The scale dependent nature of coarse-graining is investigated by deriving the collision term of the Boltzmann equation. Numerical evaluation of the collision term is carried out in a one-dimensional mass sheet system. This shows that the collision term contains only the phase-mixing effect if the number of particles in a coarse-grained volume is sufficiently large. This result also indicates that $N$-body simulations can trace the original system exactly over a time scale proportional to the number of particles in the simulation, after which the scattering effect becomes important.

§ 1. Introduction

The dynamics of stellar systems is completely determined by the equations of motion of the individual stars. However, in practice we do not have complete information about the individual stars, and so we are usually interested in macroscopic characteristics.

There are two main procedures to obtain macroscopic quantities. One is long time averaging. This is appropriate for systems in a quasi-static state, such as globular clusters. Long time averaging is often equivalent to statistical averaging over an ensemble of all possible micro-states that can produce the given macroscopic properties. The other procedure is coarse-graining, which is defined as averaging within small but finite volumes in phase space. This becomes necessary when the system evolves dynamically, so that we are interested in its behaviour over short time scales.

Coarse-graining does not change the original system as observation does not, but the coarse-grained quantities may behave qualitatively differently from the microscopic ones. One of the most important examples of this is the relaxation of collisionless stellar systems. It is difficult to discuss relaxation for collisionless systems, since the Boltzmann entropy$^1$ defined in the microscopic state does not change in time. Lynden-Bell’s idea$^2$ was that the coarse-grained system can have increasing entropy, and hence can relax. However, numerical simulations have shown that some equilibrium states do not have the distribution that Lynden-Bell argued$^{3,4}$. Other authors have shown that coarse-grained evolution does not always increase the Boltzmann entropy.$^{5-8}$ Thus, it still is necessary to investigate the effects of coarse-graining.

In a previous paper$^9$ (we refer to it as Paper I) we showed that the coarse-grained Boltzmann entropy changes due to two main effects: scattering, which is a familiar effect, e.g., in molecular dynamics, and phase-mixing, which is induced by the long range nature of gravity. These effects must depend on the scale of the coarse-graining. In this paper we investigate the scale dependent characteristics of coarse-graining.
As in Paper I, we discuss the effects of coarse-graining by looking at the collision term in the Boltzmann equation. The coarse-graining procedure and the induced collision term are given in § 2. We coarse-grain a one-dimensional mass-sheet system and give a numerical estimate of the collision term in § 3. We discuss the reliability of N-body simulations of one-dimensional systems in § 4.

§ 2. Coarse-graining

Let us consider a system with a distribution function (DF) \( f(x, v, t) \) which obeys the collisionless Boltzmann equation

\[
\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0,
\]

where repeated indices are summed. The coarse-graining is defined by the expression

\[
\langle f \rangle(w, t) = \int d^6 w' D(w') f(w + w', t),
\]

where \( w = (x, v) \) are phase space coordinates. \( D \) is a smoothing function which is normalized by \( \int D(w') d^6 w' = 1 \) and assumed to be even. It determines the scale of the smoothing region. This coarse-graining procedure is applicable to any quantity such as the potential. The coarse-grained quantities are denoted by the symbol \( \langle \cdots \rangle \).

The evolution of the coarse-grained DF is given by Eqs. (11) and (12) in Paper I,

\[
\left\{ \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} - \frac{\partial \langle \Phi \rangle}{\partial x_i}(x) \frac{\partial}{\partial v_i} \right\} \langle f \rangle = \Gamma_{CGx} + \Gamma_{CGv},
\]

where

\[
\Gamma_{CGx} = -\int d^6 w' D(w') v_i \frac{\partial}{\partial x_i} f(w + w', t),
\]

\[
\Gamma_{CGv} = \int d^6 w' D(w') \frac{\partial}{\partial x_i} \left[ \langle \Phi(x + x_i) - \langle \Phi \rangle(x) \rangle \frac{\partial}{\partial v_i} f(w + w', t) \right].
\]

In Paper I we showed that the effective collision terms contain both the phase-mixing and scattering effects. The former comes from the tidal force made by distant particles; i.e., it is produced by the mean field. On the other hand the latter is caused by closer particles that are inside the smoothing region. We also showed \( \Gamma_{CGx} \) contains only the phase-mixing effect and \( \Gamma_{CGv} \) contains both the phase-mixing and scattering effects.

