Effective Interaction Theory and Unitary-Model-Operator Approach to Nuclear Saturation Problem

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The structure of the effective interaction is reviewed on the similarity-transformation theory. The Hermitian effective interaction is constructed, and its formal relation with the usual non-Hermitian form is discussed. As an application of the Hermitian theory, the unitary-model-operator approach (UMOA) is formulated. The two-body effective interaction is introduced and shown to be used for a basic ingredient in describing nuclear properties. The relationship between the G-matrix theory and the present approach is discussed. The theory is applied to the calculation of the ground-state properties of \(^{16}\text{O}\) with various nucleon-nucleon potentials.

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§ 1. Introduction

For many years the nuclear many-body theory has been developed to reproduce theoretically nuclear data starting with a realistic nucleon-nucleon (NN) force. The saturation property of finite nuclei, namely, the relation between binding energy and density, has been one of the most difficult problems in the nuclear structure theory. The origin of the strong spin-orbit potential in the nuclear one-body potential has also
been a long-standing problem.

There would be some possibilities to be considered for further development of the nuclear many-body theory. The first one is the relativistic effects that may come from mesonic degrees of freedom, virtual excitations of nucleons, anti-nucleon components in a single-nucleon state and so on. The second one is the ambiguity in the NN force used in the nuclear structure calculation. In recent years, a modern version of the meson-exchange potential, such as the Bonn potential, has been proposed, but in spite of highly theoretical derivation there still remain some ambiguities. The third one is the difficulty in solving the nuclear many-body problem. Many approaches to the nuclear structure theory have been proposed and applied in the past forty years, but there would be room for developing the nuclear many-body theory.

In principle the disagreement between the experimental data and the theoretical predictions could be attributed to the combinations of all the three possibilities. However it seems that almost all the arguments have been addressed to the former two possibilities, that is, the relativistic effects and the NN force. As to the nuclear many-body theory, it would be a common belief that it is hardly possible to go beyond the basic framework of Brueckner’s reaction matrix (or $G$-matrix) theory.\(^1\)

The main purpose of the present work is to give a deeper insight into the nuclear structure calculation by introducing an effective interaction as a basic ingredient of the nuclear many-body theory. The concept of the effective interaction has been used widely in nuclear,\(^2\) atomic and molecular physics.\(^3\) The effective interaction to be discussed here is defined as an interaction that acts in a certain model space (or $P$ space) and yields some of the same eigenvalues as those of an original Hamiltonian. In this respect the effective interaction should have a decoupling property between the model and excluded spaces. Many efforts have been directed to theoretical derivation of the effective interaction, and its development has been reviewed by several authors, namely, by Brandow,\(^4\) Barrett and Kirson,\(^5\) Kuo\(^6\) and Ellis and Osnes,\(^7\) and Ando, Bando and Nagata.\(^8\)

In general the effective interaction cannot be determined uniquely. The possible effective interactions can be classified into some categories according to $E$-dependence or Hermiticity. Since the folded-diagram theory succeeded in removing the $E$-dependence, which has been developed mainly by Brandow,\(^4\) Kuo, Lee, and Ratcliff\(^9\) and Lindgren,\(^10\) the $E$-independent and non-Hermitian effective interaction has been used widely in the nuclear many-body calculation. However, it has been shown by several authors\(^11\)–\(^14\) that the non-Hermitian effective interaction can be converted to a Hermitian form.

The effective interactions can also be classified into two types, namely, those for valence nucleons outside a core (or a closed shell) and those for a total many-body system including a core. The folded-diagram theory, which has been developed in the time-dependent approach by Kuo and his collaborators,\(^2,\)\(^9\) has been designed for the effective interactions acting among valence nucleons. This type of the effective-interaction theory has been widely applied to the actual calculations of two-valence nucleon systems.\(^15\)–\(^18\) On the other hand, it seems that the effective interaction for describing a total many-body system has not always been clarified satisfactorily. Recently some attempts for this kind of the effective interactions have been proposed
by Barrett and his collaborators.\textsuperscript{19,20}

In the nuclear many-body theory the \( G \)-matrix, initiated by Brueckner and Bethe and their collaborators\textsuperscript{1,21,22} was introduced for the nuclear matter calculation. The application of the \( G \)-matrix theory to finite nuclei was made by Kuo and Brown.\textsuperscript{23} One might consider that the \( G \)-matrix is a kind of the effective interaction. However, rigorously speaking, the \( G \)-matrix is not an effective interaction, because it has no decoupling property between the model space and its complement. The properties of the \( G \)-matrix may be summarized as follows: It is \( E \)-dependent, not decoupled and non-Hermitian (the \( G \)-matrix is Hermitian only for a constant starting energy). To say from a formal point of view, the \( G \)-matrix is the \( Q \)-box in the space of two-particle states in the folded-diagram theory of Kuo et al.,\textsuperscript{2,9} where the \( Q \)-box is defined as the sum of irreducible and non-folded diagrams. The many-body theory based on the \( G \)-matrix has been established and applied to many actual problems as a standard method, but it would be much more desirable if there could be a many-body theory described in terms of \( E \)-independent and Hermitian effective interactions with the decoupling property.

The use of the Hermitian effective interaction in the nuclear structure calculation has been proposed by the present authors on the basis of the unitary-model-operator approach referred to shortly as UMOA. The UMOA was first formulated by Providencia and Shakin,\textsuperscript{24} and followed by several authors.\textsuperscript{25} In this approach a unitary transformation was introduced to describe short-range correlations of two nucleons, and a transformed two-body interaction was derived. About ten years ago the authors introduced a new principle for determining the unitary transformation by applying the effective-interaction theory of the Hermitian form.\textsuperscript{26-28} It has been shown that the unitary transformation can be determined from the decoupling condition between the model space and its complement. The new version of UMOA enables us to use the Hermitian, \( E \)-independent and decoupled effective interaction in place of the \( G \)-matrix. Advantages of the use of this type of the effective interaction would be stated as follows: The decoupling property of the effective interaction works to reduce the number of diagrams to be evaluated, because one can determine the effective interaction decoupled between the occupied two-particle and unoccupied two-particle states, and as a result two-particle-two-hole excitations do not occur any more. It is noted that the usual Hartree-Fock (HF) calculation leads to transformed one-body states which decouple one-body interaction, including self-consistent average potential, between occupied and unoccupied states. The essence of UMOA is to introduce transformed two-body states which decouple the two-body interactions. Another advantage is that in UMOA a self-consistent average potential can be introduced without any approximation. This situation is quite different from that in the \( G \)-matrix theory. Due to the \( E \)-dependence, the definition of the self-consistent average potential is, in principle, impossible in the \( G \)-matrix theory.

In the present work we discuss the formal structure of the effective-interaction theory, in particular, of the Hermitian form. As an application of the general effective-interaction theory, we formulate UMOA in which an Hermitian two-body effective interaction is introduced through the unitary transformation of the Hamiltonian. We apply UMOA to the calculation of the ground-state properties of
with various NN potentials, and see to what extent the results would be improved in comparison with the usual approaches.

The organization of the present paper is as follows: In § 2 we discuss a general theory of the effective interaction of both non-Hermitian and Hermitian forms. In § 3 UMOA is formulated and some important formulae are derived. The calculation procedures and some approximations, which are necessary for actual calculations, are discussed in § 4. In § 5, the results of some theoretical predictions for $^{16}$O are reported with various NN forces, namely, the Reid soft-core, Super soft-core, Paris and Bonn potentials. The concluding remarks are given in § 6.

§ 2. General theory of the effective interaction

2.1. Derivation of the effective interaction by means of similarity transformation

In the nuclear structure calculation, it is common practice to solve the Shr"odinger equation in a restricted Hilbert space referred to as a model space. The effective-interaction theory has aimed at theoretical derivation of an equivalent interaction acting in this model space. The rigorous definition of the effective interaction is that it should act within the model space and the model-space eigenvalue equation yields some of the same eigenvalues as those of the original Hamiltonian. However, in general, these requirements cannot determine the effective interaction uniquely.

The most general definition of the effective Hamiltonian, denoted by $\mathcal{H}$, is given through the eigenvalue equation in the model space, $\mathcal{H}\phi_k = E_k \chi \phi_k$, where $\chi$ is called the metric (or overlap) and $E_k$ should be one of the eigenvalues of the original Hamiltonian. In a long history of the theory of the effective interaction, many types of the solutions have been derived. We refer to some important solutions for $\mathcal{H}$ without explicit expressions. In 1929, Van Vleck\(^3\) already proposed an idea of constructing $\mathcal{H}$ by introducing a unitary transformation of the Hamiltonian. Kato\(^4\) derived a solution for $\mathcal{H}$ which was characterized as the Hermitian form with a non-diagonal metric. Fukuda, Sawada and Taketani\(^5\) and Okubo\(^6\) gave a Hermitian form with the elementary metric ($\chi=1$) in the study of the field-theoretical derivation of the meson-exchange NN potential. A simple non-Hermitian and $E$-dependent form of $\mathcal{H}$ was derived by Lödin\(^7\) and Feshbach\(^8\). The $E$-independent counterpart of the non-Hermitian form was given by Bloch\(^9\) and its structure was extensively studied by Bloch and Horowitz\(^10\).

The perturbation expansion of the effective interaction, especially for the non-Hermitian form, was also studied in connection with the linked-cluster expansion by many authors, namely, Brueckner\(^11\) Brandow\(^12\), Lindgren\(^13\), Kvasnicka\(^14\),\(^15\) and Poves and Zucker\(^16\) in the time-independent approach.

The structure of the Hermitian effective interaction was clarified by des Cloizeaux\(^17\) and Brandow\(^18\) and some Hermitian forms for $\mathcal{H}$ were derived. However, the mutual relation among them had not been known for a long time until the general relation between the Hermitian and non-Hermitian forms was found by Shavitt and Redmon\(^19\), Westhaus\(^20\) and one of the authors (K. S.).\(^21\) It has been proved that all the Hermitian effective interactions can be reduced to the Van Vleck form given in
terms of the unitary transformation of the Hamiltonian.

All the studies mentioned above were made in the time-independent approach. There was also a long history of the derivation of $\mathcal{H}$ in the time-dependent approach. In this approach the problem of calculating the effective interaction for valence particles outside a core was extensively studied. The non-Hermitian and $E$-independent effective interaction was constructed diagrammatically by introducing the folded diagrams. The original idea of the folded diagram was due to Morita, and was followed by Oberlechner, Owono-N'Guema, and Richert, and Kuo, Lee and Ratcliff. In particular, Kuo et al., proved that the folded diagrams were generated from the energy derivatives of the sum of the non-folded diagrams, called the $\tilde{Q}$-box, and they obtained a general expression of the effective interaction in terms of only the $\tilde{Q}$-boxes.

The formal interrelations among various interactions have been discussed by Klein and Brandow. It has been shown by the present authors that a general effective interaction can be derived by means of similarity transformation and the variety of the effective interactions is due to the variety of possible transformations. We here review briefly this unified description of the effective interaction on the basis of the similarity transformation theory.

We define two projection operators $P$ and $Q$ that project a state onto the model space and its complement, respectively. The operators $P$ and $Q$ satisfy $P^2 = P$, $Q^2 = Q$, $PQ = 0$ and $P + Q = 1$. Hereafter we shall refer to the model space and its complement (the excluded space) simply as the $P$ and $Q$ spaces, respectively.

We introduce an operator $\omega$ which acts as a mapping between the $P$ and $Q$ spaces, i.e.,

$$|q\rangle = \omega |p\rangle, \quad (|q\rangle \in Q, |p\rangle \in P). \quad (2.1)$$

The operator $\omega$ satisfies

$$\omega = Q \omega P, \quad (2.2)$$
$$\omega Q = 0, \quad (2.3)$$
$$P \omega = 0 \quad (2.4)$$

and

$$\omega^2 = 0. \quad (2.5)$$

With $\omega$ we define an operator $X(n)$ by

$$X(n) = (1 + \omega)(1 + \omega^* \omega + \omega \omega^*)^n, \quad (2.6)$$

where $n$ is a real number. The inverse of $X(n)$ is

$$X^{-1}(n) = (1 + \omega^* \omega + \omega \omega^*)^{-n}(1 - \omega). \quad (2.7)$$

The above relation can be verified by using the fact that $(1 - \omega)(1 + \omega) = 1$ that comes from Eq. (2.5).

