J} \left( \delta_{v0} + \frac{1}{\sqrt{6}} \delta_{v,\pm 2} \right), \]
\[ Q_v[L_y] = Q_v[L_z] = 0, \]
\[ L_x[L_x] = -q^2 \mathcal{J}, \quad L_y[L_y] = L_z[L_z] = -\mathcal{J}, \]
\[ L_k[L_k] = 0 \quad \text{for} \quad k \neq l = x, y, z. \quad (\text{B.4}) \]

B.2. Effective forms of \( \bar{\mathcal{L}} \mathcal{D} \) and \( \bar{\Phi} \mathcal{D} \)

We write down below the effective forms of \( \bar{\mathcal{L}}_i \mathcal{D} \) (\( i=1 \sim 3 \)). The \( \mathcal{D} \) function can be unity or a linear combination of \( D_{\nu}^\nu \) or \( D_\nu \) with respect to the \( \nu \) indices. Results for \( \bar{\Phi} \mathcal{D} \) can be obtained by exchanging the commutator (the outermost one in the second-order case) with the anticommutator and vice versa.

First, the \( \mathcal{V}_i \) operation gives the following:
\[ (\bar{\mathcal{L}}_i \mathcal{D})_{\text{eff}}[1, \mathcal{V}_i] = \]
\[\begin{align*}
  i=1 & \quad \frac{1}{2q} [\bar{R}_x, \mathcal{D}] , \\
  i=2 & \quad \frac{1}{2} u [\bar{R}_y, \mathcal{D}] - \frac{1}{2} iv [\bar{R}_z, \mathcal{D}] , \\
  i=3 & \quad \frac{1}{2} u [\bar{R}_z, \mathcal{D}] - \frac{1}{2} iv [\bar{R}_y, \mathcal{D}] , \quad (\text{B.5})
\end{align*}\]
\[ (\bar{\mathcal{L}}_i \mathcal{D})_{\text{eff}}[2, \mathcal{V}_i] = \]
\[\begin{align*}
  i=1 & \quad iqu^2 [\bar{R}_x, [\bar{R}_x, \mathcal{D}]] + iqv^2 [\bar{R}_z, [\bar{R}_z, \mathcal{D}]] \\
  & \quad - quv [\bar{R}_y, [\bar{R}_y, \mathcal{D}]] + quv [\bar{R}_x, [\bar{R}_z, \mathcal{D}]] , \\
  i=2 & \quad u [R_x, \mathcal{D}] + iv [R_z, \mathcal{D}] , \\
  i=3 & \quad -iu [R_y, [\bar{R}_x, \mathcal{D}]] - v [\bar{R}_z, [\bar{R}_z, \mathcal{D}]]. \quad (\text{B.6})
\end{align*}\]

In the special case \( D=1 \), these reduce to
\( \hat{L}_{\text{eff}}[1, V_1] = \left( \frac{1}{q} \hat{R}_x, u \hat{R}_y, u \hat{R}_z \right), \)

\( \hat{\Phi}_{\text{eff}}[1, V_1] = (0, -iv \hat{R}_x, -iv \hat{R}_y), \)

\( \hat{L}_{\text{eff}}[2, V_1] = \frac{u}{2qS_2} \hat{R}_z \delta_{i2}, \quad \hat{\Phi}_{\text{eff}}[2, V_1] = \frac{iv}{2qS_2} \hat{R}_z \delta_{i2}. \) \( \text{(B.7)} \)

Next, we have the following first-order effects from \( V_q^{(2)} \) and \( V_c^{(2)} \)

\( (\hat{L}_D)_{\text{eff}}[1, V_q^{(2)}] = \frac{1}{2} \tilde{g} \delta_{ii} \sum_k W_k (\hat{R}_k^2, D), \) \( \text{(B.8)} \)

\( (\hat{L}_D)_{\text{eff}}[1, V_c^{(2)}] = \)

\( i = 1 \quad -\frac{1}{2} qf_z \{ \hat{I}_x, D \}, \)

\( i = 2 \quad -\frac{1}{2} uf_z (\hat{I}_y, D) + \frac{1}{2} iv f_z [\hat{I}_x, D], \)

\( i = 3 \quad -\frac{1}{2} uf_z (\hat{I}_y, D) + \frac{1}{2} iv f_z [\hat{I}_x, D]. \) \( \text{(B.9)} \)

Here \( W_k \) and \( f_k \) are given by Eqs. (3·17) and (3·25), respectively.

