Riemannian Structure Induced by Parameter-Dependent Quantum State Vectors

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(Received January 24, 1994)

The quantum distances introduced recently by Provost-Vallee, Berry and Anandan-Aharonov are formulated and generalized with the help of projection operators and their Hilbert-Schmidt norms. As an application, a time-energy uncertainty relation is discussed.

§ 1. Introduction

The Berry phase manifests itself in diverse branches of physics. It attracted much attention mainly because of its gauge theoretic structure. To explain what we intend to investigate in this paper, we briefly recapitulate some aspects of the Berry phase. Let $|\psi(s)\rangle$ be a normalized quantum state vector depending on a set of parameters $s=(s_1, s_2, \cdots, s_P)$. If we define a field on the parameter space $M=\{s\}$ by

$$a_\mu(s) = i \langle \psi(s) | \partial_\mu \psi(s) \rangle, \quad \partial_\mu = \frac{\partial}{\partial s_\mu},$$

it behaves like a $U(1)$ gauge potential under the gauge transformation

$$|\psi(s)\rangle \rightarrow e^{ia(s)} |\psi(s)\rangle, \quad a(s) \in \mathbb{R}.$$ (1.2)

For a closed contour $C$ in $M$, the Berry phase $\gamma(C)$ is given by

$$\gamma(C) = \oint_C a_\mu(s) ds^\mu = \frac{1}{2} \int_S f_{\mu\nu}(s) ds^\mu \wedge ds^\nu,$$ (1.3)

where $S$ is an (arbitrary) area bounded by $C$ and $f_{\mu\nu}(s)$ is the gauge field associated with the potential $a_\mu(s)$:

$$f_{\mu\nu}(s) = \partial_\mu a_\nu(s) - \partial_\nu a_\mu(s).$$ (1.4)

An interesting complex tensor field related to $f_{\mu\nu}(s)$ was introduced by Provost and Vallee as early as in 1980:

$$t_{\mu\nu}(s) = \langle \partial_\mu \psi(s) | \partial_\nu \psi(s) \rangle - \langle \partial_\nu \psi(s) | \psi(s) \rangle \langle \psi(s) | \partial_\mu \psi(s) \rangle.$$ (1.5)

The tensor $t_{\mu\nu}(s)$ is invariant under the transformation (1.2) and its imaginary part is given by

$$\text{Im}(t_{\mu\nu}(s)) = -\frac{1}{2} f_{\mu\nu}(s).$$ (1.6)

On the other hand, defining another gauge invariant field $g_{\mu\nu}(s)$ by

$$g_{\mu\nu}(s) = \text{Re}(t_{\mu\nu}(s)),$$ (1.7)
we have

\[ g_{\mu\nu}(s)ds^\mu ds^\nu = 2 - 2|\phi(s + ds)|\phi(s)|^2, \quad (1.8a) \]

\[ = 1 - |\langle\phi(s + ds)|\phi(s)\rangle|^2 \quad (1.8b) \]

up to the second order of \( ds^\mu \). The expressions (1.8a) and (1.8b) were discussed by several authors as the quantum distance between the states \( |\phi(s)\rangle \) and \( |\phi(s + ds)\rangle \). The field \( g_{\mu\nu}(s) \) can be regarded as the metric tensor to measure the distance in the set of \( s \)-dependent state vectors.

The purpose of this paper is to generalize the above-introduced metric structure of the set of \( s \)-dependent state vectors to the case of the set of \( N \)-dimensional linear subspaces spanned by \( N \) \( s \)-dependent vectors. An \( N \)-dimensional linear subspace is specified by fixing an orthonormal set of \( N \) vectors, i.e., an \( N \)-frame. The metric tensor on the set of \( N \)-dimensional linear subspaces should be invariant under unitary transformations of \( N \) basis vectors defining the \( N \)-frame. In § 2, we discuss how such a metric tensor is constructed. Our definition of the metric tensor coincides with the one recently discussed by Tanimura.\(^6\) We find that the discussion is very much elucidated by introducing a projection operator which projects the whole Hilbert space onto the linear subspace concerned. For finitely separated two \( N \)-dimensional linear subspaces, we define three types of distance. In § 3, we discuss the detail of the derivation of the formula given in § 2. We consider in § 4 an application of the quantum distance. We obtain a generalized version of the Anandan-Aharonov uncertainty relation, which is described in terms of the geodesic distance. By making use of the distance other than the geodesic one, we are led to a new type of time-energy uncertainty relation. Although the last relation is less stringent than the Anandan-Aharonov type one, it supplies us, with a simple method to estimate the lower bound of the time-energy uncertainty. The final section is devoted to a summary and an outlook.