We consider a transformation of the Hamiltonian
\[ H(n) = X^{-1}(n) H X(n). \] (2.8)

We require that the \( P \)-space operator \( P \overline{H}(n)P \) be an effective Hamiltonian, that is, it should have the same eigenvalues as those of the original Hamiltonian \( H \). This requirement is equivalent to the following decoupling condition:

\[ Q X^{-1}(n) H X(n) P = 0. \] (2.9)

Substituting \( X(n) \) in Eq. (2.6) into Eq. (2.9), we have

\[ Q H P + Q H Q \omega - \omega P H P - \omega P H Q \omega = 0. \] (2.10)

This equation is the basic equation for determining \( \omega \).\(^{49,50}\) Note that the solution to Eq. (2.9) is independent of the number \( n \). This implies that if \( \omega \) is a solution to Eq. (2.10), the \( P \)-space operator

\[ \mathcal{H}(n) = P \overline{H}(n) P = (P + \omega^\dagger \omega)^{-n} H (1 + \omega) (P + \omega^\dagger \omega)^{-n}, \] (2.11)

can be an effective Hamiltonian for any real number \( n \).

We discuss the meaning of the operator \( \omega \) more in detail. Let \( d \) be the dimension of the \( P \) space. Suppose that \( |\Phi_k\rangle \) be one of \( d \) eigenstates of \( H \) that we want to solve with the effective interaction. We decompose \( |\Phi_k\rangle \) into the \( P \)-space component \( P|\Phi_k\rangle \) (\( \equiv |\phi_k\rangle \)) and the \( Q \)-space component \( Q|\Phi_k\rangle \). It has been shown\(^{47}\) that the \( Q \)-space component can be generated by the operation of \( \omega \) onto \( |\phi_k\rangle \), i.e., \( Q|\Phi_k\rangle = \omega |\phi_k\rangle \). Therefore, we may state that the operator \( \omega \) induces a mapping of the model-space state onto a subspace of the \( Q \)-space states which have overlaps in the \( d \) eigenstates \( \{|\Phi_k\rangle\} \). We may write as

\[ |\Phi_k\rangle = |\phi_k\rangle + \omega |\phi_k\rangle, \quad (|\phi_k\rangle = P|\Phi_k\rangle). \] (2.12)

From Eq. (2.10), we have an equation for \( \omega^\dagger \) as

\[ PHQ + \omega^\dagger QHQ - PHP^\dagger - \omega^\dagger QHP \omega = 0. \] (2.13)

Eliminating the \( QHQ \) term from Eqs. (2.10) and (2.13), we have

\[ P(1 + \omega^\dagger) HP(P + \omega^\dagger \omega) = (P + \omega^\dagger \omega) PH (1 + \omega) P. \] (2.14)

Using Eqs. (2.11) and (2.14), we obtain a relation

\[ \mathcal{H}(n) = \mathcal{H}(-n-1). \] (2.15)

From the above relation we see that \( \mathcal{H}(n) \) becomes Hermitian when \( n = -1/2 \).

For a practical purpose two effective interactions would be of special importance. One is \( \mathcal{H}(0) \) that is written explicitly as

\[ \mathcal{H}(0) = PHP + PHQ \omega. \] (2.16)

The above form has been used most widely in the actual calculations. If we use \( \Omega = P + \omega \), which has been called the wave operator, \( \mathcal{H}(0) \) is simply written as \( \mathcal{H}(0) = PH \Omega \). The effective Hamiltonian \( \mathcal{H}(0) \) is non-Hermitian, but it seems to be the simplest. This effective Hamiltonian agrees with that derived in the time-dependent and diagrammatical approach.\(^{21}\)
Another important effective Hamiltonian is $\mathcal{H}(-1/2)$, as stated previously, which is Hermitian. It is written explicitly as

$$\mathcal{H}(-1/2) = (P + \omega^* \omega)^{1/2} H(P + \omega)(P + \omega^* \omega)^{-1/2}.$$  \hfill (2·17)

The above expression does not look apparently Hermitian, but it can be proved to be Hermitian.\(^{(47,48)}\) In recent years, the Hermitian effective-interaction theory has been studied with an active interest by several authors.\(^{(14,51,52)}\) The Hermitian theory is much more complicated than the standard non-Hermitian theory. However, in an actual case, the nuclear effective interactions determined empirically are Hermitian. Although the non-Hermiticity in the nuclear effective interactions has been shown to be small,\(^{14}\) the Hermitian form would have some advantages over the non-Hermitian theory.

2.2. Structure of the Hermitian effective interaction

We here discuss the structure of the Hermitian effective Hamiltonian. It has been proved\(^{(11-13)}\) that the expression of $\mathcal{H}(-1/2)$ in Eq. (2·17) is converted to

$$\mathcal{H}(-1/2) = Pe^{-z}He^zP$$  \hfill (2·18)

with

$$z = \text{arctanh}(\omega - \omega^*)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (\omega^* \omega)^n - \text{h.c.}.$$  \hfill (2·19)

The operator $z$ satisfies $z^* = -z$ and therefore $e^z$ is a unitary operator. The equivalence of two expressions of $\mathcal{H}(-1/2)$ in Eqs. (2·17) and (2·18) can be verified from the relation\(^{48}\)

$$e^zP = (1 + \omega)(P + \omega^* \omega)^{-1/2}.$$  \hfill (2·20)

Since $\mathcal{H}(-1/2)$ is a unitary-transformed Hamiltonian, it is evidently Hermitian.

We introduce an effective interaction $\mathcal{V}(n)$ through

$$\mathcal{H}(n) = PH_0P + \mathcal{V}(n),$$  \hfill (2·21)

where $H_0$ is the unperturbed Hamiltonian. We divide the original Hamiltonian $H$ into

$$H = H_0 + V,$$  \hfill (2·22)

where $V$ is the perturbation. We here do not assume that $PH_0P$ is degenerate. The assumption of degeneracy in $PH_0P$ makes the problem simplify extensively. However, in actual problems, there are many systems with non-degenerate unperturbed energies. Recently, the perturbation theory for a non-degenerate system has been in progress.\(^{(10,53,54)}\)

From Eqs. (2·15) and (2·21), the effective interaction $\mathcal{V}(0)$, which hereafter we shall denote by $R$, becomes

$$\mathcal{V}(0) = R$$
The Hermitian effective interaction $\mathcal{C}V(-1/2)$, denoted by $W$, is given with the non-Hermitian form $R$ by \(^{(11)}\)

\[
\mathcal{C}V(-1/2) = W
\]

\[
= Pe^{-Z(H_0 + V)e^ZP - PH_0P}
\]

\[
= \sum_{\omega=0}^{\infty} \sum_{\omega=0}^{\infty} F(\omega, \omega)(\omega^\dagger \omega)^p R(\omega^\dagger \omega)^q + h.c.,
\]

where

\[
F(\omega, \omega) = \begin{cases} 
1/2 & \text{for } \omega = \omega = 0, \\
\frac{(-1)^{p+q}(q-\omega)(2p-1)!!(2q-1)!!}{2(p+q)(p+q+1)(2p)!!(2q)!!} & \text{otherwise}.
\end{cases}
\]

The first few terms in $W$ are

\[
W = \frac{1}{2} (R + R^\dagger) + \frac{1}{8} \{((\omega^\dagger \omega)(R - R^\dagger) + h.c.) - \frac{1}{16} ((\omega^\dagger \omega)^2(R - R^\dagger) + h.c.) + \cdots.
\]

The general convergence condition for the expansion of $W$ in Eq. (2.26) has not been known. However, we can show that if the operators $R$ and $\omega^\dagger \omega$ are known, the expansion of $W$ can be summed up to be \(^{(13), (14)}\)

\[
\langle a_i | W | a_j \rangle = \frac{\sqrt{1 + \mu_i^2} \langle a_i | R | a_j \rangle + \sqrt{1 + \mu_j^2} \langle a_i | R^\dagger | a_j \rangle}{\sqrt{1 + \mu_i^2} + \sqrt{1 + \mu_j^2}},
\]

where $|a_i\rangle$ and $\mu_i^2$ are defined through the eigenvalue equation

\[
\omega^\dagger \omega |a_i\rangle = \mu_i^2 |a_i\rangle.
\]

Using the above equation, we have a simple expression of $z$ as

\[
z = \sum_{k=1}^{d} \text{arctanh} \mu_k |\nu_k\rangle \langle a_k| - h.c.,
\]

where $|\nu_k\rangle$ is a normalized $Q$-space state defined by $|\nu_k\rangle = (\omega/\mu_k)|a_k\rangle$. If we use the basis states $|a_i\rangle$, the calculation of the Hermitian effective interaction $W$ is simplified to a large extent. In actual applications we often use these nonperturbative expressions of $W$ and $z$.

2.3. The $Q$-box expansion of the effective interaction

The perturbation expansion of the effective interaction has long been studied by many authors. There have been two approaches to the effective-interaction theory, namely, the time-independent and time-dependent approaches. We here discuss mainly the time-independent approach that would be appropriate for deriving the effective interaction for a general quantum system. The perturbation expansion has been established for a system with degenerate unperturbed energies. \(^{(4), (10), (35)}\) However,
most realistic physical systems have non-degenerate unperturbed energies. If we want to derive effective interactions in a rather wide model space, we need to know a general expansion formula and a method of summing up the series. Recently a general theory for a non-degenerate model space has been formulated by several authors, in both the time-independent\(^{53,54}\) and time-dependent\(^{55}\) ways.

In the time-independent approach, we define the \(\tilde{Q}\)-box as a function of a starting energy \(\varepsilon\) by

\[
\tilde{Q}(\varepsilon) = PV P + PV Q - \frac{1}{\varepsilon - Q H Q} Q V P .
\]

We also define the multi-energy \(\tilde{Q}\)-box by

\[
\tilde{Q}_n(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1})
= (-1)^{n+1} PV Q \left(\frac{1}{\varepsilon_1 - Q H Q} \cdots \frac{1}{\varepsilon_{n+1} - Q H Q}\right) Q V P .
\]

We note that \(\tilde{Q}_n(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1})\) is defined for \(n \geq 1\) and it does not contain the term \(P V P\). The operators \((\varepsilon_i - Q H Q)\) are commutable so that the multi-energy \(\tilde{Q}\)-box does not depend on the order of the energy denominators.

Suppose that the unperturbed term \(P H_0 P\) has non-degenerate energies, i.e.,

\[
\sum_{\alpha} \varepsilon_{\alpha} P_{\alpha} ,
\]

where \(P_{\alpha}\) is the projection operator onto a model-space state \(|\alpha\rangle\). The non-Hermitian effective interaction \(R\) given in Eq. (2.23) is expanded into\(^{53,54}\)

\[
R = F_1 + F_2 + F_3 + \cdots ,
\]

where

\[
F_1 = \sum_{\alpha} \tilde{Q}(\varepsilon_{\alpha}) P_{\alpha} ,
\]

\[
F_2 = \sum_{\alpha\beta} \tilde{Q}_1(\varepsilon_{\alpha}, \varepsilon_{\beta}) P_{\alpha} \tilde{Q}(\varepsilon_{\beta}) P_{\beta} ,
\]

\[
F_3 = \sum_{\alpha\beta\gamma} \{ \tilde{Q}_2(\varepsilon_{\alpha}, \varepsilon_{\beta}, \varepsilon_{\gamma}) P_{\alpha} \tilde{Q}(\varepsilon_{\beta}) P_{\beta} \tilde{Q}(\varepsilon_{\gamma}) P_{\gamma} + \tilde{Q}_1(\varepsilon_{\alpha}, \varepsilon_{\gamma}) P_{\alpha} \tilde{Q}_1(\varepsilon_{\beta}, \varepsilon_{\gamma}) P_{\beta} \tilde{Q}(\varepsilon_{\gamma}) P_{\gamma} \}.
\]

If the unperturbed energies \(\{\varepsilon_{\alpha}\}\) are degenerate, that is,

\[
\sum_{\alpha} \varepsilon_{\alpha} P_{\alpha} = \varepsilon_0 P ,
\]

where \(\varepsilon_0\) is the degenerate unperturbed energy, the expansion of \(R\) becomes

\[
R = \tilde{Q} + \tilde{Q}_1 \tilde{Q} + \tilde{Q}_2 \tilde{Q} \tilde{Q} + \tilde{Q}_1 \tilde{Q}_1 \tilde{Q} + \cdots ,
\]

where \(\tilde{Q}\) is \(\tilde{Q}(\varepsilon_0)\) and \(\tilde{Q}_1, \tilde{Q}_2\) are the energy derivatives of the \(\tilde{Q}\)-box defined by
In order to obtain an expansion formula for the Hermitian effective interaction \( W \) in Eq. (2·26), we need to express \( \omega^* \omega \) in terms of the \( \tilde{Q} \)-boxes. The expression of \( \omega^* \omega \) has been given by

\[
\omega^* \omega = -\left\{ \sum_{\alpha} P_{\alpha} \tilde{Q}_s(\epsilon, \epsilon) P_{\alpha} + \sum_{\alpha, \beta, \gamma} [P_{\alpha} \tilde{Q}_s(\epsilon, \epsilon, \epsilon) P_{\beta} \tilde{Q}_s(\epsilon, \epsilon, \epsilon) P_{\gamma} + \text{h.c.}] \right\}
\]

