Determination of elastic parameters of transversely isotropic media by seismic techniques

F.M. Lyakhovitskiy Department of Geology, Moscow State University, Moscow 117234, USSR

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Summary. Three techniques for determining the elastic parameters of a transversely isotropic rockmass are discussed. These can yield estimates of Young’s modulus and the shear modulus for engineering applications.

1 Introduction

The determination of elastic parameters of in situ rocks by seismic techniques is important for the solution of a number of practical problems in the engineering, mining, oil, and coal industries. The seismic data often reveal that the rockmass is transversely isotropic. Such cases present several problems in measurement techniques, selection of numerical models, and data interpretation.

2 Interpretation methods

Evaluation of the elastic constants $C_{ij}$ of an anisotropic rockmass requires measurements of velocity in a number of directions using quasi P- and quasi shear-waves. The following methods are available: cross-hole shooting (Fig. 1a); vertical-seismic-profiles (Fig. 1b); and reflection and refraction measurements (particularly appropriate for studying azimuthal anisotropy)(Fig. 2). White et al. (1983) describe the method of vertical-seismic-profiles using four geophones mounted in two nearby holes and a remote surface source.

A transversely-isotropic model (TIM) of the massif will be used for interpreting the data, and it is necessary to determine five elastic constants $C_{11}$, $C_{33}$, $C_{44}$, $C_{66}$, and $C_{13}$, where the z-axis is in the symmetry direction. Five independent P- and shear-wave velocity determinations can produce sufficient information for the evaluation of these parameters. However, in the course of this evaluation, considerable data redundancy is desirable in order to eliminate distortions caused by local inhomogeneities and to reduce the effect of errors of individual measure-
The graphic data approximation can then be carried out from discrete determinations of the arrival times or the P- and shear-wave velocities. We shall analyse a few examples.

White et al. (1983) suggest the following approximate techniques. The initial data are the values of the phase-velocities $V_p$, $V_{SV}$, $V_{SH}$ (Fig. 3) experimentally determined for the different angles $i$ between the vertical axis (the axis of symmetry) and the normal to the wave surface. The density $\rho$ is also assumed to be known. On the basis of the formula:

$$\rho(V_{SH})^2 = [C_{44} + (C_{66} - C_{44}) \sin^2 i];$$

(1)

and the coordinate system $X = \sin^2 i$, $Y = (V_{SH})^2$, the regression line gives $C_{44}$ and $(C_{66} - C_{44})$.

The following approximate equations are used to obtain the values $V_p(i)$ and $V_{SV}(i)$, where the anisotropy is assumed to be small:

$$(V_p)^2 = [C_{33} + \xi C_{33} \sin^2 i + \xi_m (C_{33} - C_{44}) \sin^2 i \cos^2 i]/\rho;$$

(2)
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\[ (V_{SV})^2 = \frac{[C_{44} + \xi_m (C_{33} - C_{44}) \sin^2 \theta \cos^2 \theta]}{\rho}; \]

where

\[ \xi_m = \frac{C_{33} [2 - 4(C_{13} + C_{44})/(C_{11} + C_{33}) - C_{33}]}{C_{33}^2 / (4A)}/A; \]

\[ \xi_1 = \frac{(C_{11} - C_{33})}{C_{33}}; \text{ and } A = C_{33} - C_{44}. \]

From equation (3), the \( C_{44} \) and \( (C_{33} - C_{44}) \). \( \xi_m \) values are determined for the pairs of values \( Y = (V_{SV})^2 \) and \( X = \sin^2 \theta \cos^2 \theta \) from the regression line. The \( C_{44} \) value is taken as the average of the two determinations and \( C_{66} \) can be determined. The \( C_{33} \) and \( \xi_1, C_{33} \) values are determined from the regression line for the pairs of values \( X = \sin^2 \theta, Y = (V_p)^2 + \xi_m (C_{33} - C_{44}) \sin^2 \theta \cos^2 \theta \), and \( \xi_1, \xi_m \), and \( C_{11}, C_{13} \) calculated consecutively. The results of this approximate method are shown in Fig. 3.