### § 2. Metric on the set of linear subspaces

#### 2.1. Metric tensor

As in the previous section, we consider quantum state vectors which depend on a set of parameters \( s = (s^1, s^2, \ldots, s^\rho) \). We express a set of \( s \)-dependent orthonormal vectors by \( \Phi(s) \):

\[ \Phi(s) = \{|\phi_i(s)\rangle: i = 1, 2, \ldots, N, \langle\phi_i(s)|\phi_j(s)\rangle = \delta_{ij}\} \quad (2.1) \]

and call it an \( N \)-frame. The \( \Phi(s) \) defines a linear subspace \( h(s) \) spanned by the \( |\phi_i(s)\rangle \)'s:

\[ h(s) = \bigoplus_{i=1}^N C|\phi_i(s)\rangle. \quad (2.2) \]

\( \Phi(s) \) also defines a projection operator

\[ P(s) = \sum_{i=1}^N |\phi_i(s)\rangle\langle\phi_i(s)| \quad (2.3) \]
satisfying
\[ P(s)^* = P(s), \quad P(s)^2 = P(s). \]  
\[ (2.4) \]
Both of \( h(s) \) and \( P(s) \) are invariant under unitary transformations
\[ |\psi_i(s)\rangle \to \sum_{j=1}^{N} U_{ij}(s) |\psi_j(s)\rangle, \]
\[ \sum_{j=1}^{N} U_{ij}(s) U_{kj}(s)^* = \delta_{ik}, \quad i, k = 1, 2, \ldots, N . \]  
\[ (2.5) \]
We hereafter call a transformation of the type of (2.5) a gauge transformation. We consider in this subsection how a gauge invariant metric tensor can be defined so as to measure the distance between \( h(s) \) and \( h(s + ds) \) or between \( P(s) \) and \( P(s + ds) \) or between \( s \) and \( s + ds \), \( ds \) being infinitesimal.

If we define an \( N \times N \) matrix \( A_{\mu}(s) \) by
\[ (A_{\mu}(s))_{ij} = i \langle \psi_i(s) | \partial_{\mu} \psi_j(s) \rangle, \quad i, j = 1, \ldots, N, \]  
\[ (2.6) \]
we have \( (A_{\mu}(s))^* = (A_{\mu}(s))_{\mu} \), i.e.,
\[ A_{\mu}(s)^* = A_{\mu}(s). \]  
\[ (2.7) \]
\( A_{\mu}(s) \) transforms like a \( U(N) \) gauge potential under (2.5). Generalizing the fields \( t_{\mu\nu}(s) \) and \( g_{\mu\nu}(s) \) introduced in § 1, we define \( T_{\mu\nu}(s) \) and \( G_{\mu\nu}(s) \) by
\[ T_{\mu\nu}(s) = \sum_{i=1}^{N} \langle \partial_{\mu} \psi_i(s) | \partial_{\nu} \psi_i(s) \rangle - \sum_{i,j=1}^{N} \langle \partial_{\mu} \psi_i(s) | \psi_j(s) \rangle \langle \psi_j(s) | \partial_{\nu} \psi_i(s) \rangle \]  
\[ (2.8) \]
and
\[ G_{\mu\nu}(s) = \text{Re} \{ T_{\mu\nu}(s) \} . \]  
\[ (2.9) \]
The geometric meaning of \( G_{\mu\nu}(s) \) has been discussed in Ref. 6). The imaginary part of \( T_{\mu\nu}(s) \) can be written as
\[ \text{Im} \{ T_{\mu\nu}(s) \} = -\frac{1}{2} \sum_{i=1}^{N} (F_{\mu\nu}(s))_{ii}, \]  
\[ (2.10) \]
where \( F_{\mu\nu}(s) \) is the \( U(N) \) gauge field:
\[ F_{\mu\nu}(s) = \partial_{\mu} A_{\nu}(s) - \partial_{\nu} A_{\mu}(s) - i[A_{\mu}(s), A_{\nu}(s)]. \]  
\[ (2.11) \]
Note that the r.h.s. of (2.10) is essentially the one in the \( U(1) \) case. It can be seen that \( T_{\mu\nu}(s) \) is gauge invariant and so is \( G_{\mu\nu}(s) \). It turns out that \( T_{\mu\nu}(s) \) can be rewritten as
\[ T_{\mu\nu}(s) = \text{Tr} [ P(s) \{ \partial_{\mu} P(s) \} \{ \partial_{\nu} P(s) \} ], \]  
\[ (2.12) \]
where \( \text{Tr} \) denotes the trace in the Hilbert space:
\[ \text{Tr} A = \sum_{a} \langle \varphi_{a} | A | \varphi_{a} \rangle, \]  
\[ (2.13) \]
\( \{ \varphi_{a} \} \) being an orthonormal complete set of the Hilbert space. Similarly, \( G_{\mu\nu}(s) \) is
given by
\[ G_{\mu\nu}(s) = \frac{1}{2} \text{Tr} \left[ \{ \partial_\mu P(s) \} \{ \partial_\nu P(s) \} \right]. \] (2.14)

The gauge invariance of \( T_{\mu\nu}(s) \) and \( G_{\mu\nu}(s) \) is manifest in the expressions (2.12) and (2.14). Note that the gauge invariant quantities such as \( P(s) \), \( T_{\mu\nu}(s) \) and \( G_{\mu\nu}(s) \) can be considered as functions on the set \( \{ h(s) \} \) rather than on \( \{ \Phi(s) \} \).