Using Eqs. (2·26) and (2·40), we have an expression of \( W \) as

\[
W = W_1 + W_2 + \cdots ,
\]

\[
W_1 = \frac{1}{2} \sum_{\alpha} \left\{ \tilde{Q}_s(\epsilon) P_{\alpha} + \text{h.c.} \right\},
\]

\[
W_2 = \left\{ \sum_{\alpha} \left[ \frac{1}{2} \tilde{Q}_s(\epsilon, \epsilon, \epsilon) P_{\alpha} \tilde{Q}_s(\epsilon, \epsilon, \epsilon) P_{\alpha} + \frac{1}{8} \tilde{Q}_s(\epsilon) P_{\alpha} \tilde{Q}_s(\epsilon, \epsilon, \epsilon) P_{\alpha} \right] \right\}
\]

\[
-\sum_{\alpha, \beta, \gamma} \frac{1}{8} P_{\alpha} \tilde{Q}_s(\epsilon, \epsilon, \epsilon) P_{\beta} \tilde{Q}_s(\epsilon, \epsilon, \epsilon) P_{\gamma} + \text{h.c.}
\]

We have seen that the non-Hermitian and Hermitian effective interactions, \( R \) and \( W \), can be expressed in terms of the multi-energy \( \tilde{Q} \)-boxes. Therefore, the construction of the effective interaction is reduced to calculating the multi-energy \( \tilde{Q} \)-boxes. It has been shown\(^{54,55}\) that these multi-energy \( \tilde{Q} \)-boxes can be given by the difference quotients and/or the energy derivatives of the single-energy \( \tilde{Q} \)-box. For example, \( \tilde{Q}_s(\epsilon_1, \epsilon_1, \epsilon_1) \) for \( \epsilon_1 \neq \epsilon_2 \) becomes

\[
\tilde{Q}_s(\epsilon_1, \epsilon_1, \epsilon_1) = \frac{1}{\epsilon_2 - \epsilon_1} \left\{ \tilde{Q}(\epsilon_2) - \tilde{Q}(\epsilon_1) \right\} d\tilde{Q}(\epsilon) \bigg|_{\epsilon=\epsilon_1}.
\]

Some partial summation methods for \( R \) in Eq. (2·33) have been introduced to calculate the effective interactions.\(^{54,55}\) If the series expansion is divergent or slowly convergent, we can use the resummation methods.\(^{56,57}\) However, we here do not discuss the detail of these problems, because we think that the expansion formulae of \( R \) and \( W \) in terms of the \( \tilde{Q} \)-boxes would be enough for discussing the physical meanings of the structure of the effective interactions in actual problems.

We here want to emphasize that the effective interactions, both of the non-Hermitian and Hermitian forms, can be constructed in a unified way by using the mapping operator \( \omega \) as given in Eq. (2·11). The general series expansion of the effective interaction has been given with an expression in terms of only the \( \tilde{Q} \)-boxes regardless of degeneracy of the unperturbed Hamiltonian.

\section{3. Formulation of unitary-model-operator approach}

\subsection{3.1. Unitary transformation of Hamiltonian and its cluster expansion}

We consider a many-body system which is described by the Hamiltonian
where $t_i$ is the kinetic energy and $v_{ij}$ the two-body interaction. The second-quantization form of $H$ is written in the usual notations as

$$H = \sum_{\alpha \beta} \langle \alpha | t_i | \beta \rangle c_{\alpha}^{\dagger} c_{\beta} + \frac{1}{4} \sum_{\alpha \beta \gamma \delta} \langle \alpha | v_{ij} | \gamma \delta \rangle c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta},$$

where $\langle \alpha | v_{ij} | \gamma \delta \rangle$ denotes the anti-symmetrized and normalized matrix element. An important problem in the nuclear many-body theory is how to describe short-range correlations in a two-nucleon system, because the NN force contains strong short-range repulsive components. For this purpose the unitary-model-operator approach (UMOA) introduces a unitary transformation of the Hamiltonian,

$$\tilde{H} = e^{-S}He^S,$$

where $S$ is a two-body operator written as

$$S = \sum_{i<j} S_{ij}$$

and the second-quantization form of $S$ is

$$S = \frac{1}{4} \sum_{\alpha \beta \gamma \delta} \langle \alpha | S_{ij} | \gamma \delta \rangle c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta}.$$

The operator $S$ will be called the correlation operator. We require that $e^S$ be unitary so that $S$ satisfies

$$S^\dagger = -S$$

or

$$\langle \alpha | S_{ij} | \gamma \delta \rangle^* = - \langle \gamma | S_{ij} | \alpha \beta \rangle.$$

The operator $S$ is a two-body operator, but $e^S$ is not a two-body operator but it contains three-or-more-body terms. Therefore, the transformed Hamiltonian $\tilde{H}$ in Eq. (3.3) contains three-or-more-body terms.

Using the well-known expansion formula, $\tilde{H}$ is written as

$$\tilde{H} = H + [H, S] + \frac{1}{2} [[H, S], S] + \cdots.$$

In general each commutator term contains two-, three-body operators and so on as shown by Providencia and Shakin.24) If one collects all the two-, three-body terms, etc., on the r.h.s. of Eq. (3.8), $\tilde{H}$ can be written in a cluster-expansion form. In order to derive a general cluster-expansion form of $\tilde{H}$, we consider a transformation of $H$ in a coordinate representation. We write

$$\tilde{H} = e^{-S}(\sum_i h_i + \sum_{i<j} v_{ij})e^S - e^{-S}(\sum_i u_i)e^S,$$

where

$$h_i = t_i + u_i.$$
The one-body term $u_i$ is an auxiliary potential. At this stage $u_i$ is arbitrary, but it will be used for describing a self-consistent one-body potential. The transformed Hamiltonian $\tilde{H}$ can be decomposed into

$$\tilde{H} = \sum_i h_i + \sum_{i<j} \tilde{v}_{ij} + \sum_{i<j<k} \tilde{v}_{ijk} + \cdots - \{ \sum_i u_i + \sum_{i<j} \tilde{u}_{ij} + \sum_{i<j<k} \tilde{u}_{ijk} + \cdots \}, \quad (3.11)$$

where, for example,

$$\tilde{v}_{12} = e^{-S_{12}}(h_1 + h_2 + v_{12})e^{S_{12}} - (h_1 + h_2), \quad (3.12)$$

$$\tilde{v}_{123} = e^{-S_{123}}(h_1 + h_2 + h_3 + v_{12} + v_{13} + v_{23})e^{S_{123}} - (h_1 + h_2 + h_3 + \tilde{v}_{12} + \tilde{v}_{13} + \tilde{v}_{23}), \quad (3.13)$$

$$\tilde{u}_{12} = e^{-S_{12}}(u_1 + u_2)e^{S_{12}} - (u_1 + u_2), \quad (3.14)$$

$$\tilde{u}_{123} = e^{-S_{123}}(u_1 + u_2 + u_3)e^{S_{123}} - (u_1 + u_2 + u_3), \quad (3.15)$$

where

$$S_{123} = S_{12} + S_{13} + S_{23}. \quad (3.16)$$

It has been shown that $\tilde{v}_{ij}$ ($\tilde{u}_{ij}$) and $\tilde{v}_{ijk}$ ($\tilde{u}_{ijk}$) contain only two- and three-body operators, respectively. The second-quantization form of $\tilde{H}$ is then given by

$$\tilde{H} = \sum_{ab} \langle a| h_1 \beta \rangle c_a^\dagger c_\beta + \left( \frac{1}{2!} \right)^2 \sum_{a'b'c'd'} \langle a' \beta | \tilde{v}_{12} | \gamma \delta \rangle c_{a'}^\dagger c_\delta c_\gamma c_\beta + \cdots$$

$$+ \left( \frac{1}{3!} \right)^2 \sum_{a'b'c'd'e'f'} \langle a' \beta \gamma | \tilde{v}_{123} | \lambda \mu \nu \rangle c_{a'}^\dagger c_\delta c_\gamma c_\lambda c_\mu c_\nu c_\beta + \cdots$$

$$- \left\{ \sum_{ab} \langle a| u_1 \beta \rangle c_a^\dagger c_\beta + \left( \frac{1}{2!} \right)^2 \sum_{a'b'c'd'} \langle a' \beta | \tilde{u}_{12} | \gamma \delta \rangle c_{a'}^\dagger c_\delta c_\gamma c_\beta + \cdots \right\}. \quad (3.17)$$

For the determination of $u_i$, we require that $u_i$ be an average potential with the interaction $\tilde{v}_{ij}$, that is,

$$\langle a| u_i | \beta \rangle = \sum_{a' \beta} \langle a| \tilde{v}_{12} | \beta \rangle \langle a' \beta | \tilde{v}_{12} | \beta \rangle, \quad (3.18)$$

where $\rho_\beta$ denotes the quantum number of the upper most occupied state (the Fermi level). Since the transformed interaction $\tilde{v}_{12}$ is a function of the one-body field $u_i$, which may be clear from Eqs. (3.10) and (3.12), $u_i$ should be determined self-consistently. If the potential $u_i$ satisfies the self-consistency condition in Eq. (3.18), the one-body field $u_i$ is canceled by the bubble diagram of $\tilde{v}_{12}$. Then the transformed Hamiltonian $\tilde{H}$ may be written in the normal-product form with respect to particles and holes as

$$\tilde{H} = E_0 + \sum_{ab} \langle a| h_1 | \beta \rangle : c_a^\dagger c_\beta : + \left( \frac{1}{2!} \right)^2 \sum_{a'b'c'd'} \langle a' \beta | \tilde{u}_{12} | \gamma \delta \rangle : c_{a'}^\dagger c_\delta c_\gamma c_\beta : + \tilde{H}^{(0)} + \cdots, \quad (3.19)$$

where $E_0$ is the constant term given by

$$E_0 = \sum_{\lambda < \rho_\nu} \langle \lambda | t_1 | \lambda \rangle + \frac{1}{2} \sum_{\lambda \mu < \rho_\nu} \langle \lambda \mu | \tilde{v}_{12} | \lambda \mu \rangle \quad \quad (3.20)$$
Effective Interaction Theory and Unitary-Model-Operator Approach

which may be considered to be the first-order ground-state energy in terms of the transformed interaction $\bar{v}_{12}$. The $\bar{H}^{(3)}$ is the three-body cluster term given by

$$\bar{H}^{(3)} = \left( \frac{1}{3!} \right)^2 \sum_{\alpha \beta \lambda \mu \nu} \langle \alpha \beta \gamma | \bar{v}_{123} | \lambda \mu \nu \rangle c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} - \left( \frac{1}{2!} \right)^2 \sum_{\alpha \beta \gamma \delta} \langle \alpha \beta | \bar{u}_{12} | \gamma \delta \rangle c_{\alpha}^{\dagger} c_{\beta}^{\dagger} c_{\gamma} c_{\delta} .$$

$$(3 \cdot 21)$$

The second term in the above is a two-body term, but this term is canceled completely by the bubble diagram of $\bar{v}_{123}$ and only the three-body terms remain in $\bar{H}^{(3)}$.

3.2. Decoupling equation for determining the correlation operator

A central problem in UMOA is how to determine the correlation operator $S_{12}$. A different choice of $S_{12}$ leads to a different many-body theory. A new principle for determining $S_{12}$ has been introduced by the authors that the transformed interaction $\bar{v}_{12}$ should be a Hermitian effective interaction constructed in a space of two-particle states with a two-body Hamiltonian $H_{12} = h_1 + h_2 + \nu_{12}$.

We define two projection operators $P$ and $Q$ that project a two-particle state onto the model space (low momentum space) and the excluded space (high momentum space), respectively. We assume that the one-body part $h_1 + h_2$ is decoupled between $P$ and $Q$ spaces, i.e.,

$$Q(h_1 + h_2)P = P(h_1 + h_2)Q = 0 .$$

$$(3 \cdot 22)$$

The condition that $\bar{v}_{12}$ be an effective interaction is equivalent to the following decoupling equation:

$$Q \bar{v}_{12} P = 0 .$$

$$(3 \cdot 23)$$

From Eqs. (3 \cdot 12) and (3 \cdot 22), the above equation is written explicitly as

$$Qe^{-S_{12}}(h_1 + h_2 + \nu_{12})e^{S_{12}}P = 0 .$$

$$(3 \cdot 24)$$

We note that, in general, the correlation operator $S_{12}$ cannot be determined uniquely from the above decoupling condition. The unique solution is obtained under the following restrictions:

$$PS_{12}P = QS_{12}Q = 0 .$$

$$(3 \cdot 25)$$

These conditions have been called the minimal effect requirements, which mean that the transformation $\exp(S_{12})$ does not induce any unnecessary transformation within each of the $P$ and $Q$ spaces.

