Quantum Nonlinear Resonance*)

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To visualize complex structure of the excited states in the many-fermion system, a profile function of the eigenstate within the quantum phase space is defined by using the boson expansion theory. When the anharmonic interaction is sufficiently weak where a corresponding semi-quantum system described by the time-dependent Hartree-Fock (TDHF) theory shows isolated resonant structure, there exists a remarkable similarity between the Poincaré section map of the TDHF trajectories and the profile functions of the eigenstates. It is shown that these eigenstates are well understood by using a new concept of quantum nonlinear resonance. It is also shown that a pair of newly appeared states due to the quantum resonance is not associated with any dissolution in a number of the good quantum numbers.

§ 1. Introduction

In recent years, it has been established in the nuclear structure physics that there coexist many stationary mean-fields in one nucleus.1) Since these mean-fields ought to be considered as stationary points of the time-dependent Hartree-Fock (TDHF)-manifold, a semi-quantum dynamics in a quantum phase space2) (viz., in the TDHF-manifold) described by the TDHF theory is expected to give a basic understanding for the excited states in the many-fermion system. Under these situations, it is of special importance to clarify an intricate relation between the semi-quantum and full-quantum many-body theories, whose mutual relation seems to remain still in a formal level.2,3) On the other hand, a quantum-classical correspondence for chaotic system4) is one of the current topics in the theoretical physics. Since the nonlinear resonance is known to play a key role in exploring complexity of the classical phase space,5) the quantum-classical correspondence between the classical nonlinear resonance and the quantum Fermi resonance has been studied in the molecular physics for understanding the quantum excited states.6)

In a context of the nuclear structure physics, however, it is an interesting subject to study whether the semi-quantum nonlinear resonance appearing in the TDHF-manifold gives some suggestion for understanding the excited states of the many-fermion system. It has been an objective of the general theory of nonlinear dynamics to analytically understand complex structure of the classical phase space, i.e., to get an analytic information on adiabatic invariants characterizing individual trajectories, to explore why there appear new local adiabatic invariants as the system gets more

* The basic idea of this paper has been reported in Refs. 11) and 16).
energy, and how the regular motion specified by some adiabatic invariants turns into the chaotic motion without any constants of motion. Thus the above subject may be stated in a more general way; can one analytically understand complex structure of quantum excited states in a way parallel to the TDHF-manifold where the nonlinear theory is applicable? What one has to explore in the quantum system is how to get a set of good quantum numbers characterizing the individual eigenstates, how it undergoes a change as the eigenenergy increases, how there appears a new type of eigenstates characterized by another set of good quantum numbers, and how there appear eigenstates without any good quantum numbers except for the eigenenergy. The objective of the present paper is to introduce a concept of a quantum nonlinear resonance, which indicates a rearrangement of good quantum numbers so as to generate a set of new good quantum numbers, and to show its validity in understanding complex structure of the excited states. It is shown by means of numerical calculation that there appear characteristic eigenstates whose structures are well understood with the quantum nonlinear resonance, when the anharmonic interaction is sufficiently weak and the corresponding TDHF-manifold shows isolated resonant island structure.

This paper is organized as follows: By using a simple model Hamiltonian, in § 2, the structure of quantum phase space is discussed with the TDHF theory. To visualize complex structure of the quantum excited states, the fermion system is first mapped to the boson system by the boson expansion theory. Then a profile function of each eigenstate obtained in the boson representation of the fermion system is introduced. Since the TDHF theory is known to be a c-number version of the boson expansion theory, the profile function is expected to expose complicated structure of the excited states, like the Poincaré section map of the TDHF trajectories in the quantum phase space. To understand similar structure between the Poincaré map and the profile function, in § 3, new notions of quantum nonlinear resonance and quantum resonant states are introduced in a way quite similar to the classical nonlinear resonance. In § 4, complex structures of the exact eigenstates are analyzed by using the quantum nonlinear resonance.

§ 2. Profile function of eigenstates in quantum phase space

Since an essential dynamics involving the classical nonlinear resonance is discussed within a simple model with two degrees of freedom, the correspondence between the semi- and full-quantum systems under small nonlinear (anharmonic) effects may be generally explored by using a modification of the SU(3) model Hamiltonian used by Li, Klein and Dreizler:

\[ \hat{H} = \sum_{i=0}^{2} \varepsilon_{i} \hat{K}_{i} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} V_{ij}(\hat{K}_{i0}\hat{K}_{j0} + \text{h.c.}), \quad \hat{K}_{ij} = \sum_{m=1}^{N} \hat{c}_{im} \hat{c}_{jm}. \]  

(2.1)