2.2. Distance

Among many possible definitions of a norm of an operator, we here quote the Hilbert-Schmidt norm
\[ \| A \| = \sqrt{\text{Tr}(A^* A)} . \] (2.15)

Then the distance \( D_1(A_1, A_2) \) between two operators \( A_1 \) and \( A_2 \) can be defined by
\[ D_1(A_1, A_2) = \| A_1 - A_2 \| . \] (2.16)

Noticing that \( \text{Tr} P(s) \) equals \( N \) for the projection operator \( P(s) \) corresponding to an \( N \)-frame, we find that the distance \( D_1(P_1, P_2) \) between two projection operators \( P_1 \) and \( P_2 \) is given by
\[ D_1(P_1, P_2) = \sqrt{N_1 + N_2 - 2 \text{Tr}(P_1 P_2)} , \] (2.17)

where \( N_i = \text{Tr} P_i, \ i=1, 2, \) are positive integers. For the case that \( N_1=N_2=1, \ P_1=|\phi><\phi|, \ P_2=|\phi><\phi|, \) (2.17) reduces to \( D_1(P_1, P_2)/\sqrt{2} = \sqrt{1 - |\langle \phi | \phi \rangle|^2} \), which is the distance discussed in Refs. 4) and 5). Provost and Vallee's discussion\(^b\) on the quantum distance suggests that the distance defined by
\[ D_2(P_1, P_2) = \inf_{W: \text{unitary}} \| P_1 - WP_2 \| \] (2.18)
would be convenient. In the next section, we will discuss the inequality
\[ D_2(P_1, P_2) \geq \sqrt{N_1 + N_2 - 2 \text{Min}(N_1, N_2) \text{Tr}(P_1 P_2)} . \] (2.19)

If one of \( N_1 \) and \( N_2 \) is equal to 1, we can replace \( \geq \) in (2.19) by \( = \). If both of \( N_1 \) and \( N_2 \) are equal to 1, we obtain \( D_2(|\phi><\phi|, \ |\phi><\phi|) = (2 - 2|\langle \phi | \phi \rangle|)^{1/2} \), which is the quantum distance discussed in Ref. 3). Noticing \( 0 \leq \text{Tr}(P_1 P_2) \leq \text{Min}(N_1, N_2) \), we have
\[ \frac{1}{\sqrt{2}} D_1(P_1, P_2) \leq D_2(P_1, P_2) \leq D_1(P_1, P_2) . \] (2.20)

The property
\[ D_1(P_1, P_2) \leq D_1(P_1, P_3) + D_1(P_3, P_2) \] (2.21)
yields
\[ D_2(P_1, P_2) \leq D_2(P_1, P_3) + D_2(P_3, P_2) , \] (2.22)

since
2.3. Geodesic distance

Since we have a Riemannian metric $G_{\mu\nu}(s)$ defined by \(2\cdot14\), we can define a geometric distance between two points $s_1$ and $s_2$ in the parameter space. Let $C(s_1, s_2)$ be the minimal geodesic which starts at $s_1$ and ends at $s_2$. The minimal geodesic $C(s_1, s_2)$ is defined as the solution of the geodesic equation

$$\frac{d^2 s^\mu(\tau)}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(s(\tau)) \frac{ds^\alpha(\tau)}{d\tau} \frac{ds^\beta(\tau)}{d\tau} = 0, \quad \mu = 1, 2, \ldots, p,$$

\(2\cdot24\)

$$\Gamma^\mu_{\alpha\beta}(s) = \frac{1}{2} G^{\mu\sigma}(s) \{ \partial_\alpha G_\beta^\sigma(s) + \partial_\beta G_\alpha^\sigma(s) - \partial_\sigma G_\alpha^\beta(s) \},$$

\(2\cdot25\)

$$d\tau = \sqrt{G^{\mu\nu}(s)} ds^\mu ds^\nu,$$

\(2\cdot26\)

\((G^{\mu\nu}(s))\) being the inverse matrix of \((G_{\mu\nu}(s))\), which realizes the minimum distance between $s_1$ and $s_2$:

$$D(s_1, s_2) = \int_{C(s_1, s_2)} \sqrt{G_{\mu\nu}(s)} ds^\mu ds^\nu.$$

\(2\cdot27\)

We note that Eqs. \(2\cdot24\) and \(2\cdot27\) can be rewritten as

$$\text{Tr} \left( \frac{d^2 P(s)}{d\tau^2} \partial_\mu P(s) \right) = 0, \quad \mu = 1, 2, \ldots, p,$$