B.3. Second order effective \( \hat{S}_kD \) operators

Effective forms of \( \hat{S}_kD \) \((k=1 \sim 3)\) operators are zero in first order in \( V_1 \). In second order we can arrange them as

\( (\hat{S}_kD)_{\text{eff}}[2, V_i] = \sum_{\text{in}} c_{ik} \hat{R}_i. \) \( \text{(B.10)} \)

Table III lists the operators \( \hat{R}_i \) and coefficients \( c_{ik} \), where \( \Delta S_{ki} \) is

\( \Delta S_{ki} = \Delta S_{ki} | \hat{S}_k; \Delta S_{ki} = \langle \hat{S}_k \rangle_i | - S_k. \) \( \text{(B.11)} \)

Here \( \langle \hat{S}_k \rangle_i \) represents the average of \( \hat{S}_k \) in the state \( |i\rangle = E_{25} \rangle, E_{13} \rangle, E_{12} \rangle \) for \( i = 1, 2, 3 \), respectively, and \( \Delta S_{ki} \) are given in Table IV.

In the special case \( D=1 \) and \( k=0, 1 \), we can write

\( \hat{S}_{k, \text{eff}}[2, V_i] = \sum_{m=x-z} A_{km} \hat{R}_m^2 + B_k \hat{I}_x, \) \( \text{(B.12)} \)

whose coefficients are given in Table V.
Rapidly Rotating Nuclei in the SU₃ Model

Table III. \((S,D)_{\text{ef}}[2, \gamma]\)

<table>
<thead>
<tr>
<th>(i)</th>
<th>(R_i)</th>
<th>(C_{\gamma\omega})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>([\bar{R}_z, [\bar{R}_z, D]])</td>
<td>(-\frac{S_z}{8q^2 S_1})</td>
</tr>
<tr>
<td>2</td>
<td>([\bar{R}_y, [\bar{R}_y, D]])</td>
<td>(-\frac{S_z}{8}(\frac{u^2 + v^2}{S_2}))</td>
</tr>
<tr>
<td>3</td>
<td>([\bar{R}_x, [\bar{R}_x, D]])</td>
<td>(-\frac{S_z}{8}(\frac{u^2 + v^2}{S_2}))</td>
</tr>
<tr>
<td>4</td>
<td>([\bar{R}_y, [\bar{R}_x, D]])</td>
<td>(-\frac{1}{8} iuv S_x S_z)</td>
</tr>
<tr>
<td>5</td>
<td>([\bar{R}_z, [\bar{R}_x, D]])</td>
<td>(\frac{1}{8} iuv S_x S_z)</td>
</tr>
<tr>
<td>6</td>
<td>(D R_x^2 + [\bar{R}_x, D] R_x)</td>
<td>(\frac{\Delta S_{\gamma\omega}}{4q^2})</td>
</tr>
<tr>
<td>7</td>
<td>(D R_y^2 + [\bar{R}_y, D] R_y)</td>
<td>(\frac{1}{4} (u^2 \Delta S_{\gamma\omega} + v^2 \Delta S_{\omega\omega}))</td>
</tr>
<tr>
<td>8</td>
<td>(D R_z^2 + [\bar{R}_z, D] R_z)</td>
<td>(\frac{1}{4} (u^2 \Delta S_{\gamma\omega} + v^2 \Delta S_{\omega\omega}))</td>
</tr>
<tr>
<td>9</td>
<td>([\bar{R}_y, D] [\bar{R}_x\gamma\omega\omega])</td>
<td>(-\frac{1}{4} iuv (S_x S_z + \Delta S_{\gamma\omega} - \Delta S_{\omega\omega}))</td>
</tr>
<tr>
<td>10</td>
<td>([\bar{R}_z, D] [\bar{R}_x\gamma\omega\omega])</td>
<td>(-\frac{1}{4} iuv (S_x S_z + \Delta S_{\gamma\omega} - \Delta S_{\omega\omega}))</td>
</tr>
<tr>
<td>11</td>
<td>(D [\bar{R}_y, \bar{R}_x\gamma\omega\omega])</td>
<td>(-\frac{1}{4} iuv (\Delta S_{\gamma\omega} - \Delta S_{\omega\omega}))</td>
</tr>
</tbody>
</table>

Table IV. \(\Delta S_{\gamma\omega}\)

<table>
<thead>
<tr>
<th>(k)</th>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

Table V. Coefficients in the expansion of \(\bar{S}_{\text{ef}}[2, \gamma]\)

<table>
<thead>
<tr>
<th>(k=0)</th>
<th>(k=1)</th>
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</thead>
<tbody>
<tr>
<td>(A_{\gamma\omega})</td>
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</tr>
<tr>
<td>(A_{\gamma})</td>
<td>(-\frac{3}{8} (qS_+ + S_+))</td>
</tr>
<tr>
<td>(A_{\omega})</td>
<td>(-\frac{3}{8} (qS_+ + S_-))</td>
</tr>
<tr>
<td>(B_k)</td>
<td>(-\frac{3}{4} uvS_-)</td>
</tr>
</tbody>
</table>

References