There are three orbits with energies \( \varepsilon_{0} < \varepsilon_{1} < \varepsilon_{2} \) and each level has \( N \)-fold degeneracy. Hereafter, we treat a system with even particle number \( N \). The lowest energy state without interaction is given by \( |\phi_{0}\rangle \) where the lowest orbit \( i=0 \) is completely occupied. The semi-quantum treatment is introduced by employing a time-dependent single
Slater-determinant given by
\[ |f\rangle = e^{i\hat{F}} |\phi_0\rangle = |q_1, p_1, q_2, p_2\rangle; \quad \hat{F} = \sum_{l=1}^{N} f_l \hat{K}_l - \text{h.c.}, \]
\[ p_i = \frac{i}{\sqrt{2}} (c_i^* - c_i), \quad q_i = \frac{1}{\sqrt{2}} (c_i^* + c_i), \quad c_i = \sqrt{N} f_i \sin |f||/|f|, \quad |f| = \sqrt{\sum_{k=1}^{N} f_k^* f_k}. \quad (2.2) \]

The TDHF equation is expressed\(^*)\) as
\[ 0 = \delta \left( \hat{H} - i \frac{d}{dt} |f\rangle \right). \quad (2.3) \]

From Eq. (2.3), one gets the following canonical equations of motion:
\[ \dot{q}_i = -\frac{\partial \hat{h}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \hat{h}}{\partial q_i}; \quad i = 1, 2. \quad (2.4) \]
\[ \hat{h}(q_1, p_1, q_2, p_2) = \langle f|\hat{H}|f\rangle - \langle \phi_0|\hat{H}|\phi_0\rangle = h_0 + h_{\text{coup}}, \quad h_0 = \sum_{l=1}^{N} h_l, \]
\[ h_l = \frac{1}{2} (\varepsilon_i - \varepsilon_0) (q^2_i + p^2_i) + \frac{1}{2} V_l (N-1)(q^2_i - p^2_i) + \frac{V_l}{2} (N-1)(q^4_i - p^4_i), \]
\[ h_{\text{coup}} = \frac{1}{2} V_1 (N-1)(q^2_i - p^2_i)(q^2_i + p^2_i) + \frac{1}{2} V_2 (N-1)(q^4_i - p^4_i)(q^4_i + p^4_i). \quad (2.5) \]

Equation (2.4) exhibits a symplectic structure of the TDHF-manifold \(\mathcal{M} = (q_1, p_1, q_2, p_2)^2\) (which just corresponds to the quantum phase space discussed in Ref. 2), and determines a TDHF trajectory for a given initial condition. The complex structure of the TDHF-manifold organized by the Hamiltonian (2.5) has been studied elsewhere.\(^10\)

Since the quantum phase space has the symplectic structure like the classical phase space, its structure is understood with the classical nonlinear theory.

As is well known, an analytic understanding of an isolated resonance provides us with a starting point in exploring the phase space structure.\(^5\) In studying the correspondence between the semi- and full-quantum theories, it is preferable to start with a case where the quantum phase space shows an isolated resonant island. To this end, we choose the following parameters; \(N = 30, \varepsilon_0 = 0, \varepsilon_1 = 1, \varepsilon_2 = 2\) and \(V_1 = V_2 = -0.01\). The Poincaré section of the TDHF-trajectories on a plane \((p_1, q_1)\) is constructed with a condition \(q_2 = 0\). In the present case with weak interaction, the Poincaré sections with different total energies usually indicate only regular motion expressed by concentric circles centered at the origin.

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*\(^*)\) Throughout this paper, we adopt the convention of using \(\hbar = 1\).
The Poincaré section with $E=40$ is illustrated in Fig. 1, where the nonlinear interaction starts to generate a typical structure which is not dominated by the concentric circles alone. There appear five kinds of regular motions; Three regular motions expressed by the inner most, middle and outer most concentric circles, and two additional regular motions by inner four-crescent and outer four-crescent structures. Although the separatrix lines are not explicitly shown in Fig. 1, each four-crescent structure is surrounded by a respective separatrix. The stable fixed point is situated just in the center of the crescent, whereas the separatrix is running through the unstable fixed point. To analytically understand the complex structure of the TDHF-manifold in Fig. 1, a new method of using the Lie transformation with the Deprit perturbation method has been proposed elsewhere.\(^\dagger\)

The first step to study the correspondence between the semi- and full-quantum theories may be given\(^\ddagger\) by the Holstein-Primakoff boson expansion method.\(^\S\) Let us introduce the boson operators satisfying

$$[B_i, B_j]=\delta_{ij}, \quad [B_i, B_j^\dagger]=0. \quad (i, j=1, 2)$$  \hspace{1cm} (2.6)