\(2\cdot24'\)

and

$$D(s_1, s_2) = \frac{1}{\sqrt{2}} \int_{C(s_1, s_2)} \|dP(s)\|,$$

\(2\cdot27'\)

respectively. Indeed, we have

$$\text{Tr} \left( \frac{d^2 P(s)}{d\tau^2} \partial_\mu P(s) \right)$$

$$= \text{Tr} \left( \frac{d}{d\tau} \left( (\partial_\nu P(s)) s^\nu \right) \partial_\mu P(s) \right), \quad s^\nu = \frac{ds^\nu(\tau)}{d\tau},$$

$$= \dot{s}^\nu \text{Tr} \{(\partial_\nu P(s))(\partial_\mu P(s))\} + \dot{s}^\nu \dot{s}^\sigma \text{Tr} \{(\partial_\mu \partial_\nu P(s))(\partial_\sigma P(s))\}$$

$$= 2 G_{\mu\nu}(s) \{ \dot{s}^\nu + \Gamma^\nu_{\mu\sigma}(s) \dot{s}^\sigma \dot{s}^\nu \},$$

\(2\cdot24''\)

\((2\cdot27')\) being the direct result of \(2\cdot14\). From \(2\cdot27'\), \(2\cdot17\), \(2\cdot14\) and
\[
\text{Tr}[P(s)P(s+ds)]=\text{Tr}\left[ P(s)\left\{ P(s) + \partial_\mu P(s) ds^\mu + \frac{1}{2} \partial_\mu \partial_\nu P(s) ds^\mu ds^\nu + \cdots \right\} \right]
\]
\[
= N + 0 - \frac{1}{2} \text{Tr}[\{\partial_\mu P(s)\}{\partial_\nu P(s)}] ds^\mu ds^\nu + \cdots , \tag{2.28}
\]
we obtain
\[
\sqrt{G_{\mu\nu}(s)} ds^\mu ds^\nu = D(s, s+ds)
\]
\[
= \frac{1}{\sqrt{2}} D_1(P(s), P(s+ds))
\]
\[
= D_2(P(s), P(s+ds)) . \tag{2.29}
\]
The last equality in (2.29) is due to (2.20) and the fact that
\[
\inf_{W \text{ unitary}} \text{Tr}[\{-P(s)WP(s+ds) + P(s+ds) W^* P(s)\}] \leq -2 \text{Tr}(P(s)P(s+ds)).
\]

2.4. Inequality for \(D\) and \(D_2\)

We already have the inequality (2.20) for \(D_1(P_1, P_2)\) and \(D_2(P_1, P_2)\). We now show that \(D_2(P(s_1), P(s_2)) = D_2(s_1, s_2)\) cannot exceed \(D(s_1, s_2)\):
\[
D_2(s_1, s_2) \leq D(s_1, s_2) . \tag{2.30}
\]
To show (2.30), we consider \(K+1\) points \(s_0=s_1, s_1, s_2, \cdots, s_{K-1}\) and \(s_K=s_2\) which lie on \(C(s_1, s_2)\). Then, the inequality (2.22) yields \(D_2(s_1, s_2) \leq \sum_{k=0}^{K} D_2(\sigma_k, \sigma_{k+1})\). Taking an appropriate \(K \to \infty\) limit and making use of (2.29), we obtain (2.30). To illustrate how the inequality (2.30) is realized, we here consider two simplest examples with \(N=1\).

**Example 1** Coherent state

The coherent state \(|\alpha\rangle\) is defined by
\[
|\alpha\rangle = \exp\left(\frac{-1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle , \tag{2.31}
\]
where \(\alpha\) is a complex parameter and \(|n\rangle\) denotes the eigenstate of the harmonic oscillator. The parameter space is the complex plane. We readily obtain
\[
D(\alpha, \beta) = |\alpha - \beta| \tag{2.32}
\]
from the metric calculated in Ref. 3). From the inner product property of \(|\alpha\rangle\), we obtain
\[
D_2(\alpha, \beta) = \sqrt{2} \sqrt{1 - \exp\left(\frac{-1}{2} |\alpha - \beta|^2\right)} . \tag{2.33}
\]
The inequality \(1 - e^{-x} \leq x, \ x \geq 0\), ensures (2.30).

**Example 2** \(Z\)-state of the harmonic oscillator

The \(Z\)-state of the harmonic oscillator is defined by
\[
|Z\rangle = (1 - |Z|^2)^{1/2} \sum_{n=0}^{\infty} Z^n |n\rangle , \quad |Z| < 1 , \tag{2.34}
\]
\[
|Z\rangle = (1 - |Z|^2)^{1/2} \sum_{n=0}^{\infty} Z^n |n\rangle , \quad |Z| < 1 ,
\]
where $Z$ is given by $Z = e^{i\varphi} \tanh(\theta/2)$, $\varphi, \theta \in \mathbb{R}$. The metric structure is obtained from (2·29) to be

$$G_{\mu\nu}(s)ds^\mu ds^\nu = \frac{1}{4} \left\{ \sinh^2 \theta (d\varphi)^2 + (d\theta)^2 \right\}, \quad (2·35)$$

corresponding to the Lobatchevski plane. Then the geodesic distance is known to be given by

$$D(Z_1, Z_2) = \frac{1}{2} \log \frac{1 + W}{1 - W}, \quad (2·36)$$

where $W$ is defined by

$$W = \left| \frac{Z_1 - Z_2}{1 - Z_1 Z_2} \right| \quad (2·37)$$

and satisfies $0 \leq W < 1$. On the other hand, $D_2(Z_1, Z_2)$ is calculated to be

$$D_2(Z_1, Z_2) = \sqrt{2} \sqrt{1 - \sqrt{1 - W^2}}. \quad (2·38)$$

The inequality $D(Z_1, Z_2) \geq D_2(Z_1, Z_2)$ can be shown easily.