By applying the general theory of the effective interaction discussed in the preceding section, the solution $S_{12}$ is obtained as

$$S_{12} = \text{arctanh}(\omega_{12} - \omega_{12}^{-1}) ,$$

$$(3 \cdot 26)$$

where $\omega_{12}$ is given by

$$\omega_{12} = \sum_{k=1}^{d} Q|\phi_k\rangle \langle \phi_k|P ,$$

$$(3 \cdot 27)$$

where $d$ is the dimension of the $P$ space and $|\phi_k\rangle$ is the eigenstate satisfying
\[ (h_1 + h_2 + v_{12})|\Phi_k\rangle = E_k|\Phi_k\rangle, \]  
and \( |\phi_k\rangle \) is the bi-orthogonal states of \( |\phi_k\rangle = P|\Phi_k\rangle \). The solution \( \omega_{12} \) in Eq. (3.27) is derived from a general relation between the \( P \)-space component \( |\phi_k\rangle \) and the true eigenstate \( |\Phi_k\rangle \) in Eq. (2.12).

In UMOA, we apply the effective-interaction theory to a system of two particles moving in a self-consistent potential. The most important part of the calculation is to obtain the mapping operator \( \omega_{12} \) which is determined from Eqs. (3.27) and (3.28). If \( \omega_{12} \) is given, the calculations of the non-Hermitian and Hermitian effective interactions will be made in a straightforward way without any approximations according to Eqs. (2.23) and (2.27).

By applying the general theory formulated in § 2, we obtain the matrix element of \( \tilde{v}_{12} \) as
\[
\langle \alpha\beta|\tilde{v}_{12}|\gamma\delta\rangle = \sum_i \{ D_i \langle \alpha\beta|\chi_i\rangle \langle \chi_i|\omega_{12}|\chi_i\rangle \langle \chi_i|\gamma\delta\rangle \\
+ D_i \langle \alpha\beta|\chi_i\rangle \langle \chi_i|\omega_{12}|\chi_i\rangle \langle \chi_i|\gamma\delta\rangle \} \}
\]
with
\[
D_i = \sqrt{1 + \mu_i^2},
\]
where \( \mu_i^2 = \mu^2 \) and \( |\chi_i\rangle \) are given through
\[
\omega_{12}|\chi_i\rangle = \mu_i^2 |\chi_i\rangle
\]
and \( R_{12} \) is the non-Hermitian effective interaction given by
\[
R_{12} = P\nu_{12}P + P\nu_{12}Q\omega_{12}.
\]
With the solution \( \omega_{12} \) in Eq. (3.27), we have for \( \omega_{12} \)
\[
\omega_{12} = \sum_i \{ P|\tilde{\phi}_i\rangle \langle \Phi_i|Q|\Phi_i\rangle \langle \tilde{\phi}_i|P.
\]
We easily see that \( \omega_{12} \) is a Hermitian \( P \)-space operator that has \( d \) positive-or-zero eigenvalues, and the eigenstates \( \{ |\chi_i\rangle \} \) are mutually orthogonal.

In this formulation of UMOA, the construction of the transformed interaction \( \tilde{v}_{12} \) is reduced to solving the eigenvalue equation for a two-particle system, although \( h_1 \) contains the self-consistent potential \( u_1 \) and, therefore, \( \tilde{v}_{12} \) and \( u_1 \) have to be determined self-consistently. Once the eigenvalue problem is solved, we do not need other approximations any more in the calculation of \( \tilde{v}_{12} \).

3.3. The three-body-cluster term and its approximate treatment

We discuss the physical meaning of the cluster expansion of the transformed Hamiltonian. Before doing this, we shall refer to the meaning of the operator \( \omega_{12} \). The following equation will be useful, which is derived from Eq. (3.33),
\[
\langle \phi_k|\omega_{12}\omega_{12}|\phi_k\rangle = \langle \Phi_k|Q|\Phi_k\rangle.
\]
The above relation means that the expectation value of \( \omega_{12}\omega_{12} \) is equal to the \( Q \)-space overlap in the true eigenstate \( |\Phi_k\rangle \) that is often called the \( \text{wound integral} \) in the
G-matrix theory. In the nuclear many-body system the wound integral has been known to be small.\textsuperscript{57} We, therefore, may say that the norm of $\omega_{12}$ is small. If the matrix element of $\omega_{12}$ is small, the matrix element of $S_{12}$ should also be small, which may be clear from the relation between $S_{12}$ and $\omega_{12}$ in Eq. (3.26).

The cluster expansion in UMOA is essentially the expansion in order of $S_{12}$, or equivalently in order of $\omega_{12}$. The three-body-cluster term $\tilde{H}^{(3)}$ contains the three-body interactions that have at least one factor of $S_{12}$, and the four-body-cluster term $\tilde{H}^{(4)}$ contains the terms with two factors of $\{S_{12}\}$. Generally, the higher-cluster term in $\tilde{H}$ is of higher order in $S_{12}$. We here assume that the four-or-more-body-cluster terms are sufficiently small and negligible.

The three-body-cluster term $\tilde{H}^{(3)}$ is defined in Eq. (3.21). The expression of $\bar{v}_{123}$ in Eq. (3.13) is given in terms of the bare interaction $v_{12}$, so it is not always convenient, because $v_{12}$ is not well-behaved due to the presence of the strong short-range repulsion. We represent $\bar{v}_{123}$ in terms of the effective interaction $\tilde{v}_{12}$. To do this we use the Campbel-Hausdorf factorization formula

$$e^{\tilde{S}_{123}} = e^{\tilde{S}_{12}} e^{\tilde{S}_{123}} \quad (3.35)$$

with

$$S_{(123)} = S_{23} + S_{31} - \frac{1}{2} [S_{12}, S_{23} + S_{31}] - \frac{1}{12} [S_{12}, S_{23} + S_{31}, S_{(123)} + S_{12}] + \cdots , \quad (3.36)$$

where $S_{(123)}$ is given in Eq. (3.16). From the above relation $\bar{v}_{123}$ becomes

$$\bar{v}_{123} = \sum_{(123)} \left[ (e^{-\tilde{S}_{123}} \tilde{v}_{12} e^{\tilde{S}_{123}} - \bar{v}_{12}) + (e^{-\tilde{S}_{123}} (h_1 + h_2) e^{\tilde{S}_{123}} (h_1 + h_2)) \right]$$

$$+ (h_1 + h_2 + h_3) - e^{-\tilde{S}_{123}} (h_1 + h_2 + h_3) e^{\tilde{S}_{123}} , \quad (3.37)$$

where $\sum_{(123)}$ denotes the summation of exchange terms for a function $f(123)$

$$\sum_{(123)} f(123) = f(123) + f(231) + f(312) . \quad (3.38)$$

From Eqs. (3.36) and (3.37), $\bar{v}_{123}$ becomes

$$\bar{v}_{123} = \bar{v}_{12}^{(2)} + \bar{v}_{12}^{(3)} , \quad (3.39)$$

where

$$\bar{v}_{12}^{(2)} = \sum_{(123)} \left[ \bar{v}_{12}, S_{23} + S_{31} \right] - \frac{1}{2} [ \bar{v}_{12}, [S_{12}, S_{23} + S_{31}] ]$$

$$+ \frac{1}{12} [ [ \bar{v}_{12}, S_{23} + S_{31} ], S_{23} + S_{31} ] + \cdots \quad (3.40)$$

and

$$\bar{v}_{12}^{(3)} = \sum_{(123)} \left[ -\frac{1}{2} [[h_1 + h_2, S_{12}], S_{23} + S_{31}] + \frac{1}{6} [[[h_1 + h_2, S_{12}], S_{12}], S_{23} + S_{31}]$$

$$- \frac{1}{3} [[[h_1 + h_2, S_{12}], S_{23} + S_{31}], S_{(123)}] \right] + \cdots . \quad (3.41)$$
As seen in the above, the three-body-cluster term can be expanded into a series in order of \( \{S_u\} \). We introduce a certain truncation scheme and evaluate the three-body-cluster terms in an approximate way.

We next discuss the cancellation mechanism of \( \bar{u}_{12} \) on the r.h.s. of Eq. (3·21). This problem has already been discussed by Providencia and Shakin,\(^24\) although the definition of the average potential \( u_1 \) is slightly different from that in the present approach. The cancellation mechanism is stated as follows: When we write \( \bar{H}^{(3)} \) in the normal-product form with respect to particles and holes, \( \bar{H}^{(3)} \) is decomposed into a constant, one-, two- and three-body operators. In this expression of \( \bar{H}^{(3)} \), the terms which come from the interaction \( \bar{v}_{123} \) contain all the terms which cancel the second term \( \bar{u}_{12} \). We can prove this fact in an algebraic manner. It will be simple and clear to show that the three-body terms \( \sum \bar{v}_{ijk} \) in \( \bar{H} \) in Eq. (3·11) contain the terms

\[
\sum_k (e^{-S_1}(\bar{v}_{1k} + \bar{v}_{2k})e^{S_1} - (\bar{v}_{1k} + \bar{v}_{2k}))
\]  

(3·42)

which is derived from Eq. (3·13) for \( \bar{v}_{123} \). The summation over the third particle denoted by \( k \) brings about the average potential. We assume that the third particle \( k \) is in one of the orbits below the Fermi surface while two particles 1 and 2 move freely. We then may write

\[
\sum_k \bar{v}_{1k} = u_1 \quad \text{and} \quad \sum_k \bar{v}_{2k} = u_2
\]  

(3·43)

and, as a result, the term in Eq. (3·42) becomes

\[
e^{-S_1}(u_1 + u_2)e^{S_1} - (u_1 + u_2).
\]  

(3·44)

This agrees with \( \bar{u}_{12} \), and then \( \bar{u}_{12} \) in \( \bar{H}^{(3)} \) is canceled. The derivation of Eq. (3·43) was made somewhat in a classical manner, but it is rigorous even in the quantum-mechanical treatment.

3.4. Relation between the G-matrix theory and UMOA

We shall discuss the relation between the G-matrix theory and the present approach. In the nuclear many-body theory, the G-matrix theory, pioneered by Brueckner,\(^1\) has been used as an important ingredient in the calculation of nuclear properties.

The G-matrix is defined by the sum of the ladder diagrams expressed as

\[
G(\epsilon) = v + v \frac{Q}{e} v + v \frac{Q}{e} v \frac{Q}{e} v + \ldots
\]

\[
= v + v \frac{Q}{e - QvQ} v,
\]  

(3·45)

where \( \epsilon \) is the starting energy, and \( v \) the two-body interaction \( v_{12} \) and

\[
e = \epsilon - Q(h_1 + h_2)Q.
\]  

(3·46)

In a formal sense of the effective interaction theory, the \( P \)-space part of \( G(\epsilon) \) corresponds to the \( \bar{Q} \)-box that is the sum of the non-folded diagrams. It should be noted that we here consider only a two-particle system, not a many-body system. By
applying the effective interaction theory to this two-particle system, we can construct the effective interactions in terms of the G-matrix. From the general expansion formula in Eqs. (2.33)~(2.36) the non-Hermitian effective interaction \( R_{12} \) in Eq. (3.32) is given by

\[
R_{12} = \sum_{\alpha} G(\varepsilon_\alpha) P_\alpha + \sum_{\alpha \beta} G_1(\varepsilon_\alpha, \varepsilon_\beta) P_\alpha G(\varepsilon_\beta) P_\beta + \sum_{\alpha \beta \gamma} \left[ G_1(\varepsilon_\alpha, \varepsilon_\beta) P_\alpha G(\varepsilon_\beta) P_\beta G(\varepsilon_\gamma) P_\gamma \right] + \cdots, \tag{3.47}
\]

where \( \varepsilon_\alpha, \varepsilon_\beta \) and \( \varepsilon_\gamma \) are the two-particle starting energies, and \( G_1(\varepsilon_\alpha, \varepsilon_\beta) \) and \( G_2(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma) \) are the multi-energy G-matrices defined generally by

\[
G_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}) = (-1)^{n+1} P Q \frac{1}{\varepsilon_1 - Q H_{12} Q} \cdots \frac{1}{\varepsilon_{n+1} - Q H_{12} Q} Q v P, \tag{3.48}
\]

where \( Q H_{12} Q = Q (h_1 + h_2 + \nu) Q \). As was discussed in § 2, \( G_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}) \) can be given by calculating the difference quotients and/or derivatives of the single-energy G-matrix \( G(\varepsilon) \).