In order to make a simple evaluation of the effect of phase-mixing, we write the collision term in a symbolic form

\[
\Gamma = \frac{1}{\mathcal{A}^6} \int d^6 w' [S_i(w + w') - \langle S_i(w) \rangle] \frac{\partial f}{\partial w_i} (w + w'),
\]

where we have taken \( \int d^6 w' D(w') \) to be integration over the region centred at the origin with volume \( \mathcal{A}^6 \). Since the phase-mixing effect is induced only by the tidal
force of the mean field, the variation of $S_i$ and $f$ within the region is small enough to be expanded in a Taylor series

$$S_i(w + w') \approx \langle S_i \rangle(w) + S_i^{(1)} w^i + S_i^{(2)} w^i w^k,$$

$$\frac{\partial f}{\partial w_i}(w + w') \approx \left( \frac{\partial f}{\partial w_i} \right) + g_{i}^{(1)} w^i,$$

where $|w'| < \Delta$ and $\langle S_i \rangle$, $\langle \partial f/\partial w_i \rangle$, $S_i^{(1)}$, $S_i^{(2)}$, and $g_{i}^{(1)}$ are constant in the patch. Then,

$$\Gamma = \frac{1}{6} \int d^6 w' (S_i^{(1)} w^i + S_i^{(2)} w^i w^k) \left[ \left( \frac{\partial f}{\partial w_i} \right) + g_{i}^{(1)} w^i \right] \propto \Delta^2.$$

Thus, the time variation of the phase space density induced by distortion, i.e., phase mixing, is proportional to the square of the dimensions of the smoothing region.

As for the scattering effect, the variation of $S$ and $f$ within the smoothing region can be very large. In this case, it is useful to use the Fourier analysis. The Fourier components of the DF are

$$\tilde{f}(k, t) = \int f(w, t) \exp[\text{i} k w] d^6 w.$$

As a smoothing function, we employ the Gaussian function

$$D(w; q) = \frac{q^6}{(2\pi)^3} \exp[-q^2 w^2/2],$$

where the wave number $q$ gives the scale of the coarse-graining. The collision terms in the Fourier space are given by

$$X_q(k, t) = \int \Gamma_{\text{col}} \exp[\text{i} k w] d^6 w,$$

$$V_q(k, t) = \int \Gamma_{\text{col}} \exp[\text{i} k w] d^6 w.$$

By using Eqs. (4) and (5), we obtain the expressions for $X_q$ and $V_q$ as follows:

$$X_q(k, t) = \tilde{f}(k, t) \frac{k_x k_y}{2q} \exp[-k^2/2q^2],$$

$$V_q(k, t) = \frac{1}{(2\pi)^3} \exp[-k^2/2q^2] \int d^3 k' \tilde{f}(k', 0, t) \tilde{f}(k_x - k'_x, k_y, t)$$

$$\times \frac{k'_x k'_y}{|k'_x|^2} \{1 - \exp[-k'_2(k'_x - k_x)/q^2]\},$$

where 0 is the zero vector. By definition, $X_q$ contains only the phase-mixing effect, and $V_q$ contains both the phase-mixing and scattering effects.

§ 3. Application to a one-dimensional mass sheet system

In this section we only consider the simple case of a one-dimensional mass sheet system, because it is complicated to evaluate $V_q$ in three-dimensions. However, mass
sheet systems are often used to study characteristics of gravitational many body systems, so that it is useful to investigate such systems.

We examine a system in which the particles are equally separated in phase space to form a water-bag (uniform density) configuration (see Fig. 1). We fix the units of length and time by setting half of the size of the system and $4\pi G$ to be unity. We also set the maximum velocity to be unity, which means that the system is in virial equilibrium. The total number of particles is $N \times N$. The Fourier transform of the DF is

$$f(k_x, k_v) = \frac{\sin(k_x N/(N-1)) \sin(k_v N/(N-1))}{N \sin(k_x/(N-1)) N \sin(k_v/(N-1))}.$$ (16)

Substituting Eq. (16) into Eqs. (14) and (15), we obtain exact expressions for the Fourier transform of the collision terms,

$$X_o(k) = \frac{k_x k_v}{2q^2} \exp\left[-(k_x^2 + k_v^2)/2q^2\right] \frac{\sin(k_x N/(N-1)) \sin(k_v N/(N-1))}{N \sin(k_x/(N-1)) N \sin(k_v/(N-1))}$$ (17)

and

$$V_o(k) = \frac{k_v \exp\left[-(k_x^2 + k_v^2)/2q^2\right]}{2\pi} \frac{\sin(k_v N/(N-1))}{N \sin(k_v/(N-1))}$$

$$\times \int dk_x' \frac{\sin(k_x N/(N-1)) \sin((k_x - k_x') N/(N-1))}{N \sin((k_x - k_x')/(N-1))}$$

$$\times \frac{1 - \exp[-k_x'(k_x' - k_x)/q^2]}{k_x'}.$$ (18)

For large $N$,

Fig. 1. The phase space distribution of our model. Note that the total number of the particles is $N \times N$.

Fig. 2. The maximum of $V_o$ with respect to $q$ for various $N$. 
Effects of the Coarse-Graining Scale

\[ X_\alpha (k) \approx \frac{1}{2q^2} \exp \left[ -\frac{(k_x^2 + k_y^2)}{q^2} \right] \sin \left( \frac{k_x N}{N - 1} \right) \sin \left( \frac{k_y N}{N - 1} \right) \sin^2 \left( \frac{\pi N}{N - 1} \right). \]  

(19)

This has the maximum value \( 1/2q^2 \). This result was expected from Eq. (9) because this term contains only the phase-mixing effect.