§ 3. **Proof of (2·19)**

This section deals with the derivation of (2·19).

3.1. **General discussion**

To obtain the $\text{Inf}_{\|P_1 - WP_2\|}$ of $\|P_1 - WP_2\|$, we have to maximize the quantity

$$J(W) = \frac{1}{2} \text{Tr} (P_1 WP_2 + P_2 W' P_1). \quad (3·1)$$

For given projection operators

$$P_1 = \sum_{j=1}^N |\varphi_j\rangle \langle \varphi_j|, \quad \langle \varphi_i \mid \varphi_j \rangle = \delta_{ij},$$

$$P_2 = \sum_{k=1}^M |\psi_k\rangle \langle \psi_k|, \quad \langle \psi_k \mid \psi_{\ell} \rangle = \delta_{k\ell}, \quad (3·2)$$

we define real coefficients $C_\beta, \beta = 1, 2, \cdots, 2N_1N_2$ by

$$C_\beta = \begin{cases} 
\text{Re} \langle \varphi_j \mid \psi_k \rangle, & \beta = (j-1)N_2 + k = 1, 2, \cdots, N_1N_2, \\
\text{Im} \langle \varphi_j \mid \psi_k \rangle, & \beta = N_1N_2 + (j-1)N_2 + k
\end{cases}$$

$$= N_1N_2 + 1, N_1N_2 + 2, \cdots, 2N_1N_2 = M. \quad (3·3)$$

The $W$-dependent real variables $\xi_\beta, \beta = 1, 2, \cdots, 2N_1N_2$ are defined by

$$\xi_\beta = \begin{cases} 
\text{Re} \langle \varphi_j \mid W \mid \psi_k \rangle, & \beta = 1, 2, \cdots, N_1N_2, \\
\text{Im} \langle \varphi_j \mid W \mid \psi_k \rangle, & \beta = N_1N_2 + 1, \cdots, M.
\end{cases} \quad (3·4)$$

Then we have
By making use of the Schwarz inequality, we have
\[ |J(W)| \leq (L(P_1, P_2)K(W))^{1/2}, \]
where \( L(P_1, P_2) \) is given by
\[ L(P_1, P_2) = \text{Tr}(P_1P_2) = \sum_{\beta=1}^{M} (C_{\beta})^2. \]  

Noticing that the unitarity of \( W \) implies \( K(W) \leq \text{Min}(N_1, N_2) \), we obtain the inequality \((2.19)\). To realize the equality in \((2.19)\), we must have \( \xi_{\beta} = \kappa C_{\beta}, \beta = 1, 2, \ldots, M, \) and \( K(W) = \text{Min}(N_1, N_2) \), which are respectively equivalent to
\[ P_1 WP_2 = \kappa P_1 P_2 \] 
and
\[ WP_2 W^* = P_1 + Q, \]
where \( Q \) is an operator satisfying
\[ \text{Tr}(P_1 Q) = 0. \]

In \((3.9)\) we assumed that \( N_1 \leq N_2 \). The existence of a unitary \( W \) satisfying \((3.9)\) can be proved by noticing the theorem\(^7\) if \( A \) and \( A' \) have the same point spectrum including multiplicities, then they are unitarily equivalent. However, \((3.9)\) associated with \((3.8)\) is valid only in the case that all of \( \sum_{i=1}^{N_1} |\langle \phi_i | \psi_k \rangle|^2 \), \( i=1, 2, \ldots, N_1 \), are equal. This severe condition disappears in the \( N_1 = 1 \) case, which will be discussed in the next subsection.

3.2. Explicit form of \( W \) realizing \( D_2 \) for \( N_1 = 1 \)

In the case that \( N_1 = 1 \) and \( N_2 \) arbitrary, we can realize the equality of \((2.19)\):
\[ D_2(P_1, P_2) = \sqrt{N + 1 - 2\sqrt{\text{Tr}(P_1P_2)}}, \]
\[ P_1 = |\phi \rangle \langle \phi |, \quad P_2 = \sum_{k=1}^{N} |\psi_k \rangle \langle \psi_k |, \]
which generalizes the distance of Provost and Vallee.\(^9\) To show this fact, it is sufficient to construct the unitary operator \( W \) satisfying \((3.8)\) and \((3.9)\). We begin with defining an operator \( B \) of the form
\[ B = i(X - Y) = B^*, \]
\[ X = P_1 P_2, \quad Y = P_2 P_1 = X^*. \]
We readily obtain formulae
\[ X k(X) = k(\lambda) X, \]
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\[ Yk(Y) = k(\lambda) Y, \quad (3.15) \]

where \( k(x) \) is any power series of \( x \) and \( \lambda \) is defined by

\[ 0 \leq \lambda = \text{Tr}(P_1 P_2) = \sum_{k=1}^{N} |\langle \phi| \phi_k \rangle|^2 \leq 1. \quad (3.16) \]