By the use of the boson operators, the fermion Hamiltonian in Eq. (2.1) is transformed into the boson Hamiltonian expressed as

$$H(B^\dagger, B)=H_0(B^\dagger, B)+H_{\text{coup}}(B^\dagger, B),$$

$$H_0(B^\dagger, B)=\epsilon_1 B_1^\dagger B_1+(\epsilon_2-\epsilon_0)B_2^\dagger B_2,$$

$$H_{\text{coup}}(B^\dagger, B)=\frac{V_1}{2} (B_{1}^\dagger \sqrt{N-B_1^\dagger B_1-B_2^\dagger B_2} B_1^\dagger \sqrt{N-B_1^\dagger B_1-B_2^\dagger B_2} + \text{h.c.})$$

$$+\frac{V_2}{2} (B_{2}^\dagger \sqrt{N-B_1^\dagger B_1-B_2^\dagger B_2} B_2^\dagger \sqrt{N-B_1^\dagger B_1-B_2^\dagger B_2} + \text{h.c.}).$$  \hspace{1cm} (2.7)

Diagonalizing the Hamiltonian in Eq. (2.7) within a set of $mn$-basis states given by

$$|m, n\rangle = \frac{1}{\sqrt{m!}} \frac{1}{\sqrt{n!}} (B_1^\dagger)^m (B_2^\dagger)^n |0\rangle, \quad (B_i |0\rangle = 0), \quad m+n\leq N,$$  \hspace{1cm} (2.8)

one obtains a set of eigenstates $|\lambda\rangle$,

$$H(B^\dagger, B) |\lambda\rangle = E_\lambda |\lambda\rangle, \quad \lambda=1, 2, \ldots.$$  \hspace{1cm} (2.9)

Here, the physical boson space having a one-to-one correspondence with the space of $N$-fermion states is characterized by $m+n\leq N$. The Hamilton matrix is block diagonalized in $(m, n)=$ (even, even)-, (even, odd)-, (odd, even)- and (odd, odd)-sectors because of its symmetry. Hereafter, we restrict ourselves to an (even, even)-sector. In the present case with $N=30$, a dimension of the Hamilton matrix to be diagonalized is 136. Exploring geometrical structure of the eigenfunctions, one usually introduces the following coherent states:

$$|q_1, p_1, q_2, p_2\rangle = |c_1, c_2\rangle = \exp\{\sum_{i=1}^{2} c_i B_i^\dagger - \text{h.c.}\} |0\rangle;$$
\[ p_i = \frac{i}{\sqrt{2}} (c_i^* - c_i), \quad q_i = \frac{1}{\sqrt{2}} (c_i^* + c_i). \]  
\[ (2.10) \]

Since there holds a relation

\[ \langle q_1, p_1, q_2, p_2 | \hat{H} | q_1, p_1, q_2, p_2 \rangle = \langle q_1, p_1, q_2, p_2 | H(B^*, B) | q_1, p_1, q_2, p_2 \rangle, \]

\[ (2.11) \]
a point in the TDHF-manifold \( \mathcal{M} : \{ q_1, p_1, q_2, p_2 \} \) defined in Eq. (2.2) just corresponds to the coherent state denoted by \( | q_1, p_1, q_2, p_2 \rangle \), viz.,

\[ \text{Fig. 2. Profiles of the eigenstates in the TDHF-manifold.} \]
which provides us with a basic relation in studying the correspondence between the semi- and full-quantum theories.

In order to illustrate geometrical structure of the exact eigenstate $|\lambda\rangle$ in the quantum phase space, one may introduce the following profile function for each eigenstate $|\lambda\rangle$,

$$G^\lambda(q_1, p_1) = \int dp_2 \mathcal{G}^\lambda(q_1, p_1; q_2 = 0, p_2), \quad \mathcal{G}^\lambda(q_1, p_1; q_2, p_2) = |(c_1, c_2|\lambda\rangle|^2,$$

which may be regarded as a quantum analog of the Poincaré section map in Fig. 1. In Fig. 2, the profile functions for six eigenstates with eigenenergies $E_1 \approx 40$ are shown. A similarity between the profile function and the Poincaré section map is remarkable; the inner pair of elliptic and hyperbolic points in Fig. 1 shows almost the same structure as the profiles of exact eigenfunctions in Figs. 2(b) and (c) (e.g., compare the maximum points of profile function in Fig. 2(b) with the centers of inner four crescents), and the outer pair in Fig. 1 shows the same as in Figs. 2(d) and (e). The outermost and innermost concentric circles in Fig. 1 have almost the same structure as the profiles of eigenfunctions in Figs. 2(a) and (f). Although we are discussing a similar structure between the TDHF trajectories and the profile functions, it should be noticed that there has been pointed out remarkable resemblance between the scar structure of some eigenfunctions and the classical trajectories running through the fixed points in studying the classical-quantum correspondence of the chaotic system.\(^\text{19}\)