The dependence of \( V_\alpha \) on \( q \) is more complicated so we try to evaluate it numerically. Figure 2 shows the variation of the maximum of \( V_\alpha \) in \( k \)-space, which we refer to as \( V_{\text{max}} \), with respect to \( q \) for different \( N \) values. The wave number \( q = \pi \) corresponds to the size of the system, and the wave number \( q = N\pi \) corresponds to the mean separation of the particles. The three curves with different \( N \) show similar features. Each curve has its minimum at \( q = q_{\text{min}} \sim 0.1 N \pi \). This value of \( q \) corresponds to the smallest coarse-graining scale which still has many particles in each smoothed region. Thus for \( q > q_{\text{min}} \) the coarse-grained DF suffers from discreteness. In fact the steep increasing of \( V_{\text{max}} \) for \( q > q_{\text{min}} \) comes directly from the motions of the individual particles. Thus we can discuss meaningful characteristics of macroscopic quantities only for \( q < q_{\text{min}} \). In this case, the behaviour of \( V_{\text{max}} \) is independent of \( N \). This means that the macroscopic evolution obtained by smoothing over areas which contain sufficiently many particles is determined by the same physical processes as in the large \( N \) limit. The slope of the curve approaches \( q^{-2} \), therefore we can understand that the collision term arises only from the phase-mixing effect, because as shown in Eq. (9), the dependence \( V_{\text{max}} \sim q^{-2} \sim L^{-2} \) directly corresponds to the smooth variation of the force within the smoothing region.

§ 4. Discussion

The \( N \)-body simulation, which is one of the most powerful tools to study astrophysical objects, is usually much smaller system in number of particles than realistic objects. Thus, it is a very important matter how well a smaller system can simulate a larger system.

The only way to compare two systems with the different numbers of particles is to compare the coarse-grained DF with the same coarse-graining scale. If the coarse-grained DF of the smaller system evolves in the same way as that of the larger system, we can say that the simulation is reliable.

We can discuss the reliability of \( N \)-body simulations but in one-dimension, by using the results of the previous section.

Let us consider two systems with the number of particles \( N_1 \) and \( N_2 \), where \( N_1 < N_2 \), and give the same coarse-grained DFs with \( q \leq 0.1 N_1 \pi \). Until the collision term is not important these two systems evolve exactly in the same way because the left-hand side of Boltzmann equation (3) contains only coarse-grained quantities. The time scale in which the collision term becomes important can be given roughly inverse of \( V_{\text{max}} \). Then with \( q \sim 0.1 N_1 \pi \) this time scale, \( t_{N_1} \), is the longest, which gives

\[ t_{N_1} \sim V_{\text{max}}^{-1} \sim 10 \left( q/\pi \right) \sim 0.1 N_1^2. \]  

(20)

After this time scale these two systems may evolve differently because the finer structures are different. With \( q \leq 0.1 N_1 \pi \) the collision term becomes important in
much shorter time scale than $t_{N_1}$, but the two systems do give the same coarse-grained DFs because the finer-grained DFs are the same before $t_{N_1}$. In the result, $t_{N_1}$ is the time scale in which the system with $N_1^2$ particles evolves in the same way as that with $N_1^2$, where $N_1 < N_2$, irrespective of the coarse-graining scale.

Even though we have examined only one special system, the results we obtained seem more general. One of the reasons for this is that the evolution of a quantity coarse-grained at a scale larger than the mean separation of the particles is completely determined by the physical processes which are approximated in the large particle number limit, i.e., the continuous limit. Thus the effect of the neat arrangement of the particles is irrelevant. Also the global shape of the system affects $V_{\text{max}}$ only at $q \sim \pi$, which is not so interesting.

One important assumption we made is that the system has uniform density. In fact, the clear dependence of $V_{\text{max}}$ on $q$, $V_{\text{max}} \propto q^{-2}$, is due to this homogeneity of the distribution. The steep increase of $V_{\text{max}}$ occurs at $q \sim 0.1N\pi$, which is the scale at which the coarse-grained distribution deviates from homogeneity. Thus if a system has another scale of inhomogeneity other than the mean separation of the particles, the system may show an increase of $V_{\text{max}}$ at that scale.

To conclude, we give a remark about the reliability of N-body simulations in one-dimension. Let us consider a system, which is inhomogeneous but with a large number of particles so that we can regard the distribution as smooth. This means that the local density is homogeneous at a sufficiently small scale. In this case we can trace the system by a simulation with much fewer particles, because, as we have shown, the evolution over a time scale shorter than $t_{N_1}$ is independent of the number of particles. $t_{N_1}$ is proportional to the number of particles and so we can trace the system, e.g., ten times longer with ten times more particles.

Acknowledgements

It is a pleasure to thank Dr. N. Gouda for many stimulating discussions. The author also thanks Dr. E. D. Stewart and Dr. Y. Yamada for a careful reading of the manuscript of this paper.

References