Applying the expansion formula in Eqs. (2.41)~(2.43) for the Hermitian effective interaction, \( \tilde{v}_{12} \) can be written in terms of the G-matrix as

\[
\tilde{v}_{12} = \frac{1}{2} \sum_{\alpha} \left[ G(\varepsilon_\alpha) P_\alpha + \text{h.c.} \right] + \frac{1}{8} \sum_{\alpha \beta} \left[ G_1(\varepsilon_\alpha, \varepsilon_\beta) P_\alpha G(\varepsilon_\beta) P_\beta + \text{h.c.} \right] + \frac{1}{8} \sum_{\alpha \beta \gamma} \left[ P_\alpha G_1(\varepsilon_\alpha, \varepsilon_\beta) P_\beta G(\varepsilon_\gamma) P_\gamma + \text{h.c.} \right] + \cdots. \tag{3.49}
\]

In the degenerate case that \( \varepsilon_\alpha, \varepsilon_\beta \) and \( \varepsilon_\gamma \) are equal to a constant, namely, \( \epsilon_0 \), the non-Hermitian and Hermitian effective interactions \( R_{12} \) and \( \tilde{v}_{12} \) are much simplified as

\[
R_{12} = G + G_1 G + G_1 G_1 G + G_2 G G + \cdots \tag{3.50}
\]

and

\[
\tilde{v}_{12} = G + \frac{1}{2} (G_1 G + \text{h.c.}) + \frac{3}{8} (G_1 G_1 G + \text{h.c.}) + \frac{1}{2} (G_2 G G + \text{h.c.}) + \frac{1}{4} G_1 G G + \cdots, \tag{3.51}
\]

where we have used \( G = G(\varepsilon) \), and \( G_1 \) and \( G_2 \) are defined generally by

\[
G_m = \frac{1}{m!} \left. \frac{d^m G(\varepsilon)}{d\varepsilon^m} \right|_{\varepsilon = \epsilon_0}. \tag{3.52}
\]

The graphical representation of \( G \) and \( R_{12} \) will be helpful for understanding their mutual relation. The diagrams of \( G \) and \( R_{12} \) are given in Fig. 1. The G-matrix consists of the ladder diagrams, and \( R_{12} \) contains the G-matrix (\( Q \)-box) and the associated folded diagrams. The arrow indicates the time constraint in the time-dependent perturbation theory or the location of the energy derivative in the time-
independent theory. The $G$-matrices with one and two arrows correspond to the first and second energy derivatives, respectively. The block $G'$ is defined by $G' = G - PuP$. The general diagram rules have been given by Kuo and his collaborators\textsuperscript{2,3)}

The inclusion of the folded diagrams brings about two properties of the effective interaction, namely, the decoupling between the $P$ and $Q$ spaces and the energy independence. The effective interaction $\bar{v}_{12}$ is just the Hermitian counterpart of $R_{12}$. Thus $\bar{v}_{12}$ has three properties of being decoupled between the $P$ and $Q$ spaces, $\epsilon$-independent and Hermitian.

The interactions, $G$, $R_{12}$ and $\bar{v}_{12}$ are determined dependently on the choice of the $P$ and $Q$ spaces. When we want to solve a closed-shell system, we assume, as usual, that the states of two particles in occupied and unoccupied orbits are included, respectively in the $P$ and $Q$ spaces. In such a case, $R_{12}$ satisfies the decoupling condition in (a) of Fig. 2. Due to the non-Hermiticity, the conjugate term of $R_{12}$ does not satisfy the decoupling condition. On the other hand, the Hermitian form $\bar{v}_{12}$ satisfies both the decoupling conditions as shown in (b) of Fig. 2. If we use $R_{12}$ or $\bar{v}_{12}$, two-particle excitations from the core state are forbidden. As a result, the diagrams to be evaluated are much reduced in the perturbation expansion based on $R_{12}$ or $\bar{v}_{12}$. This decoupling property should be compared with the HF condition, that is, the decoupling of the one-body part of the Hamiltonian including the kinetic energy and the self-consistent one-body potential, as shown in (c) of Fig. 2.

In general we may construct three kinds of many-body theories based on the interactions, $G$, $R_{12}$ and $\bar{v}_{12}$. The many-body theory based on the $G$-matrix has been
well established. It has been proved recently \cite{68} that the use of the non-Hermitian effective interaction $R_{12}$ leads to the coupled-cluster method (CCM) which has been developed by Coester \cite{59}, Kümmel, Zabolitzky and Lührmann \cite{60} in nuclear physics and by Sinanoglu, \cite{61} Cizek, \cite{62} Arponen \cite{63} and Bartlett \cite{64} in quantum chemistry. The many-body theories based on $G$, $R_{12}$ and $\tilde{v}_{12}$ may be summarized as in Table I.

\section*{§ 4. Calculation procedure}

4.1. Three-step method for the calculation of the Hermitian effective interaction

The effective interaction $\tilde{v}_{12}$ is determined dependently on the choice of the $P$ and $Q$ spaces. The theory formulated in the preceding section can be applied to a problem with any $P$ and $Q$ spaces. However, in UMOA we intend to obtain an effective interaction without the interaction inducing two-particle-two-hole (2P-2h) excitations. For this purpose we divide a space of two-body states $|a\beta\rangle$ into three subspaces according to whether $a$ and $\beta$ are occupied or not in the unperturbed ground state, i.e.,

\begin{align}
|a\beta\rangle \in & \begin{cases} 
D_{ep} & \text{if } a \text{ and } \beta \text{ are unoccupied}, \\
D_{ph} & \text{if one of } a \text{ and } \beta \text{ is occupied}, \\
D_{2h} & \text{if } a \text{ and } \beta \text{ are occupied}.
\end{cases} 
\end{align}

The graphical representation of these subspaces is given in Fig. 3. The condition that $\tilde{v}_{12}$ should not contain the 2p-2h excitation-inducing terms is equivalent to that $\tilde{v}_{12}$ should be decoupled between the $D_{ep}$ and $D_{2h}$ spaces.

In principle, the occupied states should be the eigenstates of $h_1 = h + u_1$ with the self-consistent potential $u_1$ as in the usual HF calculation. However, in the present calculation, so as to avoid complication, we introduce fixed basis states, the harmonic oscillator (h.o.) states. If we define the projection operators $P$ and $Q$ with the h.o. basis, the term $(h_1 + h_2)$ in Eq. (3·12) is generally not decoupled and the assumption in Eq. (3·22) made in § 3 is not satisfied. However, it should be noted that, even though

\begin{table}
\centering
\caption{Summary of the many-body theories based on the $G$-matrix, $R_{12}$ and $\tilde{v}_{12}$.}
\begin{tabular}{|c|c|c|c|}
\hline
many-body theory & $G$-matrix theory & UMOA & CCM \\
\hline
basic element & $G =$ sum of ladder diagrams & $R_{12} =$ folded diagrams & $\tilde{v}_{12} =$ Hermitian counterpart of $R_{12}$ \\
\hline
Hermiticity & non-Hermitian \footnote{The $G$-matrix is non-Hermitian if the $P$ space is non-degenerate.} & non-Hermitian & Hermitian \\
\hline
$E$-dependence & $E$-dependent & $E$-independent & $E$-independent \\
\hline
decoupling property & No & QR_{12}P = 0 & P\tilde{v}_{12}Q = 0 \\
& & PR_{12}Q = 0 & Q\tilde{v}_{12}P = 0 \\
\hline
self-consistency in the single-particle energies & generally impossible & possible & possible \\
\hline
\end{tabular}
\end{table}
we neglect the term \( P(h_1 + h_2)Q \), we can construct the effective interaction decoupled between the spaces \( D_{2h} \) and \( D_{2p} \) in Eq. (4·1). The reason is as follows: By definition of \( D_{2h} \) and \( D_{2p} \), \( h_1 + h_2 \) is, in itself, decoupled as \( D_{2h}(h_1 + h_2)D_{2p} = 0 \), because \( h_1 \) and \( h_2 \) are one-body operators, and the spaces \( D_{2h} \) and \( D_{2p} \) do not have a common one-body state. In other words, the term of \( P(h_1 + h_2)Q \) does not contain the term which violates the decoupling between the spaces \( D_{2h} \) and \( D_{2p} \) for any \( P \) and \( Q \) spaces.

Before solving the decoupling equation for \( \tilde{\omega}_{12} \), we should note the following two points:

(a) **How to determine the self-consistent potential.**

The self-consistent one-body potential \( u_1 \) has been defined in Eq. (3·18). In principle the present approach has no difficulty for determining \( u_1 \) for any one-body states. However, in the nuclear many-body problem, we have to treat a two-body potential with a short-range repulsive core. The effective interaction theory is designed for obtaining the eigenvalues for the model-space states. Therefore, the effective interaction \( \tilde{\omega}_{12} \) behaves well in the \( P \) space, but it is not always so in the \( Q \) space. For this reason it is generally difficult to determine \( u_1 \) for high-momentum states belonging to the \( Q \) space. It is noted that if we consider an average potential with a finite width and depth such as the Wood-Saxon potential, the one-body potential \( u_1 \) for extra high-momentum states should be small compared with the kinetic energy. We thus assume that the potential \( u_1 \) becomes zero for extra high-momentum states.

(b) **How to choose the \( P \) and \( Q \) spaces.**

If a state in the \( P \) space mixes strongly with the \( Q \)-space states in the eigenstate \( |\Phi_k\rangle \) in Eq. (3·28), the matrix element of \( \omega_{12} \) (or \( S_{12} \)) becomes large and, as a result, the higher-order-cluster terms in \( \tilde{H} \) in Eq. (3·17) have large contributions. In general, the mixing between the \( P \)- and \( Q \)-space states takes place when the \( P \) and \( Q \) spaces are quasi-degenerate in energy. For this reason we should choose the \( P \) and \( Q \) spaces that are well separated in energy. With due regard to points (a) and (b), we solve the problems according to the following three-step procedure.

(1) **First-step decoupling.**

Let us write a two-particle state as a product of the h.o. wave functions with the usual notations as \( |n_{a1}l_{a1}m_a, n_{b1}l_{b1}m_b\rangle \). We define the \( P_1 \) and \( Q_1 \) spaces with a number \( \rho_1 \) as

\[
\begin{align*}
\rho_F & \left| \begin{array}{c}
D_{2h} \\
D_{2p}
\end{array} \right|
\end{align*}
\]
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\[ |\alpha\beta\rangle \in \begin{cases} P_1 & \text{if } 2n_a + l_a + 2n_b + l_b \leq \rho_1, \\ Q_1 & \text{otherwise}. \end{cases} \quad (4\cdot2) \]

The graphical representation of the \( P_1 \) and \( Q_1 \) spaces is given in Fig. 4. The number \( \rho_1 \) is supposed to be sufficiently large compared with the Fermi level \( \rho_F \).

The definitions of the \( P_1 \) and \( Q_1 \) spaces allow to divide the two-body states written in relative and center-of-mass (c.m.) coordinates into the \( P_1 \) and \( Q_1 \) spaces as

\[ |n\ell SJ, NL\rangle \in \begin{cases} P_1 & \text{if } 2n + \ell + 2N + L \leq \rho_1, \\ Q_1 & \text{otherwise}. \end{cases} \quad (4\cdot3) \]

where \((n, \ell)\) and \((N, L)\) are the h.o. quantum numbers of the relative and c.m. states, respectively. The quantum numbers, \( S, T \) and \( J \) are, respectively, spin, isospin and angular momentum defined by \( J = l + S \).

With the \( P_1 \) and \( Q_1 \) spaces, we solve the decoupling equation under the following assumptions.

(i) The self-consistent one-body potential introduced in this step, which is denoted by \( u_{i_1}^{(1)} \), is calculated according to

\[ \langle \alpha|u_{i_1}^{(1)}|\beta\rangle = \sum_{\lambda \leq \rho_1} \langle \alpha\lambda|P_1 \bar{v}_i^{(1)} P_1|\beta\lambda\rangle \quad (4\cdot4) \]

which implies that

\[ Q_1(u_{i_1}^{(1)} + u_{i_2}^{(1)})Q_1 = 0, \quad (4\cdot5) \]

where \( \bar{v}_i^{(1)} \) is the effective interaction to be determined in the first step.