We define a unitary operator \( W \) by

\[ W = e^{i\zeta \lambda}, \quad (3.17) \]

\( \zeta \) being a real parameter specified below. Although the operator \( W \) itself is a complicated function of \( X \) and \( Y \), with the help of the properties

\[ B^{2m} P_2 = P_2 (1 - \lambda)^m \lambda^{m-1} X, \quad m = 1, 2, 3, \ldots, \]
\[ B^{2m+1} P_2 = i (1 - P_2) (1 - \lambda)^m \lambda^m X, \quad m = 0, 1, 2, \ldots, \quad (3.18) \]

we obtain a simple expression for \( WP_2 \):

\[ WP_2 = P_2 \left( 1 + \frac{1}{\lambda} (\cos \xi - 1) X \right) - (1 - P_2) X \frac{\sin \xi}{\sqrt{\lambda(1 - \lambda)}}, \quad (3.19) \]

\[ \xi = \zeta \sqrt{\lambda(1 - \lambda)} \cdot \quad (3.20) \]

Then we have (3.8) with \( \kappa \) given by

\[ \kappa = \frac{1}{\sqrt{\lambda}} \left( \sqrt{\lambda} \cos \xi - \sqrt{1 - \lambda} \sin \xi \right). \quad (3.21) \]

After some manipulations, we have

\[ \text{Tr}(WP_2 W^* P_1) = \cos^2 (\xi - \delta), \quad (3.22) \]

where \( \delta \) is fixed by \( \tan(2\delta) = 2 \sqrt{\lambda(1 - \lambda)} / (1 - 2\lambda) \). If we fix the parameter \( \zeta \) by

\[ \zeta = \frac{1}{2 \sqrt{\lambda(1 - \lambda) \text{Arctan}}} \left[ 2 \sqrt{\lambda(1 - \lambda)} \right], \quad (3.23) \]

the r.h.s. of (3.22) is equal to one, implying that (3.9) with (3.10) is attained. For the \( \zeta \) of (3.23), \( \kappa \) in (3.21) is equal to \( (\sqrt{\lambda})^{-1} \), inducing \( J(W) = \sqrt{\lambda} \). We then have (3.11).

\section*{§ 4. Time-energy uncertainty relations}

Let us consider a conventional situation that any quantum state vector develops in time obeying the Schrödinger equation

\[ i \hbar \frac{d}{dt} |\phi(t)\rangle = H |\phi(t)\rangle, \quad (4.1) \]

where \( H \) is the Hamiltonian which may depend on time. Recently, Anandan and Aharonov\(^b\) derived quite an interesting time-energy uncertainty relation

\[ \int_{t_1}^{t_2} (\langle \phi(t)|H^2|\phi(t)\rangle - \langle \phi(t)|H|\phi(t)\rangle^2)^{1/2} dt \geq \hbar \text{Arccos}(|\langle \phi(t_1)|\phi(t_2)\rangle|). \quad (4.2) \]
In their discussion, the dynamics is not restricted and a single state vector $|\phi(t)\rangle$ is considered. We here consider a set of vectors $\{|\psi_i(t)\rangle; i=1, 2, \ldots, N\}$ and assume that, at any instant of time, the $|\psi_i(t)\rangle$'s are described by a set of time-dependent continuous parameters ($s^1(t), s^2(t), \ldots, s^N(t)$). Then we can use the machinery developed in § 2. The case of this kind of dynamics was discussed by Perelomov and Provost-Vallee.

The projection operator (2·3) with $s$ replaced by $t$ satisfies

$$i\hbar \frac{d}{dt} P(t) = [H, P(t)] \tag{4·3}$$

and

$$(i\hbar)^2 \frac{d^2}{dt^2} P(t) = i\hbar \left[ \frac{dH}{dt}, P(t) \right] + [H, [H, P(t)]] \tag{4·4}$$

By a calculation similar to (2·27), we have

$$\text{Tr}(P(t)P(t+dt)) = N - \frac{(dt)^2}{2\hbar^2} \text{Tr}(P(t)[H, [H, P(t)]])) \tag{4·5}$$

up to the second order of $dt$. If we define $\Delta E(t)$ by

$$\Delta E(t) = \sqrt{\frac{1}{2} \text{Tr}(P(t)[H, [H, P(t)]]))}$$

$$= \sqrt{\sum_{i=1}^{N} |\langle \psi_i(t)|H^2|\psi_i(t)\rangle| - \sum_{i,j=1}^{N} |\langle \psi_i(t)|H|\psi_j(t)\rangle|^2} \tag{4·6}$$

we are led to the relation

$$\Delta E(t) dt = \hbar D(t, t + dt) \tag{4·7}$$

From (2·28), (4·7) and the property of distance, we see that the $\Delta E(t)$ integrated over the interval $[t_1, t_2]$ satisfies the inequality