Another method of calculating \( C_{ij} \) directly from the travel-times of waves is suggested by Crampin & Radovich (1982). In this process the travel-times of \( P \)- and shear-waves are used in the (squared) coordinate system \( X = x^2, Y = t^2(x) \). The travel-time curve for the SH wave is a hyperbola and the approximation is made for a straight line in the \( (X,Y) \)-space. The velocity, determined by the dip \( a_k \) of the straight line is:

\[ (V_k)^2 = 1/a_k = \Delta x^2 / \Delta t^2, \]

where the bar superscript indicates group (ray) velocity. The modulus \( C_{66} \) is determined from \( V_{SH} \) and the modulus \( C_{44} \) from the value \( Y_{SH}(0) \) of the intersection with the axis \( Y \) (the depth \( H \) of the source is known):

\[ C_{44} = \frac{\rho H^2}{Y_{SH}(0)}; \quad C_{66} = \rho \left( \frac{V_{SH}}{Y_{SH}} \right)^2. \]
The P-wave travel-time $t^2(x^2)$ is approximated by two segments: the initial part as $x \to 0$; and for large $x$. Further approximate expressions are used which are valid for small anisotropy. In the first segment, we obtain:

$$\bar{v}^2_{p1} = \frac{[2(C_{13} + 2C_{44}) - C_{33}]}{\rho};$$

and

$$y_{p1}(0) = \rho \frac{h^2}{C_{33}};$$

whereas for the second segment (for large $x$) we obtain:

$$\bar{v}^2_{p2} = \frac{[7C_{11} + 2(C_{13} + 2C_{44}) - C_{33}]}{8\rho};$$

and

$$y_{p2}(0) = \frac{8h^2\rho}{[7C_{33} - C_{11} + 2(C_{13} + 2C_{44})]}$$

It should be noted, that the dip of the right-hand segment depends on the distance from the origin of the coordinates and on the length of the segment (the base of approximation).

This approach can also be used with other variables with more general conditions. The primary data are the discrete travel-times of P-, SV-, and SH-waves in the (squared) coordinate system. Their linear approximation is obtained for the entire interval for the SH-waves or for the initial part of the P-, SV-waves. For $x \to 0$ the velocity ratios $\bar{v}_{p1}/\bar{v}_{p}(0)$ and $\bar{v}_{sv1}/\bar{v}_{sv}(0)$ determine the apparent anisotropy coefficients $\bar{v}_{p}$ and $\bar{v}_{sv}$ of P- and SV-waves, respectively (see Lyakhovitsky 1981, equation 2). As shown by Lyakhovitskiy (1984), the approximate equation for a wide class of TIM is obtained as follows:

$$\bar{v}_{p} = \frac{\bar{v}_{p1}/\bar{v}_{p}(0)}{\xi} = \frac{(C_{13} + C_{44})}{C_{33}}$$

The constants $C_{44}$, $C_{66}$ are determined from equation (4) and $C_{33}$ from equation (6). The constant $C_{13}$ is derived from (9):

$$C_{13} = \bar{v}_{p1}(C_{33} \rho)^{1/2} - 2C_{44}$$

The dip of the travel-time graph of the SV-wave is used to determine $C_{11}$:

$$C_{11} = \rho \frac{(\bar{v}_{sv1})^2 + (C_{13} + C_{44})^2}{(C_{33} - C_{44})}.$$
3 Evaluation of technical elastic parameters

Young's modulus $E$, the shear modulus $G$, and Poisson's ratio $\nu$ are often used to solve the practical problems of mining and engineering geology. We shall call them technical parameters to distinguish from the elastic constants $C_{ij}$. The direct method of their determination is based on the study of deformation of sample material under conditions of uniform stress (for example, the axial strain of a rod). The indirect method, based on the calculation of elastic constants from the elastic-wave velocities, is called the dynamic method.

Let us analyse the connection between the technical parameters and the $C_{ij}$ constants for TIM. We assume the $z$-axis (downwards) to be the symmetry axis of the medium. The three values of Young's modulus, $E_x$, $E_y$, and $E_z$, correspond to the strain along the $x$-, $y$-, and $z$-axes, respectively. In the shear modulus $G_{ij}$, the first index denotes the direction of the normal to the section along which the tangent force is applied; and the second index the direction of this force. For Poisson's ratio $\nu_{ij}$, the first index shows the direction of the crosswise deformation, and the second index the direction of the axial deformation (compression or extension).