### § 3. Quantum nonlinear resonance

As is well known in the general theory of the nonlinear dynamics,\(^\text{5}\) there occur various situations produced by the nonlinear resonance before an onset of global chaos. It is an objective of this section to introduce a quantum analog of the nonlinear resonance. Illustrating it as simple as possible, we consider a boson system with two degrees of freedom. The integrable part of the Hamiltonian is expressed as

$$H_{\text{integ}} = \omega_1 B_1^\dagger B_1 + \omega_2 B_2^\dagger B_2.$$

In the present case, the basis states are given by $|m, n\rangle$ in Eq. (2.8). Here $m$ and $n$ are the quantum numbers of the "oscillator quanta" which are eigenvalues of the following commutable operators,

$$N_1 = B_1^\dagger B_1, \quad N_2 = B_2^\dagger B_2.$$

A resonant condition is expressed as

$$k_1 \omega_1 - k_2 \omega_2 = 0, \quad k_1 \text{ and } k_2 \text{ are prime numbers}.$$

When there holds the resonant condition (3.3), one may introduce a new quantum number $\mathcal{N}$ called "resonant oscillator quantum" which is specified by the following operator:
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\[ J = k_2 N_1 + k_1 N_2. \] (3.4)

To make the following discussion transparent, we hereafter treat a special case with \( k_1 = 2 \) and \( k_2 = 1 \). However the generalization for any integer value of \( k_1 \) and \( k_2 \) is straightforward. Using the resonant oscillator quantum, one may introduce the resonant subspace \( D^N \),

\[ D^N: \left\{ \left| 0, \frac{N}{2} \right\rangle, \ldots, \left| 2k, \frac{N}{2} - k \right\rangle \right\}, \quad k = N - \frac{N}{2} \quad \text{for even} \ N, \] (3.5a)

\[ D^N: \left\{ \left| 1, \frac{N-1}{2} \right\rangle, \ldots, \left| 2k+1, \frac{N-2k-1}{2} \right\rangle \right\}, \quad k = N - \frac{N+1}{2} \quad \text{for odd} \ N, \] (3.5b)

whose elements have the same quantum number \( N \) (\( N \) represents both the \( q \)- and \( c \)-numbers). It is also noticed that the states in \( D^N \) have the same unperturbed energy \( \omega, N = \omega N/2 \).

Now let us discuss another new quantum number specifying the states when the resonant condition (3.3) is satisfied. For this aim, we consider an ideal case with \( N = 2 \),

\[ D^{N=2}: \{ |2, 0\rangle, |0, 1\rangle \} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \] (3.6)

With the aid of the last representation in Eq. (3.6), the general Hamiltonian in the subspace \( D^{N=2} \) is represented as

\[ H^{N=2} = \alpha E + \beta T, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (3.7)

The first term expresses an integrable Hamiltonian \( H_{\text{integ}} \) and \( \alpha \) means a degree of violation from a complete resonant situation. When there holds a resonant condition (3.3), \( \alpha \) takes 0. The second term denotes the anharmonic interaction

\[ H_{\text{anhar}} = \frac{\beta}{\sqrt{2}} (B_1^\dagger B_1^\dagger B_2 + B_1^\dagger B_1 B_1), \] (3.8)

within the subspace \( D^{N=2} \).

In an ideal case with \( \alpha = 0 \), one may introduce a set of new basis states

\[ \mathcal{R}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathcal{R}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \] (3.9)

which satisfy

\[ T \mathcal{R}_\pm = t_\pm \mathcal{R}_\pm, \quad t_\pm = \pm 1. \] (3.10)

The above ideal case is extended to a near degenerate case. When the subspace \( D^{N=2} \) forms an approximate invariant subspace of the total Hamiltonian, i.e., when the coupling between \( D^{N=2} \) and the rest part of the physical boson space are sufficiently small, the eigenstates of the Hamiltonian in Eq. (3.7) has a sense irrespective of the other part of the Hilbert space. The eigenvalues and eigenstates of Eq. (3.7) are
given by

\[ \mathcal{H}^{n=2} U_\pm = \Omega_\pm U_\pm, \quad \Omega_\pm = \pm \sqrt{\alpha^2 + \beta^2}, \]

\[ U_+ = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{(\alpha^2 + \beta^2) + \alpha \sqrt{\alpha^2 + \beta^2}}} \begin{pmatrix} \alpha + \sqrt{\alpha^2 + \beta^2} \\ \beta \end{pmatrix}, \]

\[ U_- = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{(\alpha^2 + \beta^2) + \alpha \sqrt{\alpha^2 + \beta^2}}} \begin{pmatrix} -\beta \\ \alpha + \sqrt{\alpha^2 + \beta^2} \end{pmatrix}. \tag{3·11} \]