$$\int_{t_1}^{t_2} \Delta E(t) dt \geq \hbar D(t_1, t_2) \tag{4·8}$$

It does not seem to be easy to obtain $D(t_1, t_2)$ which concerns a minimal geodesic under restricted dynamics. From (2·30) and (2·19), we have

$$\int_{t_1}^{t_2} \Delta E(t) dt \geq \sqrt{2N} \hbar \sqrt{1 - \sqrt{Q(t_1, t_2)}} \tag{4·9}$$

where $Q(t_1, t_2)$ is given by

$$0 \leq Q(t_1, t_2) = \frac{1}{N} \sum_{i,j} |\langle \psi_i(t_1)|\psi_j(t_2)\rangle|^2 \leq 1 \tag{4·10}$$

We observe that the quantity $\Delta E(t)$ of (4·6) is invariant under gauge transformations (2·5) and appropriate to measure the time-energy uncertainty in an $N$-dimensional linear subspace. Although the inequality (4·9) is less stringent than (4·8), it is a convenient tool to estimate the uncertainty since its r.h.s is directly calculable without recourse to the investigation of geodesics. To illustrate how the relation (4·9) works,
we consider two examples.

**Example 1 Coherent state**

In § 2, we obtained (2·32) and (2·33) for the set of coherent states. If we put $|\psi(t_1)\rangle = |\alpha\rangle$ and $|\psi(t_2)\rangle = |\beta\rangle$ in (4·2), we obtain $\int dt \Delta E(t) dt \geq \hbar d(\alpha, \beta)$ where $d(\alpha, \beta)$ is given by $d(\alpha, \beta) = \text{Arccos}(e^{-|1/2(a-\beta)|})$. We have $D(\alpha, \beta) < d(\alpha, \beta) < D(\alpha, \beta)$ for $\alpha = \beta$. $\hbar D(\alpha, \beta)$ measures the energy fluctuation when the dynamics is restricted so that only the coherent states are realized, while $\hbar d(\alpha, \beta)$ corresponds to the least time-energy uncertainty under general dynamical situations.

**Example 2 Comparison of (4·9) with (4·8) for an N=2 case**

The vectors

$$
e_1(s) = \begin{pmatrix} e^{i(x+\phi)} \cos \theta \cos \phi \\ e^{-i\phi} \sin \theta \\ e^{-i\phi} \cos \theta \end{pmatrix}, \quad e_2(s) = \begin{pmatrix} -e^{i(x+\phi)} \sin \phi \\ e^{i(x+\phi)} \cos \phi \\ 0 \end{pmatrix},
$$

$$e_3(s) = \begin{pmatrix} e^{i(x+\phi)} \sin \theta \cos \phi \\ e^{i(x+\phi)} \sin \theta \sin \phi \\ e^{-i\phi} \cos \theta \end{pmatrix} \tag{4·11}$$

constitute an orthonormal set, where the parameters $s = (\theta, \phi, \chi, \psi)$ are assumed to satisfy $\theta, \phi \in (0, \pi/2)$ and $\chi, \psi \in (0, \pi)$. We assume that the dynamics is restricted such that the vectors $e_i(s)$ keep the form (4·11) and the parameters vary with time, taking values $s_a = (\theta_a, \phi_a, \chi_a, \psi_a)$ at time $t_a$, $a = 1, 2$, with $t_2 \geq t_1$. If we put $|\psi_a(t)\rangle = e_i(s(t))$, $k = 1, 2, 3$, and consider the $N=3$ case, the relation (4·9) gives no meaningful information since its r.h.s. vanishes. On the other hand, if we consider the $N=2$ case involving only $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$, we obtain

$$Q(t_1, t_2) = \frac{1}{2} \sum_{k=1}^2 |\langle \psi_k(t_1)|\psi_k(t_2)\rangle|^2$$

$$= -\frac{1}{2} \left[ 1 + \{ \cos(\chi_2 - \chi_1) \cos(\phi_2 - \phi_1) \sin \theta \sin \theta_2 + \cos 2(\phi_2 - \phi_1) \cos \theta_1 \cos \theta_2 \}^2 \\
+ \{ \sin(\chi_2 - \chi_1) \cos(\phi_1 + \phi_2) \sin \theta \sin \theta_2 - \sin 2(\phi_2 - \phi_1) \cos \theta_1 \cos \theta_2 \}^2 \right]. \tag{4·12}$$

Putting $t_1 = t$, $t_2 = t + dt$, $\theta_1 = \theta$, $\theta_2 = \theta + d\theta$ and so on, we are led to the relation $Q(t, t + dt) = 1 - [D(t, t + dt)]^2/2$, where $D(t, t + dt)$ is given by

$$D(t, t + dt) = \left\{ (\partial \theta)^2 + \sin^2 \theta (\partial \phi)^2 + \sin^2 \theta \sin^2 2\phi (d\chi)^2 + \sin^2 2\phi \left( \frac{1}{2} \cos 2\phi d\chi \right)^2 \right\}^{1/2} \tag{4·13a}$$

$$= \langle d\psi(t) | d\psi(t) \rangle - \langle d\psi(t) | \psi(t) \rangle \langle \psi(t) | d\psi(t) \rangle. \tag{4·13b}$$