As described by Rabinovich (1946), the inequality

$$\nu_{yz} + \nu_{zx} + \nu_{xy} \leq 1.5$$

is obtained. Each of the $\nu_{ij}$ ratios can be more than 0.5, unlike the isotropic medium when the inequality $\nu \leq 0.5$ is valid.

Five technical parameters can be considered for TIM instead of the five elastic constants $C_{ij}$:

$$E_x = E_y = E_h; \quad E_z = E_v; \quad \nu_{xy} = \nu_{yz}; \quad \nu_{xz} = \nu_{yz}; \quad G_{xz} = G_{yz};$$

(13)

where $E_h$ and $E_v$ are the Young's moduli in the horizontal, and vertical directions, respectively. The remaining parameters are expressed by equations:

$$G_{xy} = (1 + \nu_{xy})E_h/2; \quad \nu_{zy} = \nu_{xz}; \quad \nu_{xz}/E_v = \nu_{zz}/E_h.$$  

(14)

The connection between the five technical parameters and $C_{ij}$ is as follows:

$$E_x = 4C_{66}[C_{33}C_{11} - C_{33}C_{66} - (C_{13})^2]/[C_{11}C_{33} - (C_{13})^2];$$
Figure 4. Test axis (z') in the case of a thin-layered two-component medium (z-axis of symmetry).

\[ E_z = \frac{C_{33}C_{11} - C_{33}C_{66} - (C_{13})^2}{C_{11} - C_{66}}; \]

\[ \nu_{yz} = 1 - \frac{2C_{33}C_{66}}{C_{11}C_{33} - (C_{13})^2}; \]

\[ \nu_{zx} = \frac{2C_{13}C_{66}}{C_{11}C_{33} - (C_{13})^2}; \]

\[ \nu_{xz} = \frac{C_{13}}{2(C_{11} - C_{66})}; \]

The formulae (15) correspond to the case where the tested sample is oriented so that the compression/strain occurs along the z-axis when determining E and \( \nu \). The tangent force is also in the same direction when determining \( G_{xz} \). Let us call this direction the test-axis z' which will not coincide with the z-axis (the axis of symmetry) but will make an angle \( \phi \) with it (Fig. 4). In the transversely isotropic medium the technical elastic parameters are a function of the angle \( \phi \) with the symmetry axis. Their graphs generate the rotation surfaces about this axis.

In the case of TIM, we shall have one graph of Young's modulus, two graphs of the shear modulus, and six graphs of Poisson's ratio, because at \( \phi \neq 0 \); \( \nu_{x'y'} \neq \nu_{yx}; \) \( \nu_{x'z'} \neq \nu_{yz}; \) \( \nu_{z'x'} \neq \nu_{z'y'} \). The equations of the graphs were obtained from the formulae in Sirotin & Shaskolskaya (1975), under the condition that the y-axis does not change its position.

The simplest method of calculation of the technical elastic parameters from the pairs of velocity values \( V_p \) and \( V_{SH} \) or \( V_p \) and \( V_{SV} \) is often used in practice. The velocities are determined from seismic data for different \( \phi \) angles. Calculation of the dynamic parameters \( E_{SH}, G_{SH}, \) and \( \nu_{SH} \) or \( E_{SV}, G_{SV}, \) and \( \nu_{SV} \) is performed by using the formulae for isotropic media.
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1.2

Figure 5. Graphs of the variation of Young's moduli $E_Z$, $E_{SH}$, $E_{SV}$ along the test axis $z'$, with respect to the angle $\phi$ between the $z'$ and the vertical symmetry axis $z$, for a model comprised of alternating thin layers of the media 1 and 2 which are described in Table 1.

Figure 6. Graphs of the variation of the shear moduli $G_{Z'X'}$, $G_{YZ'}$, $G_{SH'}$, $G_{SV}$ with angle $\phi$, for the same two-component thin-layered model as in Fig. 5.

What is the difference between these values and the true values of technical parameters in the case of TIM? This may be answered by mathematical modelling. The technical and dynamic elastic parameters - Young's modulus, the shear modulus and Poisson's ratio - derived from the two-two-component thin-layered model comprised of the media in Table 1 are shown in Figs. 5, 6, and 7 respectively.