As is easily seen, \( U_\pm \) reduce to the states in Eq. (3·6) in a case with \( \beta = 0 \), whereas they reduce to \( \mathcal{R}_\pm \) in a case with \( \alpha = 0 \). Namely, the global oscillator quanta \((m, n)\) are the good quantum numbers in a case with \( \beta = 0 \), whereas are the local quanta \((\mathcal{N}, t_\pm)\) in a case with \( \alpha = 0 \). More generally, there always exist two good quantum numbers \((\mathcal{N}, \Omega_\pm)\), when the subspace \( \mathcal{D}_{n=2} \) is an approximate invariant subspace of the total Hamiltonian. Since there does not occur any dissolution in a number of the good quantum numbers, there may still exist a pair of new regular quantum states generated by the anharmonic effects which otherwise induces a well-known divergent difficulty when one applies the perturbative treatment based on the \( mn \)-basis states. An extremely important character of new quanta \((\mathcal{N}, \Omega_\pm)\) rests on their local property. When there exists a couple of well-isolated states whose unperturbed energies are accidentally degenerate, one may introduce a local subspace \( \mathcal{D} \) spanned by these two states irrespective of the other part of the Hilbert space. Namely, the local isolated quanta are defined in a tiny region of the physical boson space. The above discussion is very similar to the classical case where an isolated nonlinear resonance generates a pair of new regular motion running through a stable and unstable fixed point, without accompanying any dissolution in a number of constants of motion. A pair of eigenstates expressed by \( U_\pm \) with \( \beta \neq 0 \) are thus called quantum resonant states.

§ 4. How to analyze complex structure of quantum space of states

4.1. The most natural representation

In the general theory of the nonlinear dynamics, an introduction of the most natural coordinate system for a given trajectory provides us with a firm basis for analyzing the nonlinear resonant structure. Since the quantum resonant condition (3·3) is specified by the integrable part of the Hamiltonian, it is decisive to develop a proper method for obtaining \( H_{\text{integ}} \) which is appropriate for a given eigenstate.

To this aim, let us briefly summarize the quantum theory of nuclear collective dynamics which has been developed under dictation of the self-consistent collective coordinate (SCC) method formulated within the TDHF theory. Since the Hamiltonian in Eq. (2·7) is defined so as to satisfy the quantum maximal decoupling condition, one may regard the \((m, n)\)-representation of using the basis states in Eq. (2·8) as a dynamical representation.

One may then introduce the \( \mu n \)-basis states which are obtained in such a way that a part of the coupling term \( H_{\text{coup}}(B^\dagger, B) \) is diagonalized within a subspace \( D_n:\{m, n\}; \)
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$m=0, \cdots, N-n$ along with $H_0(B^\dagger, B)$. Introducing a projection operator $P(n)$ onto $D_n$ through

$$P(n) = \sum_{n=0}^{N-n} |m, n\rangle \langle m, n|, \quad I = \sum_{n=0}^{N} P(n),$$

(4·1)

where $I$ denotes a unit operator of the physical boson space, one may express the total Hamiltonian in Eq. (2·7) as

$$iH(B^\dagger, B)I = H_{\text{dia}} + H_{\text{off-dia}},$$

$$H_{\text{dia}} = \sum_{n=0}^{N} P(n)H(B^\dagger, B)P(n), \quad H_{\text{off-dia}} = \sum_{n_1, n_2=0, n_1+n_2=n}^{N} P(n_1)H(B^\dagger, B)P(n_2).$$

(4·2)

By diagonalizing $H_{\text{dia}}$ in each subspace $D_n$, one gets a set of new basis states

$$P(n)H(B^\dagger, B)P(n)|\mu, n\rangle = E_{\mu}^{(n)}|\mu, n\rangle \quad \text{for given } n,$$

(4·3)

where $\mu=0, \cdots, N-n$ with $E_{\mu}^{(n)} \leq \cdots \leq E_{N-n}^{(n)}$.

In the second step, one may define the $\nu$-basis states by further diagonalizing another part of the coupling term $H_{\text{coup}}(B^\dagger, B)$ within a subspace $D_{\nu}$: $\{|\mu, n\rangle; n=0, \cdots, N-\mu\}$. With the aid of a projection operator $P(\nu)$ defined through

$$P(\nu) = \sum_{n=0}^{N-\mu} |\mu, n\rangle \langle m, n|, \quad I = \sum_{\nu=0}^{N} P(\nu),$$

(4·4)

the Hamiltonian in Eq. (4·2) is expressed as

$$iH(B^\dagger, B)I = \sum_{\mu=0}^{N-\mu} \sum_{n=0}^{N-\mu} |\mu, n\rangle E_{\nu}^{(n)}|\mu, n\rangle + \sum_{\mu=0}^{N-\mu} H_{\text{stab}}^{\mu} + \sum_{\mu_1, \mu_2=0, \mu_1+\mu_2=\nu}^{N} H_{\text{sepa}}^{\mu_1, \mu_2},$$

$$H_{\text{stab}}^{\mu} = P(\mu)H_{\text{off-dia}}P(\mu), \quad H_{\text{sepa}}^{\mu_1, \mu_2} = P(\mu_1)H_{\text{off-dia}}P(\mu_2); (\mu_1 \neq \mu_2).$$