(ii) We assume that \( \bar{v}_i^{(1)} \) is diagonal in \( K = 2N + L \). This assumption can be interpreted as that the effective interaction \( \bar{v}_i^{(1)} \) depends on the average distance from the center of the nucleus to the c.m. position of two relevant nucleons. This assumption has been confirmed to be acceptable in the G-matrix calculation.\(^{65}\)

(iii) Since the \( Q_1 \) space has an overlap with the \( D_{ph} \) space in Fig. 3, the Pauli-principle violation will take place in treating the two-body correlations. However, it has been established\(^{66}\) that the Pauli principle can well be taken into account in a finite-dimensional truncated space if the model space contains sufficiently high-momentum two-body states belonging to the \( D_{ph} \) space.

(2) Second-step decoupling.

The \( P_1 \) space considered in the first step should be taken as large as possible, but it is too large as a low-momentum model space. We further reduce the model space to a smaller one. Introducing an integer \( \rho_2 \), we divide the \( P_1 \) space into the \( P_2 \) and \( Q_2 \).
spaces according to
\[ |nlSTJ, NL\rangle \in \begin{cases} P_2 & \text{if } 2n + l + 2N + L \leq \rho_2, \\
Q_2 & \text{if } \rho_2 < 2n + l + 2N + L \leq \rho_1. \end{cases} \quad (4.6) \]

We solve the decoupling equation between the \( P_2 \) and \( Q_2 \) spaces with the interaction \( P_1 \tilde{v}\left[\frac{P_2}{2}\right]P_1 \). Let \( Q_X \) be the product space of the \( Q_2 \) and \( D_{ph} \) spaces as shown in Fig. 5. Because of the Pauli principle, \( Q_X \tilde{v}\left[\frac{P_2}{2}\right]Q_X \) should be removed in the calculation. However, the treatment of \( Q_X \) with the relative and c.m. states is generally very difficult. We here employ the angle-average approximation of Wong \(^{66}\) and Sauer \(^{67}\), which has been used in the \( G \)-matrix calculations.

The self-consistent average potential in this step denoted by \( u_1^{(2)} \), is determined from
\[ \langle \alpha|u_1^{(2)}|\beta\rangle = \begin{cases} \sum_{\lambda} \langle \alpha|\tilde{v}\left[\frac{P_2}{2}\right]|\beta\rangle & \text{if } |\alpha\rangle \in P_2, \ |\beta\rangle \in P_2, \\
\sum_{\lambda} \langle \alpha|\tilde{v}|\beta\rangle & \text{otherwise}. \end{cases} \quad (4.7) \]

In the above we have used the \( P_2 \)-space effective interaction \( \tilde{v}\left[\frac{P_2}{2}\right] \) for calculating \( u_1^{(2)} \) for low-momentum states, and the potential \( u_1^{(2)} \) for high-momentum states is assumed to be the same as in the first step. Further details of this treatment of \( u_1^{(2)} \) have been discussed in Ref. 27. With the definition of \( u_1^{(2)} \) we solve the decoupling equation between the \( P_2 \) and \( Q_2 \) spaces, and obtain the \( P_2 \)-space effective interaction \( P_2 \tilde{v}\left[\frac{P_2}{2}\right]P_2 \).

(3) Third-step decoupling.

We further divide the \( P_3 \) space into three subspaces as
\[ |\alpha\beta\rangle \in P_3 \quad \text{if} \quad |\alpha\beta\rangle \in D_{zh}. \]

![Fig. 5. Definitions of the spaces \( P_3, Q_2 \) and \( Q_X \) with the boundary number \( \rho_2 \) given in Eq. (4.6).](https://example.com/fig5.png)

![Fig. 6. Definitions of the spaces \( P_3, P_X \) and \( Q_3 \) given in Eq. (4.8).](https://example.com/fig6.png)
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\[ |a\beta\rangle \in P_x \text{ if } |a\beta\rangle \in D_{ph}, \]
\[ |a\beta\rangle \in Q_3 \text{ if } |a\beta\rangle \in D_{2p}. \]  

(4.8)

The graphical representation of the \( P_x, Q_3 \) and \( P_x \) spaces is given in Fig. 6. As seen from Figs. 3 and 6, the \( P_x \) and \( Q_3 \) spaces are the subspaces of \( D_{2h} \) and \( D_{2p} \), respectively.

In this step we solve the decoupling equation with the basis states \( \{ |a\beta\rangle \} \). The average potential \( u_1^{(3)} \) is calculated from

\[ \langle a|u_1^{(3)}|\beta\rangle = \begin{cases} 
\sum_{\lambda < \rho} \langle a\lambda|\bar{v}^{(3)}|\lambda\beta\rangle & \text{if } |a\lambda\rangle \in P_3, |\lambda\beta\rangle \in P_3, \\
\sum_{\lambda < \rho} \langle a\lambda|\bar{v}^{(3)}|\lambda\beta\rangle & \text{otherwise},
\end{cases} \]

(4.9)

where \( \bar{v}^{(3)} \) is the effective interaction to be determined in this step. In the same way as in the second step, we use \( \bar{v}^{(3)} \) for the matrix element between the \( P_x \)-space states. Removing the \( P_x \) space from the space of two-body states with consideration for the Pauli principle, the effective interaction \( \bar{v}^{(3)} \) is determined by solving the following eigenvalue equation in the \( P_3 + Q_3 \) space,

\[ (P_3 + Q_3)(h_1 + u_1^{(3)} + h_2 + u_2^{(3)} + \bar{v}^{(3)})(P_3 + Q_3)|\Phi_h\rangle = E_h|\Phi_h\rangle. \]

(4.10)

For solving the above equation we need no assumption if the number \( \rho_2 \) is properly chosen. By applying the general theory for calculating the effective interaction, we finally obtain the interaction \( \bar{v}^{(3)} \) that has the decoupling property \( Q_3 \bar{v}^{(3)} P_3 = P_3 \bar{v}^{(3)} Q_3 = 0 \). It is now clear that \( \bar{v}^{(3)} \) has no interaction that induces \( 2p - 2h \) excitations. Further discussion on the calculation procedure has been made in Refs. 27) and 28).

4.2. Three-body-cluster effects and some correction terms

As has been discussed in § 3, the transformed Hamiltonian \( \tilde{H} \) in Eq. (3.17) is written in an expansion form in powers of the correlation operator \( S_{12} \). Assuming that \( S_{12} \) is small, we evaluate the three-body cluster (TBC) terms by taking their leading contribution into account. We here calculate the ground-state energy and the one-body energies of the occupied orbits.

The leading contribution of the TBC terms to the ground-state energy is given by

\[ \Delta E_0^{(3)} = \frac{1}{3!} \sum_{\lambda, \mu, \nu, \rho} \langle \lambda \mu \nu | \bar{v}^{(3)} + \bar{v}^{(3)} | \lambda \mu \nu \rangle - \frac{1}{2!} \sum_{\lambda, \mu} \langle \lambda \mu | \bar{u}_{12} | \lambda \mu \rangle, \]

(4.11)

where \( \bar{v}^{(3)} \) and \( \bar{v}^{(3)} \) are given in Eqs. (3.40) and (3.41), respectively, and \( \bar{u}_{12} \) in Eq. (3.14). As discussed in § 3.3, we note the following facts:

(i) The contributions of \( \bar{u}_{12} \) to \( \Delta E_0^{(3)} \) are exactly canceled.
(ii) If \( \bar{u}_{12} \) is decoupled between the \( D_{2h} \) and \( D_{2p} \) spaces, the contribution of \( \bar{v}^{(3)} \) of first-order in \( S_{12} \) vanishes and \( \Delta E_0^{(3)} \) begins with the terms of second-order in \( S_{12} \).
(iii) The contribution of \( \bar{v}^{(3)} \) vanishes up to third-order terms in \( S_{12} \). Therefore, \( \bar{v}^{(3)} \) may be assumed to have smaller contributions than \( \bar{v}^{(3)} \). From the above reason, the leading terms in \( \Delta E_0^{(3)} \) are given by

\[ \Delta E_0^{(3)} = T_a + T_b, \]

(4.12)
where

\[ T_a = -\frac{1}{4} \sum_{\lambda,\mu,\nu < \rho, \sigma} \sum_{\alpha, \beta, \gamma} \bar{v}_{\lambda\mu\nu\rho} S_{\alpha\beta\alpha} S_{\gamma\beta\mu} , \]  
\[ T_b = -\sum_{\lambda,\mu,\nu < \rho, \sigma} \sum_{\alpha, \beta, \gamma} \bar{v}_{\lambda\mu\nu\rho} S_{\alpha\beta\alpha} S_{\gamma\beta\mu} . \]  

(4.13)  

(4.14)

Here we have used the notations

\[ \bar{v}_{ab\rho}\alpha = \langle \alpha | v_{12} | \gamma \rangle , \]  
\[ S_{ab\rho}\alpha = \langle \alpha | S_{12} | \gamma \rangle . \]  

(4.15)  

(4.16)

In the actual calculation, we shall use \( \bar{v}_{12} \) for \( \bar{v}_{12} \) which is obtained in the third-step decoupling procedure discussed in the preceding subsection.

Another contribution to the ground-state energy comes from the non-diagonal terms in the one-body part in \( \tilde{H} \). If we perform the Hartree-Fock calculation for determining the basis states with the interaction \( \bar{v}_{12} \), the one-body term \( h_1 \) is always diagonal in these basis states. However, in the present calculation we use fixed basis states, and therefore \( h_1 \) is not diagonalized. By taking the diagonal part in \( h_1 \) as the unperturbed energies, we evaluate all the perturbative corrections up to second order. The leading contribution to \( E_0 \), denoted by \( \Delta E_0^{(0)} \), is expressed diagrammatically as in (b) of Fig. 7. Finally the ground-state energy is given by

\[ E_0 = \sum_{\lambda < \rho, \tau} \langle \lambda | t_1 | \lambda \rangle + \frac{1}{2} \sum_{\lambda, \mu, \nu < \rho, \sigma} \langle \lambda | \bar{v}_{12} \rangle | \lambda \mu \rangle + \Delta E_0^{(0)} + \Delta E_0^{(3)} - T_{cm} . \]  

(4.17)

The first term on the r.h.s. is the kinetic energy, and the second and third terms correspond to diagrams (a) and (b) in Fig. 7, respectively. The term \( \Delta E_0^{(3)} \) is given in Eq. (4.12). The last term \( T_{cm} \) is the center-of-mass kinetic energy of the nucleus, which we take \( (3/4) \hbar \omega \).

The contribution of the TBC terms to the one-body energy of an occupied state \( | \lambda \rangle \) is given by

\[ \Delta E_\lambda^{(3)} = \frac{1}{2} \sum_{\mu, \nu < \rho, \tau} \langle \lambda \mu \nu | \bar{v}_{12} \rangle | \lambda \mu \nu \rangle - \sum_{\mu < \rho, \tau} \langle \lambda | \bar{v}_{12} | \lambda \mu \rangle . \]  

(4.18)

We follow the same procedure as in \( \Delta E_0^{(3)} \) and the leading contributions become

\[ \Delta E_\lambda^{(3)} = T_c + T_d + T_e + T_f + T_g , \]  

(4.19)

where

\[ T_c = \frac{3}{4} \sum_{\mu, \nu < \rho, \tau} \sum_{\alpha, \beta > \rho} \bar{v}_{\lambda \mu \nu \rho} S_{\alpha\beta\alpha} S_{\gamma\beta\mu} , \]  

\[ \times \bar{v}_{\lambda \mu \nu \rho} S_{\alpha\beta\alpha} S_{\gamma\beta\mu} , \]  

(4.20)

Fig. 7. Graphical representation of the diagrams evaluated for the ground-state energy. The effective interaction \( \bar{v}_{12} \) is represented by a wavy-line vertex. The \( \Delta E_0^{(3)} \) denotes the TBC terms in the ground-state energy. The \( \times \) vertices with dotted lines mean the one-body Hamiltonian \( h_1 (= t_1 + u_1) \).
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Fig. 8. Calculated diagrams for the occupied single-particle state. The effective interaction $\tilde{v}_{12}$ is represented by a wavy-line vertex. The $\times$ vertex with a dotted line denotes the one-body Hamiltonian $h_1$.

There are other corrections to the one-body energy. We evaluate them perturbatively. We take the diagrams given in Fig. 8 into account. Diagrams (b) and (c) come from the non-diagonal part in the one-body term $h_1$. Diagram (d) is a kind of core-polarization correction and always appears as one of the leading corrections. The other second-order diagrams do not appear because of the decoupling property of $\tilde{v}_{12}$.