The analysis of calculations for different models reveals that $E'(\phi)$ has a minimum between $\phi = \pi/6$ and $\pi/4$. The limits to the changes in $E'(\phi)$ and dynamic moduli $E_{SH}$ and $E_{SV}$ lie in the interval between the Young's
moduli for the first and second layers. In other words, the extreme high and low values are not observed for Young's integral modulus for the transversely isotropic medium formed by alternating thin layers. The behaviour of $E_{SV}$ graphs is quite different from that of $E'(\phi)$. $E_{SV}$ have an intermediate maximum and cusps because of the strong velocity-anisotropy. The $E_{SH}$ graphs have better correlation with $E'(\phi)$ graphs. The minimum $E_{SH}(\phi)$ and $E'(\phi)$ values are close, although they are refer to different angles (the minimum of $E_{SH} = E_{SH}(0)$). Consequently, Young's modulus $E_{SH}(0)$ may be used to estimate the minimum $E'(\phi)$ values. The maximum $E'(\phi)$ and $E_{SH}(\phi)$ values are also close and correspond to an angle $\phi = \pi/2$.

The graph of the $G_{yz}$ function increases monotonically, whereas the $G_{z''x'}$ function has a maximum at $\phi = \pi/4$ (Fig. 6). Functions $G_{yz''}, G_{z''x''}, G_{SH}, G_{SV}$ do not acquire extreme high or low values. The graphs of the "dynamic" shear modulus $G_{SH}$ exactly coincide with the graphs of the shear

<table>
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<tr>
<th>Medium</th>
<th>$E$ ($10^{10}$N/m$^2$)</th>
<th>$G$ ($10^{10}$N/m$^2$)</th>
<th>$\nu$</th>
<th>$\rho$ ($10^3$kg/m$^3$)</th>
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<td>1.54</td>
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<td>2</td>
<td>0.75</td>
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<td>0.20</td>
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Table 1. Parameters of the media comprising the two-component, thin-layered transversely isotropic model

![Graph of the variation of Poisson's ratio with angle \( \phi \) for the same two-component thin-layered model as in Fig. 5. The broken (crossed) lines (1) show \( \nu_1, \nu_2 \) for the media 1, 2, in Table 1 respectively. The lines marked by dots (2), and the solid lines (3), show the variation of the dynamic Poisson's ratios (\( \nu_{SH}, \nu_{SV} \)), and the Poisson's ratios (\( \nu_{z'x'}, \nu_{x'z'}, \nu_{x'y}, \nu_{yx'}, \nu_{z'y}, \nu_{yx} \)) with \( \phi \), respectively.](https://academic.oup.com/gji/article-abstract/91/2/439/874601)
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modulus $G_{yz'}$ and can be used for realistic estimations.

At $\phi = 0$, the following equation is obtained for Poisson's ratios: $v_{zx} = v_{zy} = v_{hh}; v_{yx} = v_{xy} = v_{hh}; v_{yz} = v_{xz} = v_{hv}$. The graphs in Fig. 7 have several peculiarities.

1) When orienting the sample along the major axes, the value $v_{hh}$ approaches the Poisson's ratio of the hardest element of the two-component model ($v_1$) (Lyakhovitsky and Nevsky 1970).

2) In the range $0^0 < \phi < 90^0$, the extreme high values ($> v_1, v_2$) are observed for $v_{x',z}, v_{z',x'}$, and the extreme low values are observed for $v_{y,z'}, v_{y,x'} (< v_1, v_2)$, where $v_1, v_2$ are Poisson's ratio for media 1, 2 respectively. Values $v_{z',y}$ and $v_{x',y}$ keep within the range $v_1$ to $v_2$. In this case, high values may exceed 0.5, and low values may approach zero.

4 Discussion

1) The variation of seismic properties with direction may be used to evaluate the elastic parameters of an anisotropic rockmass. Three variants are discussed for estimating the $C_{ij}$ constants of transversely isotropic media.

2) The dynamic elastic parameters determined from the velocity graphs can be used for approximate evaluation of Young's modulus, and the shear modulus of transversely isotropic media. But in some cases the dynamic Poisson's ratios may differ significantly from the true values.

References


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