(4·5)

Diagonalizing the Hamiltonian within the subspace $D_{\nu}$, i.e.,

$$\left\{ \sum_{n=0}^{N-\mu} |\mu, n\rangle E_{\nu}^{(n)}(\mu, n) + H_{\text{stab}}^{\mu} \right\} |\mu, \nu\rangle = E_{\mu, \nu}|\mu, \nu\rangle \quad \text{for given } \mu,$$

(4·6)

where $\nu=0, \cdots, N-\mu$ with $E_{\nu, \nu=0} \leq \cdots \leq E_{\nu, \nu=N-\mu}$, one obtains

$$iH(B^\dagger, B)I = H_{\text{Integ}} + \sum_{\mu_1, \mu_2=0, \mu_1+\mu_2=\nu}^{N-\mu} H_{\text{sepa}}^{\mu_1, \mu_2}, \quad H_{\text{Integ}} = \sum_{\mu=0}^{N-\mu} \sum_{\nu=0}^{\mu} |\mu, \nu\rangle E_{\mu, \nu}|\mu, \nu\rangle.$$
\begin{equation}
|\lambda^{(1)}\rangle = |\lambda\rangle^{(0)} + \sum_{\mu=\mu_0} f^{\mu}_{\mu,\nu}|\mu,\nu\rangle, \quad f^{\mu}_{\mu,\nu} = \frac{(\mu\nu|H_{sep}|_{\mu,\nu})}{E_{\mu,\nu} - E_{\mu,\nu}}, \quad (4.8)
\end{equation}

within the first order perturbation theory. Here, the zero-th order state $|\lambda\rangle^{(0)}$ is chosen by using the maximum overlap condition with $|\lambda\rangle$, i.e., $|\lambda\rangle^{(0)} = |\mu_0,\nu_0\rangle$ with Max$\{|(\mu\nu|\langle \lambda)|\}$. When there exists a certain state $|\mu_0,\nu_0\rangle$ satisfying $E_{\mu_0,\nu_0} - E_{\mu_0,\nu_0} \approx 0$, a perturbative expression in Eq. (4.8) does not work any more. In this case, one may introduce a local resonant subspace $D$ and the Hamiltonian $\mathcal{H}$ expressed as

\begin{equation}
\mathcal{H} = \frac{E_{\mu_0,\nu_0} - E_{\mu_0,\nu_0}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (\mu_0,\nu_0|H_{sep}|_{\mu,\nu})(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}), \quad (4.9)
\end{equation}

which have the same structure as those in Eqs. (3.6) and (3.7). In Eq. (4.9), the subspace composed by the $\mu\nu$-basis states is expressed by a bold-face $D$, whereas the subspace $D$ in § 3 consists of the $mn$-basis states.

We are now in a position to analyze the structure of the eigenstates whose profiles are illustrated in Fig. 2. Note that the system with parameters used in Figs. 1 and 2 is very near to an unperturbed system characterized by $H_0(B^*, B)$, because the interaction strength is sufficiently smaller than the single-particle level spacing. In this case, the resonant condition is approximately given by

\begin{equation}
\varepsilon_2 - \varepsilon_0 = 2(\varepsilon_1 - \varepsilon_0), \quad (4.10)
\end{equation}

and the resonant oscillator quanta $\mathcal{N}$ in Eq. (3.4) is expressed as

\begin{equation}
\mathcal{N} = m + 2n = \mu + 2\nu. \quad (4.11)
\end{equation}

When one perturbatively evaluates the effects of $H_{coup}(B^*, B)$ in Eq. (2.7) by using the $mn$-basis states, one has at least to calculate the third order terms which first give a coupling between two states with the same resonant quantum number $\mathcal{N} = m + 2n$. On the other hand, a coupling between two states having the same resonant quantum number $\mathcal{N} = \mu + 2\nu$ appears in the first order perturbation, when one applies the perturbation theory based on the $\mu\nu$-basis states. This fact shows superiority of using the $\mu\nu$-basis states over the $mn$-basis states.

Since there holds approximate resonant condition (4.10), one may divide the $\mu\nu$-basis states into the resonant subspaces like in Eq. (3.5),

\begin{equation}
D^{\mathcal{N}}: \left\{ \left| \frac{\mathcal{N}}{2} \right|, \left| 2 + \frac{\mathcal{N}}{2} \right|, \ldots, \left| 2k + \frac{\mathcal{N}}{2} - k \right| \right\}, \quad k = N - \frac{\mathcal{N}}{2}, \mathcal{N}: \text{even}, \quad (4.12a)
\end{equation}

\begin{equation}
D^{\mathcal{N}}: \left\{ \left| 1 - \frac{\mathcal{N}}{2} \right|, \left| 3 - \frac{\mathcal{N}}{2} \right|, \ldots, \left| 2k + 1, \frac{\mathcal{N}}{2} - k - 1 \right| \right\}, \quad k = N - \frac{\mathcal{N}}{2} + 1, \mathcal{N}: \text{odd}. \quad (4.12b)
\end{equation}