Since the metric (4·13a) is that of $P^2(C)$, we can obtain $D(t_1, t_2)$ in this example: $D(t_1, t_2) = \text{Arccos} (|\langle \psi(t_1)|\psi(t_2)\rangle|)$. Then (4·8) yields
The maximum of the r.h.s. of (4·14) is given by \( \pi \hbar/2 = 1.57\hbar \). On the other hand, it can be seen from (4·12) that, for a suitable pair of \( s_1 \) and \( s_2 \), the r.h.s. of (4·9) takes its maximum value \( \sqrt{4 - 2\sqrt{2}}\hbar = 1.08\hbar \).

§ 5. Summary and outlook

We have discussed how to define the quantum distance between two linear subspaces of a Hilbert space. The main tool we have made use of is the projection operator projecting the whole Hilbert space onto a linear subspace concerned. The distance has been defined with the aid of the Hilbert-Schmidt norm of the difference of two projection operators. The Hilbert-Schmidt norm turns out to be convenient to write down the Provost-Vallee distance and the Anandan-Aharonov one as well as the generalized versions of them. We have also discussed the time-energy uncertainty, the lower bound of which is given by the quantum distance multiplied by \( \hbar \).

We have considered how the metric tensor is introduced on the space of parameters describing quantum states. The quantum geodesic distance appearing in the discussion of the time-energy uncertainty concerns a one-dimensional integral on the parameter space. To envisage how physical information is extracted from multiple-integrals on the parameter space, we consider a set of orthonormal vectors \( \{ |\psi_i(s)\rangle \rangle i = 1, 2, \ldots, N \} \) and \( s = (s^1, s^2) \). We have

\[
\left( \frac{\partial s}{\partial t} \right) (s) = H(s)|\psi_i(s)\rangle, \quad \mu = 1, 2, \quad i = 1, 2, \ldots, N,
\]

where \( H(s) \) is given by

\[
H(s) = i\hbar [\partial_s U(s)] U^*(s) H(s) H(s)^*.
\]

From (2·8) and (2·9), we obtain

\[
G_{\mu\nu}(s) = \frac{1}{2\hbar^2} \left[ \sum_{i=1}^{N} \langle \psi_i(s) | H(s) H(s) | \psi_i(s) \rangle - \sum_{i,j=1}^{N} \langle \psi_i(s) | H(s) | \psi_j(s) \rangle \langle \psi_j(s) | H(s) | \psi_i(s) \rangle \right] + \{ \mu \leftrightarrow \nu \}.
\]

We here assume, for simplicity, that \( \Omega \) is a convex domain on the \( s \)-plane. We further assume that \( \varphi(s) \) is a real continuous function on the boundary \( \partial \Omega \) of \( \Omega \) satisfying

\[
\left\{ \frac{\partial \varphi(s)}{\partial s^\mu} + \frac{\partial \varphi(s)}{\partial s^\nu} \right\} \frac{ds^\mu(t)}{dt} \frac{ds^\nu(t)}{dt} = G_{\mu\nu}(s) \frac{ds^\mu(t)}{dt} \frac{ds^\nu(t)}{dt}
\]

along \( \partial \Omega \). If \( \partial \Omega \) is parametrized as \( \partial \Omega = \{(s^1(t), s^2(t)): 0 \leq t \leq 1, s^\nu(0) = s^\nu(1)\} \), \( \partial \Omega \) and \( \varphi \) define a curve \( C = \{(s^1, s^2, \varphi(s)): s \in \partial \Omega\} \) in a three-dimensional Euclidean space. Then the integral

\[
\sigma(S(C)) = \int_C \left[ \det(G_{\mu\nu}(s)) \right]^{1/2} ds^1 ds^2, \quad \overline{\Omega} = \Omega \cup \partial \Omega,
\]
can be identified with the area of a certain surface $S(C)$ bounded by the curve $C$. It is known that there exists a unique surface $S_0(C)$ bounded by $C$ which minimizes the value of $\sigma(S(C))$. The problem to obtain $S_0(C)$ for the prescribed $C$ is the famous Plateau problem which has a long history in mathematics. Since the value of $\sigma(S_0(C))$ depends only on the choice of the function $\varphi(s)$ on $\partial \Omega$, the inequality $\sigma(S(C)) \geq \sigma(S_0(C))$ would involve some nontrivial information on the matrix elements of $H_\mu(s)$ and $H_\nu(s)H_\nu(s)$. The discussion of the above type can be generalized to the case of higher-dimensional integrals and will be given in a future communication.

Acknowledgements

The authors are grateful to K. Fujii, S. Hamamoto, T. Kobayashi, K. Morita, M. Nakahara and H. Yamakoshi for valuable discussions. This work was partially supported by the Grant-in-Aid #04804014 of the Ministry of Education, Science and Culture.

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