Finally the one-body energy becomes

$$E_i = \langle \lambda | t_1 | \lambda \rangle + \sum_{\mu, \nu} \langle \lambda \mu | \tilde{v}_{12}^{(3)} | \lambda \mu \rangle + \Delta E_i^{(p)}(b) + \Delta E_i^{(p)}(c) + \Delta E_i^{(p)}(d) + \Delta E_i^{(3)},$$

where the first term is the kinetic energy, the second the potential energy of diagram (a) in Fig. 8. The terms $\Delta E_i^{(p)}(b)$, $\Delta E_i^{(p)}(c)$ and $\Delta E_i^{(p)}(d)$ are the perturbative corrections corresponding to diagrams (b), (c) and (d) in Fig. 8, respectively. The last term $\Delta E_i^{(3)}$ is the three-body-cluster contribution given in Eq. (4·19).

§ 5. Application of UMOA to $^{16}\text{O}$

5.1. Approximation procedure

We apply UMOA to the calculation of the ground-state properties of $^{16}\text{O}$, the one-body energies of occupied states and the effective interactions of the $0s0p$-shell states by following the calculation procedures discussed in the preceding section. We employ the Reid soft-core (RSC), Super-soft-core (SSC), Paris, and Bonn potentials as the original NN interaction $v_{12}$ in Eq. (3·1). We investigate the depen-
The dependence of the calculated results on the adopted NN potentials to understand some features of the above-mentioned NN potentials in the nuclear structure calculation. As is well known, these potentials have almost the same compilation of central, tensor, spin-orbit components, etc., and the similar radial dependence at distance of more than 2 fm, but have radial dependence quite different from each other in the short-range region of less than 1 fm. The strength of the tensor-force component is also different from each other. In recent years attention has been called to the dependence of the nuclear structure calculation on the tensor-force strength.

The use of theoretically derived potentials, for example, the Bonn potential, is much of interest for various reasons. This NN interaction has been known to fit the most recent two-nucleon scattering data and to yield predictions for three-nucleon systems that are much better than the results given with the other potentials. It has also been applied to the calculation of the ground-state properties of $^{16}$O in Refs. 73), 75), 78) and the energy spectrum of two-valence-particle systems such as $^{18}$O and $^{18}$F in Refs. 16)−18).

As is well known, there still remains ambiguity in determining the NN force from the study of two-nucleon systems only. The nuclear force will finally be established in solving three-or-more-nucleon systems including nuclear matter. Therefore, it will be necessary and important to know various theoretical predictions for many-nucleon systems with various nuclear forces. Accumulation of such works will provide us with deeper understanding of the nuclear force.

We discuss some important details of the calculation procedure. It is worth noting that due to the effect of the strong repulsive core in the NN force the rigorous solution to the two-body eigenvalue equation (3.28) cannot be obtained by diagonalizing the two-body Hamiltonian with use of only the finite-dimensional h.o. basis. We have observed that it is tremendously difficult to expand the eigenfunction of the bound state of deuteron into a set of the h.o. wave functions if we truncate the h.o. basis to be finite dimensional. In order to treat the two-body correlations in the short-range region, high accuracy is necessary even though the potentials are regularized at the origin. For this purpose we employ a set of Gaussian functions in addition to the h.o. basis to describe the short-range correlations. The space of the two-body states has been chosen so as to be sufficiently large for reproducing the bound-state data of deuterons. Technical details have been given in Refs. 27) and 28).

The numerical calculation for solving the decoupling equations in procedures in § 4 is carried out by giving a set of parameters $\hbar\omega$, $\rho_1$, $\rho_2$ and $l_{\text{max}}$. The $\hbar\omega$ is the unit of the h.o. energy. The $\rho_1$ and $\rho_2$ are the numbers specifying the spaces $(P_1, Q_1)$ and $(P_2, Q_2)$, respectively, as shown in Figs. 4 and 5. The $l_{\text{max}}$ is the maximum value of the angular momentum $l$ of the partial waves taken into consideration in the calculation. As has been discussed in the previous works, the contributions of the partial waves with $l \geq 5$ are quite small. We therefore make truncation $l_{\text{max}}=4$. As for the parameter $\hbar\omega$, we may say that the calculation in UMOA should not, in principle, depend on the choice of $\hbar\omega$, if we make no approximation. However, since we make some approximations, the result might depend on $\hbar\omega$. A reasonable way of choosing $\hbar\omega$ may be to determine it so that the ground-state energy takes the minimum value. As has been observed in Ref. 27), the dependence of the ground-state
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energy on $\hbar \omega$ is rather weak, but the energy minimum is realized with $\hbar \omega$ nearly equal to 14.0 MeV in $^{16}$O. Therefore, in the present calculation, we fix $\hbar \omega$ as 14.0 MeV.

The effective interaction $\tilde{v}^{[1]}$ given in the first-step decoupling procedure depends on the number $\rho_1$ in Fig. 4. Since $\rho_1$ determines the space $P_1$ in which the one-body field is supposed to be active and, of course, should be taken to be sufficiently large, it is important to examine the convergence of the calculated result as $\rho_1$ increases. We have observed that the almost convergent results are obtained if we take $\rho_1$ to be 20.\textsuperscript{27) The dependence of the effective interactions on $\rho_1$ is shown in Table II.

The parameter $\rho_2$ determines the spaces $P_3$, $Q_3$ and $P_X$ as shown in Fig. 6. The $\rho_2$ should also be taken to be sufficiently large, because in the present calculation the TBC effects and the perturbation corrections are evaluated in the spaces $P_3$, $Q_3$ and $P_X$. The dependence of the results on $\rho_2$ has been examined with special care. We have observed that the results change monotonically as $\rho_2$ does and almost convergent results have been obtained for $\rho_2=10$.\textsuperscript{27,28) The $\rho_2$ dependence of the results is shown in Table III. As shown in the table, the total energies show the convergence at $\rho_2=10$ within uncertainty of about 2%, but the perturbative corrections, $\Delta E^{(1)}$ and the TBC effect $\Delta E^{(3)}$ have still sizable dependencies on $\rho_2$. It will be of interest to know the extreme value for $\rho_2 \to \infty$. As was shown in Refs. 27) and 28), the following extrapolation formula is satisfied quite well, i.e.,

$$E(\rho_2) = E_\infty + C e^{-\gamma \rho_2^2},$$ \hspace{1cm} (5.1)

where $E_\infty$, $C$ and $\gamma$ are certain constants. We have seen from the $\chi^2$-fitting that the parameter $\gamma$ is almost constant in each group of the first-order potential energies of the ground and one-body states, the second-order perturbative corrections, the TBC terms and the correction terms in the charge radius. The values of $\gamma$ obtained with the Bonn-B potential are given in Table IV. The extreme values for $\rho_2 \to \infty$ are also given in Table III. The functions, $E(\rho_2)$, are shown for the binding energy per nucleon and the single-particle energies including only the kinetic energy (KE) and the potential energy (PE) in Fig. 9. The same functions for the perturbative corrections

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<tr>
<td>23</td>
<td>12</td>
<td>-2.3994</td>
<td>-2.4063</td>
<td>-2.4095</td>
</tr>
<tr>
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<td>01</td>
<td>-1.8770</td>
<td>-1.9093</td>
<td>-1.9141</td>
</tr>
<tr>
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<td>03</td>
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<td>-4.7674</td>
</tr>
<tr>
<td>33</td>
<td>10</td>
<td>-3.2778</td>
<td>-3.2922</td>
<td>-3.3023</td>
</tr>
<tr>
<td>33</td>
<td>12</td>
<td>-1.2444</td>
<td>-1.2494</td>
<td>-1.2516</td>
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Table II. The $\rho_1$ dependence of diagonal hole-hole matrix elements $\langle ab TJ | \tilde{v}^{[1]} | ab TJ \rangle$ for the Bonn-B potential in units of MeV, where $a$ and $b$ denote the single-particle orbits labeled as $1=0s_u2$, $2=0p_12$ and $3=0psf2$. The number $\rho_1$ is fixed as 10.
Table III. The \( \rho_z \) dependence of the ground-state (g.s.) and single-particle energies for the Bonn-B potential. Calculated results are given for a set of parameters, \( \rho_z=20 \) and \( h\omega=14.0 \) MeV. Here, KE and PE stand for the kinetic energy and the first-order potential energy, respectively. The values of \( \Delta E^{(p)} \) and \( \Delta E^{(3)} \) represent the sum of the second-order perturbative corrections and the three-body cluster contributions, respectively. The Coulomb energies are included. All energies are in units of MeV.

\[
\begin{array}{cccccc}
\hline
\rho_z & \text{g.s.} & \text{KE} & \text{PE} & \Delta E^{(p)} & \Delta E^{(3)} & \text{Total} \\
\hline
6 & 241.50 & -333.86 & -10.07 & -5.82 & -108.25 & 10.50 \\
8 & 241.50 & -340.04 & -7.64 & -5.71 & -111.89 & 10.50 \\
10 & 241.50 & -341.83 & -5.80 & -5.33 & -111.46 & 10.50 \\
\infty & 241.50 & -342.22 & -3.84 & -5.28 & -109.84 & 10.50 \\
\hline
\end{array}
\]

\[\Delta E^{(p)}\] are shown in Fig. 10.

In the present calculation all the results include the Coulomb effect. The magnitude of the Coulomb effect has been estimated to be about 13 MeV for the ground-state energy.

5.2. Ground-state energy and charge radius

We discuss the saturation property of \(^{16}\text{O}\). The results for the binding energy are given in Table V and Figs. 11 and 12 that include the kinetic energy, the first-order potential energy, the second-order perturbative corrections \( \Delta E^{(p)} \) and the TBC contributions \( \Delta E^{(3)} \) with various NN potentials.

The various contributions to the ground-state energy are given in Table VI for the Bonn-B potential. From the calculated results in Table VI we may point out the following features, which are almost common to the other potentials.

Table IV. The values of the parameter \( \gamma \) obtained from the \( \chi^2 \)-fitting in Eq. (5.1) for the Bonn-B potential. The \( \chi^2 \)-fitting is made in each group of (A) first-order potential energies of the ground and one-body states, (B) the second-order perturbative corrections, (C) the TBC terms, and (D) the correction terms in the charge radius.

\[
\begin{array}{cccc}
\hline
\text{Group} & \text{(A)} & \text{(B)} & \text{(C)} & \text{(D)} \\
\hline
\gamma & 0.048 & 0.018 & 0.024 & 0.010 \\
\hline
\end{array}
\]
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Fig. 9. The $\rho_2$ dependence of the first-order potential terms of the binding energy per nucleon and of the single-particle energies of $0s_{1/2}$, $0p_{1/2}$ and $0p_{3/2}$ states. The abscissa is measured in units of $\exp(-\gamma \rho_2^2)$ with $\gamma = 0.048$. The extrapolated value $E(\rho_2 \to \infty)$ is given by a crossing point of the graph of $E(\rho_2)$ and the vertical line. The parameters $\rho_2 = 20$ and $\hbar \omega = 14.0$ MeV are used.

Fig. 10. The $\rho_2$ dependence of the perturbative corrections $\Delta E_{\rho_2}^{(a)}$ for the ground-state energy and $\Delta E_{\rho_2}^{(b)} = \Delta E_{\rho_2}^{(a)}(b) + \Delta E_{\rho_2}^{(a)}(c) + \Delta E_{\rho_2}^{(a)}(d)$ for the single-particle energies. The parameter $\gamma$ is taken to be 0.018. The other notations are the same as in Fig. 9.

Fig. 11. Binding energies per nucleon and inverse charge radii of $^{16}$O for different potentials. The solid circles are the predictions in UMOA with the parameters $\rho_2 = 20$, $\rho_2 \to \infty$ and $\hbar \omega = 14.0$ MeV. The Paris, SSC, RSC, A and B denote the Paris, Super soft-core, Reid soft-core, Bonn-A and Bonn-B potentials, respectively. The experimental value is given by the solid circle with error bar. The results of Schmid et al. are given with the cross symbols where A, B and C denote the relativistic momentum space Bonn potentials.

Fig. 12. Binding energy per nucleon and charge radii of $^{16}$O for various NN potentials. The solid circles are the predictions in UMOA. The cross symbols are the predictions in CCM. The FBHF(2), FBHF(3) and FBHF(4) denote the two-, three- and four-body-cluster approximations in CCM. The HJ means the Hamada-Johnston potential. The other notations are given in Fig. 11.
Table V. The single-particle energies (S.P.E.), the $l$-$s$ splitting ($\Delta E_{ls}$) between the $0p_{1/2}$ and $0p_{3/2}$ states, the binding energy per nucleon ($E/A$) and the charge radius ($r_c$) of $^{16}\text{O}$ for various potentials. The parameters are taken as $\rho_1=20$ and $\rho_2=\infty$. The other parameters are the same as in Table III. The experimental single-particle energies are the average values of $^{16}\text{O}$ and $^{15}\text{N}$. All energies are in units of MeV.