The projection operator $P(\mathcal{N})$ onto the resonant subspace $D^{\mathcal{N}}$ is introduced through
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\[ P(\mathcal{N}) = \sum_{\mu=0}^{N-3/2} |2\mu, \frac{\mathcal{N}}{2} - \mu\rangle \langle 2\mu, \frac{\mathcal{N}}{2} - \mu| \] for even \( \mathcal{N} \),

\[ P(\mathcal{N}) = \sum_{\mu=0}^{N-(\mathcal{N}+1)/2} |2\mu+1, \frac{\mathcal{N}-1}{2} - \mu\rangle \langle 2\mu+1, \frac{\mathcal{N}-1}{2} - \mu| \] for odd \( \mathcal{N} \).

By diagonalizing the Hamiltonian in Eq. (4.7) within \( D^2 \), i.e.,

\[ P(\mathcal{N}) H(B^i, B) P(\mathcal{N})|\mathcal{N}, i\rangle_{\text{res}} = E_{\mathcal{N}, i}|\mathcal{N}, i\rangle_{\text{res}}, i=0, \cdots, N-(\mathcal{N}+1)/2 \text{ or } N-\mathcal{N}/2, \]

one obtains a set of states \(|\mathcal{N}, i\rangle_{\text{res}}\) which is called resonant basis states.

Since we are considering \((m, n)=(\text{even, even})\)-sector, the \( \mu\nu \)-basis states contained in \( D^{30-40} \) are \(|\mu, \nu\rangle = |(0, 20), (4, 18), (8, 16), (12, 14), (16, 12), (20, 10)\rangle\) whose energies \( E_{\mu\nu} \) defined in Eq. (4.6) are listed in Table I. As is seen from Table I, there are two pairs of the \( \mu\nu \)-basis states, \(|\mu=16, \nu=12\rangle, |\mu=12, \nu=14\rangle\) and \(|\mu=8, \nu=16\rangle, |\mu=4, \nu=18\rangle\), each of which is characterized by having the almost degenerate unperturbed energies \( E_{\mu\nu} \). On the other hand, \(|\mu=20, \nu=10\rangle\) and \(|\mu=0, \nu=20\rangle\) are rather isolated from the other states. In Table II, the resonant basis states \(|\mathcal{N}, i\rangle_{\text{res}}, i=0-5\rangle\) with \( \mathcal{N}=40 \) are expressed for the \( \mu\nu \)-basis states. It is recognized from Table II that a pair of states \(|\mathcal{N}, 1\rangle, |\mathcal{N}, 2\rangle\) is a result of a strong mixture between \(|\mu=8, \nu=16\rangle\) and \(|\mu=4, \nu=18\rangle\), whereas \(|\mathcal{N}, 3\rangle, |\mathcal{N}, 4\rangle\) are between \(|\mu=16, \nu=12\rangle\) and \(|\mu=12, \nu=14\rangle\).

In Table III, the eigenfunctions \(|\lambda\rangle; \lambda=101-106\rangle\) with \( E_{\lambda}=40 \) expressed in the \( \mu\nu \)-basis states are listed. Comparing Table II with III, one may see excellent agreement between the eigenfunctions \(|\lambda\rangle; \lambda=101-106\rangle\) and the resonant basis states \(|\mathcal{N}, i\rangle_{\text{res}}, i=0-5\rangle\). In Table IV, the overlaps between the eigenfunctions and resonant basis states are listed. Since a pair of eigenstates \(|\lambda\rangle; \lambda=102, 103\rangle\) is well described within the subspace \(|\mu=8, \nu=16\rangle, |\mu=4, \nu=18\rangle\), it forms an approximate invariant subspace of the Hamiltonian discussed in §3. The eigenstates \(|\lambda\rangle; \lambda=102, 103\rangle\) are thus described by \( U_\lambda \) in Eq. (3.11) and are regarded to be a pair of quantum resonant states characterized by the local quantum numbers. The same discussion holds for a pair of eigenstates \(|\lambda\rangle; \lambda=104, 105\rangle\). On the other hand, the eigenstates \(|\lambda=101\rangle\) and \(|\lambda=106\rangle\) are well described by the single \( \mu\nu \)-basis state, which is characterized by the global quantum numbers \( \langle \mu, \nu \rangle \) specified by \( H_{\text{integ}} \) in Eq. (4.7). The above understanding for the quantum eigenstates is very similar to those for the semi-quantum TDHF trajectories in Fig. 1 discussed in Ref. 11).

In this paper, we are restricted ourselves to a case with one isolated resonance which is well separated from the other resonances. Since the classical nonlinear resonance is known to give a starting point in studying the order-to-chaos transition, it is an interesting question what happens when the anharmonic interaction becomes large, and two eigenstates belonging to different quantum resonances start to interact.

### Table I. Unperturbed energies \( E_{\mu\nu} \) of the \( \mu\nu \)-basis states.

<table>
<thead>
<tr>
<th>( E_{\mu=20, \nu=10} )</th>
<th>( E_{\mu=15, \nu=12} )</th>
<th>( E_{\mu=12, \nu=14} )</th>
<th>( E_{\mu=8, \nu=16} )</th>
<th>( E_{\mu=4, \nu=18} )</th>
<th>( E_{\mu=0, \nu=20} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40.0107</td>
<td>40.0360</td>
<td>40.0381</td>
<td>40.0317</td>
<td>40.0311</td>
<td>40.0510</td>
</tr>
</tbody>
</table>
Table II. For a case with $\mathcal{N}=40$, energies of resonant basis states $|\mathcal{N}, i\rangle_{\text{res}}$ are listed in the first column. Their wave functions in terms of the $\mu\nu$-basis states are also listed.

| $\mathcal{N}=40$ | $|\mu=20, \nu=10\rangle$ | $|\mu=16, \nu=12\rangle$ | $|\mu=12, \nu=14\rangle$ | $|\mu=8, \nu=16\rangle$ | $|\mu=4, \nu=18\rangle$ | $|\mu=0, \nu=20\rangle$ |
|-----------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| $E_{\mathcal{N}1}=40.011$ | .99961 | -.02783 | .00124 | -.00007 | .00000 | .00000 |
| $E_{\mathcal{N}1}=40.031$ | -.00074 | -.02091 | .09315 | -.59011 | .80164 | -.00441 |
| $E_{\mathcal{N}1}=40.032$ | .00153 | .04636 | -.15495 | .78546 | .59740 | -.00353 |
| $E_{\mathcal{N}1}=40.036$ | .02599 | .91686 | -.38067 | .11610 | -.01730 | .00013 |
| $E_{\mathcal{N}1}=40.039$ | .00988 | .39497 | .90687 | .14609 | .01247 | -.00012 |
| $E_{\mathcal{N}1}=40.051$ | .00000 | .00000 | .00002 | .00020 | .00565 | .99998 |

Table III. For a case with $\mathcal{N}=40$, energies of eigenstates $\lambda=101, \ldots, 106$ are listed in the first column. Their wave functions $|\lambda\rangle$ in terms of the $\mu\nu$-basis states are also listed.

| $\mathcal{N}=40$ | $|\mu=20, \nu=10\rangle$ | $|\mu=16, \nu=12\rangle$ | $|\mu=12, \nu=14\rangle$ | $|\mu=8, \nu=16\rangle$ | $|\mu=4, \nu=18\rangle$ | $|\mu=0, \nu=20\rangle$ |
|-----------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| $E_{\lambda1}=40.011$ | .99999 | -.02795 | .00124 | -.00007 | .00000 | .00000 |
| $E_{\lambda1}=40.031$ | -.00089 | -.02546 | .10568 | -.64065 | .75959 | -.00420 |
| $E_{\lambda1}=40.032$ | .00164 | .04995 | -.15380 | .74242 | .64947 | -.00384 |
| $E_{\lambda1}=40.036$ | .02616 | .91921 | -.37254 | -.12159 | -.01992 | .00014 |
| $E_{\lambda1}=40.039$ | .00974 | .38333 | .90661 | .15032 | .01342 | -.00012 |
| $E_{\lambda1}=40.051$ | .00000 | .00000 | .00002 | .00020 | .00568 | .99996 |

Table IV. Overlap between the exact eigenstates $|\lambda\rangle$ and the resonant basis states $|\mathcal{N}, i\rangle_{\text{res}}$.

| $\mathcal{N}=40$ | $|\lambda=101\rangle$ | $|\lambda=102\rangle$ | $|\lambda=103\rangle$ | $|\lambda=104\rangle$ | $|\lambda=105\rangle$ | $|\lambda=106\rangle$ |
|-----------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| $|\lambda=101\rangle$ | .99999 | .00000 | .00000 | .00000 | .00000 | .00000 |
| $|\lambda=102\rangle$ | .99474 | .00451 | .00000 | .00000 | .00000 | .00000 |
| $|\lambda=103\rangle$ | .99459 | .00005 | .00001 | .00000 | .00000 | .00000 |
| $|\lambda=104\rangle$ | .99349 | .00007 | .00001 | .00000 | .00000 | .00000 |
| $|\lambda=105\rangle$ | .99949 | .00005 | .00001 | .00000 | .00000 | .00000 |
| $|\lambda=106\rangle$ | .99918 | .00000 | .00000 | .00000 | .00000 | .00000 |

(overlap). These subjects are under investigation.

References

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