<table>
<thead>
<tr>
<th></th>
<th>Bonn-A</th>
<th>Bonn-B</th>
<th>Paris</th>
<th>RSC</th>
<th>Exp</th>
</tr>
</thead>
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<tr>
<td>S.P.E.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0s_{1/2}$</td>
<td>-41.23</td>
<td>-38.16</td>
<td>-37.74</td>
<td>-36.83</td>
<td>-40±8</td>
</tr>
<tr>
<td>$0p_{1/2}$</td>
<td>-16.16</td>
<td>-14.48</td>
<td>-14.45</td>
<td>-14.39</td>
<td>-13.9</td>
</tr>
<tr>
<td>$0p_{3/2}$</td>
<td>-21.92</td>
<td>-19.86</td>
<td>-19.82</td>
<td>-19.47</td>
<td>-20.1</td>
</tr>
<tr>
<td>$\Delta E_{ls}$</td>
<td>5.76</td>
<td>5.38</td>
<td>5.51</td>
<td>5.37</td>
<td>5.08</td>
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<tr>
<td>$E/A$</td>
<td>7.68</td>
<td>6.86</td>
<td>6.61</td>
<td>6.77</td>
<td>6.48</td>
</tr>
<tr>
<td>$r_c$[fm]</td>
<td>2.56</td>
<td>2.62</td>
<td>2.66</td>
<td>2.64</td>
<td>2.60</td>
</tr>
</tbody>
</table>

(a) The second-order perturbative corrections yield small effects on the ground-state energy, which are estimated to be about 1% of the first-order potential energy.

(b) The contributions from the TBC terms consist of two-terms, $T_a$ and $T_b$, as given in Eq. (4·12). The results show that due to the cancellation of these terms the total contribution becomes small. The net effect of the TBC terms is estimated to be about 1.5% of the first-order potential energy. This result suggests good convergence of the cluster expansion of the transformed Hamiltonian in Eq. (3·17).

Another feature of the results is that the calculated binding energies depend weakly on the potentials used in the calculation. This feature is quite different from that of the results obtained in CCM and the Brueckner-Hartree-Fock (BHF) method.

The results for the charge radius are given in Table V for various NN potentials. Details of the calculation procedure for the charge radius has been given in Ref. 28. The relation between the binding energy per nucleon and the (inverse) charge radius are shown in Figs. 11 and 12 for various potentials. For the sake of comparison we show the results of the BHF and CCM calculations in Figs. 11 and 12, respectively. We see from these figures that the predictions given in the present work are much improved, although they are still missing the experimental point.

5.3. Single-particle energies

The results for the single-particle energies of occupied (hole) states with the Paris, RSC, SSC and Bonn potentials are given in Table V and Fig. 13. Agreement between the calculated and experimental energies is fairly good in all cases with the
Fig. 13. Calculated single-particle energies of $^{16}$O in UMOA with various NN potentials. The parameters used are the same as in Fig. 11.

Table VII. Contributions to the single-particle energies for the Bonn-B potential with $\rho_1=20$ and $\rho_2=10$. The $\Delta E^{(b)}$, $\Delta E^{(c)}$ and $\Delta E^{(d)}$ denote the second-order perturbative corrections corresponding to (b), (c) and (d) in Fig. 8. The values $T_e$ and $T_d$ are the three-body-cluster contributions defined in Eq. (4.19). The $\Delta E_{ls}$ is the $l$-$s$ splitting between the $0p_{1/2}$ and $0p_{3/2}$ states. All energies are in units of MeV:

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$0s_{1/2}$</th>
<th>$0p_{1/2}$</th>
<th>$0p_{3/2}$</th>
<th>$\Delta E_{ls}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>KE</td>
<td>10.50</td>
<td>17.50</td>
<td>17.50</td>
<td>0.00</td>
</tr>
<tr>
<td>PE</td>
<td>-54.24</td>
<td>-36.54</td>
<td>-40.07</td>
<td>3.53</td>
</tr>
<tr>
<td>$\Delta E^{(b)}$</td>
<td>-0.29</td>
<td>-0.38</td>
<td>-0.39</td>
<td>0.01</td>
</tr>
<tr>
<td>$\Delta E^{(c)}$</td>
<td>-4.43</td>
<td>-0.56</td>
<td>-1.55</td>
<td>0.99</td>
</tr>
<tr>
<td>$\Delta E^{(d)}$</td>
<td>8.25</td>
<td>4.55</td>
<td>3.59</td>
<td>0.96</td>
</tr>
<tr>
<td>$T_e$</td>
<td>-1.74</td>
<td>-1.82</td>
<td>-1.63</td>
<td>-0.19</td>
</tr>
<tr>
<td>$T_d$</td>
<td>3.90</td>
<td>2.76</td>
<td>3.06</td>
<td>-0.30</td>
</tr>
<tr>
<td>$T_s$</td>
<td>0.57</td>
<td>0.53</td>
<td>0.39</td>
<td>0.14</td>
</tr>
<tr>
<td>$T_f$</td>
<td>0.25</td>
<td>0.27</td>
<td>0.21</td>
<td>0.06</td>
</tr>
<tr>
<td>$T_e$</td>
<td>-1.97</td>
<td>-1.47</td>
<td>-1.72</td>
<td>0.25</td>
</tr>
<tr>
<td>Total</td>
<td>-39.19</td>
<td>-15.15</td>
<td>-20.60</td>
<td>5.45</td>
</tr>
<tr>
<td>Experiment</td>
<td>-40±8</td>
<td>-15.70</td>
<td>-21.80</td>
<td>6.20</td>
</tr>
</tbody>
</table>

five potentials used. Table VII shows the various components in the single-particle energies with the Bonn-B potential. The final results for $\rho_1=20$ and $\rho_2\rightarrow\infty$ are given in Table III for the Bonn-B NN potential.

In our calculation the $l$-$s$ splitting of the $0p$ orbits has been evaluated as 5.38 MeV with the Bonn-B potential, which should be compared with the experimental value, 6.2
It should be noted that the sizable contribution to the $l$-$s$ splitting comes from perturbative corrections (c) and (d) of Fig. 8. The core-polarization contribution of diagram (d) is consistent with the recent calculation\textsuperscript{74} with the same NN potential given by the $G$-matrix theory.

The TBC effects are also evaluated for the single-particle energies. The net effect of the TBC terms is not so large as seen in Table III. The calculated values for the corrections of the TBC terms with the Bonn-B potential are 1.15, 0.44, and 0.44 MeV for the $0_{3/2}$, $0p_{3/2}$, and $0p_{1/2}$ orbits, respectively. These corrections of the TBC terms are estimated to be less than 2% of the potential energies of the two-body-cluster terms. Similarly to the case of the binding energy, this fact suggests that the convergence of the cluster expansion in the present work is fairly good. It is noted that the TBC terms work to shift slightly the absolute values of the single-particle energies, but they hardly have a contribution to the $l$-$s$ splitting of the $0p$ orbits.

5.4. Effective interactions of the Op-shell states

We compare the calculated two-body matrix elements with the empirical matrix elements given by Hauge and Maripuu.\textsuperscript{79} These empirical matrix elements have been obtained on the basis of the Sussex matrix elements\textsuperscript{80} determined from the phase shift analysis in transfer reactions. Figure 14 shows the comparison of the empirical and calculated matrix elements with the Bonn-B potential. We see from the figure that the present calculation reproduces fairly well both the magnitudes and state dependence of the empirical matrix elements.

§ 6. Concluding remarks

We have discussed a general theory of effective interaction of both the non-Hermitian and Hermitian forms. We have seen that if we represent the effective interactions in terms of the mapping operator $\omega$ defined in § 2, all the effective interactions defined so far are written in a simple and unified form. We have discussed the structure of the Hermitian effective interaction and shown that the Hermitian effective interaction can be constructed from the non-Hermitian effective interaction. We have also discussed the perturbation expansion of the effective interaction for systems with degenerate and non-degenerate unperturbed energies in the time-independent approach. The expansion formulae for both the non-Hermitian and Hermitian effective interactions have been given in terms of the $Q$-boxes.
introduced by Kuo et al.\textsuperscript{2,9)}

We have derived a two-body effective interaction in the framework of the unitary-model-operator approach (UMOA) by applying the general theory of the effective interaction to the nuclear many-body theory. A unitary transformation has been introduced to describe two-body short-range correlations and determined from the condition that the effective interaction should be decoupled between high- and low-momentum states.

The characteristics of the present approach are as follows: (a) The effective interaction thus defined is $E$-independent and Hermitian. The decoupling property of the effective interaction works to reduce the number of diagrams to be evaluated. The $G$-matrix defined as the sum of ladder diagrams does not have this decoupling property.

(b) One can easily introduce a self-consistent single-particle potential for both occupied and unoccupied single-particle states. In the $G$-matrix theory, however, it has been very difficult to take the effects of selfconsistent potential into consideration for occupied and unoccupied states. It is because the $G$-matrix is $E$-dependent.

We have discussed the formal relation between the $G$-matrix theory and UMOA. We have shown that the Hermitian effective interaction in UMOA can be expressed in terms of the $G$-matrix, because the $G$-matrix corresponds to the $\hat{Q}$-box defined in the space of two-particle states. Therefore, many-body correlations considered in UMOA may be understandable from the standpoint of the $G$-matrix theory.

In general, the transformed Hamiltonian introduced in UMOA contains three-or-more-body interactions. We have made cluster expansion of the transformed Hamiltonian. We have shown that the present expression of the transformed Hamiltonian is based on the expansion in powers of the correlation operator. The leading higher-order corrections come from the three-body-cluster term. Therefore we have given an approximate method of evaluating the three-body-cluster term.

The present approach has been applied to the calculation of the ground-state properties of $^{16}$O. As the original NN interaction we have employed various types of potentials, namely, the Reid soft-core, Super-soft-core, Paris and Bonn potentials. We have calculated the ground-state energy, the single-particle energies of occupied orbits, the charge radius and the matrix elements of the effective interactions in the $0s$- and $0p$-shell states. Compared with the calculations made by the $G$-matrix theory or the coupled-cluster method, a large energy gain in the ground-state energy has been obtained for each of the NN potentials used. As for the charge radius, the calculated results are slightly small but very close to the experimental data, in contrast to those obtained in the Brueckner-Hartree-Fock calculation.\textsuperscript{75)}

We have also calculated the single-particle energies of occupied orbits. The theoretical predictions have reproduced well the experimental data. In particular, the $l$-$s$ splitting of the $0p$ states has been obtained in about 90% of the experiment.

We have calculated the two-body matrix elements of the effective interaction in the $0s$- and $0p$-shell states. The present calculation has shown that the difference between the theoretical and empirical matrix elements are considerably small.

It is noted that the dependence of the calculated result on the NN potentials is not so large as in the calculation in the Brueckner-Hartree-Fock method\textsuperscript{75)} and the
coupled-cluster method.\textsuperscript{60} However, from Table V, we can see clearly a weak dependence of the result on the NN potentials used. It would be interesting to examine this potential dependence in comparison with other calculations. In recent years the theory and method for calculating few-body systems have been in progress. Glökle and Kamada\textsuperscript{(76,77)} have solved the three-body Faddeev equation and the four-body Yakubovsky equation and obtained the rigorous solutions for the three- and four-body systems. We have observed that the dependence of the calculated binding energies of $^{16}\text{O}$ on the NN potentials resembles much of that for the results of the few-body systems. It has been known that the potential dependence comes mainly from the difference of the tensor-force strength.\textsuperscript{72,77} Many calculations have shown that the weaker tensor force brings about the larger binding energy. This trend holds also in the present calculation, which is consistent with the three- and four-body calculations.\textsuperscript{76,77}

The discrepancy between the theoretical predictions and the experiments is yet sizable. The present approach is based on a non-relativistic description of nuclei. It will be of interest to clarify other possibilities such as relativistic effects and to evaluate corrections to the non-relativistic approach. We believe that there exist certain effective interactions acting in a low-momentum model space which represent renormalization effects or corrections coming from the extended space such as high-momentum two-nucleon states, meson-nucleon states and so on. The effective-interaction theory will give us a promising way of solving the nuclear many-body problem, but there still remain many difficult tasks to be done before reaching the final goal.

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